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AN

ELEMENTARY COURSE

OF

MATHEMATICS,

PREPARED FOR THE USE OF THE

ROYAL MILITARY ACADEMY.

BY ORDER OF THE MASTER-GENERAL AND BOARD OF ORDNANCE.

Henry Drury Harness, editor.

VOLUME I.

CONTAINING

. ARITHMETIC AND ALGEBRA.

By W. Rutherford, Esq. LLD. & F.R.A.S.

APPLICATION OF ALGEBRA TO GEOMETRY, PLANE TRIGONOMETRY,
SPHERICAL TRIGONOMETRY, MENSURATION,
AND COORDINATE GEOMETRY OF TWO DIMENSIONS.

By Stephen Fenwick, F.R.A.S.

DIFFERENTIAL AND INTEGRAL CALCULUS.

By W. Rutherford, Esq. LLD. & F.R.A.S.

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LONDON:
AJESTY'S STATIONERY OFFICE:
BLISHED BY
59 HIGH HOLBORN.
1853.

(I–II.)

LONDON:
Printed by WILLIAM CLOWES and SONS, Stamford Street,
For Her Majesty's Stationery Office.

TO THE RIGHT HONOURABLE

LIEUTENANT-GENERAL LORD RAGLAN, G.C.B.,

MASTER-GENERAL OF THE ORDNANCE,

&c. &c. &c.

MY LORD,

I HAVE the honour to lay before you the course of Mathematics for the Royal Military Academy, which in compliance with the directions given by the Marquis of Anglesey, while Master-General of the Ordnance, has been compiled under my superintendence. It will, I trust, not only meet the present wants of the Cadets, but also prove useful for such references and further studies as their professional duties may render necessary after they receive their commissions. The cordial co-operation of the gentlemen employed upon it, and the care bestowed by each of them on the subjects intrusted to him, induce a confident hope that this will be the case, and that their labours will deserve your Lordship's approval.

The names of these gentlemen are published in the title-page of each volume to which they have contributed, but the valuable assistance afforded by Mr. Barlow, whose name is not thus recorded, also demands my grateful acknowledgment. The reluctance caused by a sense of my incompetency to undertake the editorship of a course extending far beyond the limits of my own information, disappeared when he consented to aid me. His great experience in the application of Mathematics to practical purposes—his long connexion with the Royal Military Academy as one of the Mathematical instructors—the numerous cases in which his former pupils have sought his assistance when their duties have compelled them to contend with difficult questions—and the interest he has evinced

in their welfare and reputation by the kindness with which he has always been ready to give that assistance—marked him out as the best adviser during the preparation of the work required; to him it has in all its stages been submitted, and to him must be attributed whatever merit its general conception and arrangement may possess.

As the time, which has elapsed since the first communication on the subject of this course was made to me, may be considered unnecessarily long, I am desirous to state that although the wishes of the Marquis of Anglesey were conveyed to me at the end of October, 1847, it was not until the middle of May, 1849, that the various official difficulties which prevented the commencement of the work were removed, and its preparation authorized.

I remain,
My Lord,
Your obedient Servant,
H. D. HARNESS,
Captain Royal Engineers.

18th October, 1852.

CONTENTS.

A COURSE

OF

MATHEMATICS.

PRELIMINARY DEFINITIONS.

MATHEMATICS is the science which treats of all quantities that can be numbered or measured. Its two great divisions are *pure mathematics* and *mixed mathematics*.

Pure mathematics consists of the three following divisions:—

1. *Arithmetic*, which treats of numbers or particular quantities;
2. *Algebra*, which treats of the relations of any quantities whatever under particular conditions, and may properly be termed Universal Arithmetic; and
3. *Geometry*, which treats of extended quantities, or continued magnitudes, as possessing three dimensions, viz., length, breadth, and thickness.

This last division embraces a much greater compass and variety of reasoning than either of the other divisions, and all of them are founded on the simplest notions of abstract quantities. The applications of these three divisions, one to another, form other important parts of pure mathematics.

Mixed mathematics is the application of the different parts of pure mathematics to those physical inquiries which are founded upon principles deduced from experiment or observation. It comprehends *Mechanics*, or the science of equilibrium and motion of bodies; *Astronomy*, in which the motions, distances, etc., of the celestial bodies are considered; *Optics*, or the theory of light, besides various other important subjects. In all these branches of mixed mathematics, if the first principles be accurately determined by experiment or observation, the results which are deduced are as certain and indisputable as those which can be deduced by geometry, or by any other part of pure mathematics, from axioms and definitions.

PRINCIPLES OF ARITHMETIC.

NUMERATION.

ART. 1. *Arithmetic* is that division of pure mathematics which treats of numbers, and of the method of performing calculations by means of them.

Number is a collection of several objects of the same kind, or of many

separate parts. It is one of the forms of *magnitude*, an attribute or quality of objects by which they are conceived to be susceptible of increase or diminution. The other form of magnitude is distinguished by the connexion or continuity of the parts,—being an entire mass without distinction of parts; whereas in number the consideration is merely how many parts it contains. The definition of number supposes the existence of *one* of the things or parts of which it is composed, taken as a term of comparison, and which, in that case, is denominated *unity*.

2. Some knowledge of numbers must have existed in the earliest ages of the world. The ten fingers with which man had been formed, the flocks and herds which he had acquired, and the variety of objects that surrounded him, would all contribute to impress his mind with a notion of number. While small numbers only were required, the ten fingers would furnish the most convenient way of reckoning them, since with his fingers any person could make those little calculations which his limited wants required. He would name all the different collections of his fingers, and frame appropriate words, in his own language, answering to *one*, *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine*, and *ten*. As his wants increased he would proceed to higher numbers, adding one continually to the former collection, as he advanced from lower numbers to higher. He would soon perceive that there is no limit to the different numbers that may be formed, and consequently that it would be impossible to express them in ordinary language by distinct names independent of each other. By arranging numbers in groups or classes, they might be expressed by a comparatively small number of words, still the continual repetition which unavoidably occurs in calculation would necessarily preclude the use of names of numbers, except in operations of the very simplest character.

3. The English names of numbers have been formed from the Saxon language, by combining the names of the first ten numbers mentioned in the preceding article.

Thus, *eleven*, signifying that one is left after ten is taken, or ten and one.
twelve, signifying that two is left after ten is taken, or ten and two.

thirteen, ten and three.	*twenty-three*, two tens and three.
fourteen, ten and four.	*thirty*, three tens.
fifteen, ten and five.	*forty*, four tens.
sixteen, ten and six.	*fifty*, five tens.
seventeen, ten and seven.	*sixty*, six tens.
eighteen, ten and eight.	*seventy*, seven tens.
nineteen, ten and nine.	*eighty*, eight tens.
twenty, two tens.	*ninety*, nine tens.
twenty-one, two tens and one.	a *hundred*, ten tens.
	a *thousand*, ten hundreds.
twenty-two, two tens and two.	a *million*, ten hundred thousand, or one thousand thousand, etc.

4. For facilitating calculations it would be found necessary to substitute short and expressive signs for words, and when some few signs or characters had been chosen, to combine them so as to represent the names of all other numbers whatever. We shall here show how this has been done by the Greeks and Romans, and then advert to the admirable system of notation which so generally prevails among different nations of the world at the present time.

5. *The Greek Notation.*—At a very early period the Greeks had recourse to the twenty-four letters of their alphabet for the representation of numbers, and by means of these, aided by the three Hebrew characters ϛ, ϟ, ϡ, they expressed the first three orders of numbers. Thus the numbers one, two, three, etc., to nine, were represented by

α, β, γ, δ, ε, ϛ, ζ, η, θ . . . (1st order, or units).

The numbers ten, twenty, thirty, etc., to ninety, by

ι, κ, λ, μ, ν, ξ, ο, π, ϟ . . . (2nd order, or tens);

and the numbers one hundred, two hundred, etc., to nine hundred, by

ρ, σ, τ, υ, φ, χ, ψ, ω, ϡ . . (3rd order, or hundreds).

Instead of multiplying distinct characters for higher numbers, they had recourse to their characters for the units, and by subscribing a small iota or dash, they denoted *one thousand* by α_{ι}, *two thousand* by β_{ι}, and so on. With these characters the Greeks could express every number under *ten thousand*. Thus

θ_{ι} ϡ ϟ θ signified nine thousand nine hundred and ninety-nine,
δ_{ι} τ π β „ four thousand three hundred and eighty-two,
γ_{ι} α „ three thousand and one.

In order to express higher numbers, they made use of the letter M, which, on being written below any character, increased its value *ten thousand* times. This contrivance enabled them to express all numbers as far as *hundreds of millions*; but instead of subscribing the letter M, it was afterwards found more convenient to write the letters Mυ., or the contraction for μυρια, ten thousand, after the character whose value was to be increased ten thousand times; and then, when lower periods failed, they repeated the letters Mυ. Thus

λδ Mυ. Mυ. Mυ. signified *thirty-four trillions.*

Archimedes, the most inventive of the Greek philosophers, divided numbers into periods of eight symbols each, which were called *octades;* and the famous Appollonius again divided them into periods of four symbols each, the first period on the left being *units*, the second *myriads*, the third *double myriads*, and so on. In this manner Appollonius was able to write any number which could be expressed by the present system of numeration. Having thus given a *local* value to his periods of four, it was remarkable that Appollonius did not perceive the advantage of making the period consist of a less number of characters. Had he done the same thing with every single character, he would have arrived at the system now in common use, and this oversight is the more remarkable as the cipher was not unknown to the Greeks, but confined exclusively to their sexagesimal operations.

6. *The Notation of the Romans.*—The traces or strokes which originally represented numbers were replaced by those characters of the Roman alphabet which most nearly resembled them. The Roman notation was much ruder than the Greek, and for the expression of number they made use of the seven following capital letters, viz.:—

I for *one;* V for *five;* X for *ten;* L for *fifty;* C for a *hundred;* D for *five hundred;* and M for a *thousand.*

By various repetitions and combinations of these they expressed all numbers. The four combined strokes which originally formed the character M for a thousand, assumed afterwards a rounded shape, fre-

quently expressed by the compound character CIↃ, consisting of the letter I inclosed on both sides by C, and by the same character reversed. This last form, by abbreviation on either side, gave two portions, one of which IↃ was condensed into the letter D and expressed five hundred. The practice of using *duodeviginti* for *octodecim*, and so on, led the Romans to the application of *deficient* numbers; and instead of writing VIIII for nine, they counted *one* back from *ten*, and placing I before X, they wrote it thus, IX. In a similar manner XIX represented *nineteen*, XL *forty*, XC *ninety*, and CM *nine hundred*. They also repeated the symbols of a thousand to denote higher numbers; thus CCIↃↃ represented *ten thousand*, and CCCIↃↃↃ an *hundred thousand*. Separating each of these, gives IↃↃ for *five thousand*, and IↃↃↃ for *fifty thousand*. Also a horizontal line drawn over any letter augmented its value *one thousand* times; thus $\overline{\text{LX}}$ signified *sixty thousand*. With this explanation the following examples will be readily understood:—

I one.	XIII thirteen.	LXX seventy.
II two.	XIV fourteen.	LXXX eighty.
III three.	XV fifteen.	XC ninety.
IV four.	XVI sixteen.	C one hundred.
V five.	XVII seventeen.	CC two hundred.
VI six.	XVIII eighteen.	D or IↃ five hundred.
VII seven.	XIX nineteen.	M or CIↃ one thousand.
VIII eight.	XX twenty.	MM two thousand.
IX nine.	XXX thirty.	$\overline{\text{V}}$ or IↃↃ five thousand.
X ten.	XL forty.	$\overline{\text{X}}$ ten thousand.
XI eleven.	L fifty.	$\overline{\text{XC}}$ ninety thousand.
XII twelve.	LX sixty.	$\overline{\text{M}}$ one million.

Thus as often as any symbol is repeated, its value is repeated as often; a symbol of less value placed after one of greater value is added to the greater, but if placed before a symbol of greater value, it is subtracted from it. Also every Ↄ added to IↃ increases its value ten times, and if a C be placed before CIↃ and a Ↄ after it, its value is increased ten times, and so does every additional C and Ↄ. This notation is still frequently employed in distinguishing dates, the chapters and sections of books, and so on.

7. Although the Greek arithmetic, as successively moulded by the ingenuity of Archimedes, of Appollonius, and of others, had attained to a high degree of perfection, and was capable of performing operations of very considerable difficulty and magnitude, still the great and radical defect of the system consisted in the entire absence of a general mark corresponding to our cipher, which without having any value in itself, should yet serve to keep the rank or power of the other characters, by occupying the vacant places in the scale of numeration. From the preceding remarks on the notation of the Greeks and Romans, the student will be able to form some idea of the great superiority of the present system, which has led to some of the most striking and remarkable scientific discoveries.

8. In the common system of numeration all numbers, however large or small, can be expressed by the ten following characters or figures, viz.:—

1 2 3 4 5 6 7 8 9 0
one, two, three, four, five, six, seven, eight, nine, nothing.

The first nine of these are called *significant figures* or *digits*, and sometimes represent units, sometimes tens, hundreds, or higher classes. When placed singly, they denote the simple numbers subjoined to the characters; when several are placed together, the first figure on the right is taken for its simple value, the next signifies so many tens, the third so many hundreds, and the others so many higher classes corresponding to the order in which they are placed. Thus 4532 signifies *four thousand*, *five hundred*, *thirty*, and *two units;* and in the number *two hundred* and *twenty-two*, which is written thus 222, the figure 2 is repeated thrice, but each has a different value; the first on the right hand is two units, the second two tens or twenty, and the third two hundreds.

9. When any of the denominations, units, tens, hundreds, etc., is wanting, it becomes necessary to supply its place with the last sign or character, viz., 0, which is termed *cipher* or *nothing*, the word cipher in the Arabic language signifying *vacuity*. This character which indicates the absence of all number, is a most important one, inasmuch as its introduction serves to preserve the proper positions of the significant figures. Thus the number *forty thousand three hundred* and *twenty* would be expressed in figures by 40320, because the denominations, units, and thousands are wanting, and the absence of each is indicated by the cipher which occupies its place. From these illustrations we may perceive that the superiority of our present system of numeration arises from a few simple signs being made to change their value as they change the position in which they are placed, and that the significant figures have a *local* as well as a *simple* value. It is thus that, in consequence of the established relative value of units and tens, *the same figure which, beginning on the right, expresses units, becomes ten times greater at each remove to the left, and by simply changing their places, the different characters become susceptible of representing successively all the different collections of units which can possibly enter into the expression of a number.* Thus we get—

10 11 12 13 14 15
ten, *eleven*, *twelve*, *thirteen*, *fourteen*, *fifteen*,
16 17 18 19
sixteen, *seventeen*, *eighteen*, *nineteen;*

where the first figure on the left signifies ten, and the second figure its simple value, or so many units. Hence 10 means ten and nothing; 11 ten and one, and so on. Again, 20 means two tens and nothing, or *twenty;* 21 two tens and one, or *twenty-one;* 30, thirty; 90, ninety; 100, ten tens or *one hundred;* and 1000, *one thousand.* The names and values of numbers will be readily acquired from the following examples.

Billions.	Hun. thou. millions.	Ten thou. millions.	Thousands of millions.	Hundreds of millions.	Tens of millions.	Millions.	Hundreds of thousands.	Tens of thousands.	Thousands.	Hundreds.	Tens.	Units.
6;	8	3	9,	2	8	7;	6.	5	1,	2	3	4

The figures which compose a large number are separated into periods and half-periods, for the more readily ascertaining the precise position which each figure occupies. The period consists of six figures, and the first, beginning on the right, is called the period of units, the second the period of millions, the third the period of billions, a contraction for millions of millions or bi-millions, and so on. Thus the number

$$\overbrace{490{,}386}^{\text{Trillions.}};\ \overbrace{407{,}135}^{\text{Billions.}};\ \overbrace{017{,}693}^{\text{Millions.}};\ \overbrace{125{,}076}^{\text{Units.}}$$

is read thus:—Four hundred and ninety thousand three hundred and eighty-six trillions; four hundred and seven thousand one hundred and thirty-five billions; seventeen thousand six hundred and ninety-three millions; one hundred and twenty-five thousand and seventy-six.

10. There does not appear to be any number naturally adapted for constituting a class of the lowest or any higher rank to the exclusion of others; though it is very probable that our present system of numeration had its origin in the practice of reckoning with the ten fingers. The number ten is called the *radix* or *scale* of the common system, because in it we ascend by collections of ten in each class to the next higher class, and though almost all nations have adopted this number as the base of their system of numeration, still it is perfectly arbitrary, and convenient as it may be for general use, there may be other scales, such as the duodenary, whose base is 12, which possess superior advantages. But whatever be the scale of notation made use of, the same principle will enable us to write all numbers in that scale. Thus in the *quinary* scale whose radix is 5, we need only the five characters, 0, 1, 2, 3, 4, and *each figure placed on the left of another will have a value five times greater than if it occupied the place of this last.* Hence in this scale, 10 means *five,* 11 *six,* 12 *seven,* and so on. In this as well as in every other scale, except the denary or decimal one, we find a difficulty in enunciating a number, because there is no longer an accordance with the decimal language which pervades the construction not only of our own, but of all civilized languages.

11. Numbers may be viewed in two ways, either by considering them without particularizing the unit to which they refer, or by designating what they are intended to enumerate. Thus two, three, five, seven are *abstract numbers,* while three men, five days, seven books are *concrete numbers.* It is evident that the formation of numbers, by the successive re-union of units, does not depend upon the nature of these units, since 5 days and 7 days together make 12 days, 5 acres and 7 acres together make 12 acres, and 5 and 7 together make 12.

12. Since numbers can only be changed by increasing or diminishing them, it follows that the whole art of arithmetic is comprehended in two operations, which are termed *Addition* and *Subtraction.* But as it is frequently required to add several equal numbers together, as well as to subtract several equal sums from a greater, till it be exhausted, other methods have been devised for facilitating the operation in these cases, and named respectively *Multiplication* and *Division.* These four rules are the foundation of all arithmetical operations whatever.

ADDITION.

13. ADDITION is the collecting together of two or more numbers, and the amount of all of them is termed their *sum.* The sign +

(*plus*) is employed to indicate addition, and $7 + 2$ signifies that 2 is to be added to 7. Also the sign $=$ (*equal*) signifies that the numbers between which it is placed are equal: thus $8 + 1 = 9$.

Examples.

1. Let it be required to find the sum of the two numbers 1724 and 4638. Take them to pieces, separating them into thousands, hundreds, tens, and units. Thus—

1724 = 1 thousand, 7 hundreds, 2 tens, and 4;
4638 = 4 thousands, 6 hundreds, 3 tens, and 8.

To each of the four parts into which the first number is separated add the part of the second which is under it, beginning at the units. Thus 8 units and 4 units are 12 units; that is, 1 ten and 2 units; again, 3 tens and 2 tens are 5 tens; 6 hundreds and 7 hundreds are 13 hundreds, or 1 thousand and 3 hundreds. Lastly, 4 thousands and 1 thousand are 5 thousands; hence the sum is either

5 thousands, 13 hundreds, 5 tens and 12 units;
or 6 thousands, 3 hundreds, 6 tens and 2 units = 6362.

2. Let it be required to find the sum of 26389, 38127, 2815, 6817, 490, 25 and 3745.

$$\begin{array}{r} 26389 \\ 38127 \\ 2815 \\ 6817 \\ 490 \\ 25 \\ 3745 \\ \hline 78408 \end{array}$$

Write the numbers, as at the side, so that the figures of the same class shall be in the same vertical columns. Then taking the sum of each class, we find there are 38 units, 27 tens, 31 hundreds, 25 thousands, and 5 tens of thousands. Now 38 units are 3 tens and 8 units; then writing 8 below the units' column, carry the 3 tens to the 27 tens, which together make 30 tens, or 3 hundreds and 0 tens. Write 0 below the column of tens, and reserve the 3 hundreds to be added to the 31 hundreds. This gives 34 hundreds or 3 thousands and 4 hundreds, and writing 4 below the column of hundreds, carry the 3 thousands to the 25 thousands, and we get 28 thousands, or 2 tens of thousands and 8 thousands. Writing 8 below the column of thousands, carry the 2 tens of thousands to the 5 tens of thousands, and finally write 7 below the column of tens of thousands, making the entire sum = 78408.

14. From these principles the following rule may be drawn:—

Rule. *Write the numbers to be added together in vertical columns, so that the units of all the numbers may be in one column, the tens in the second, the hundreds in the third, and so on. Draw a line under the last number, and, beginning with the column of units, add successively the numbers contained in each column: if the sum does not exceed nine, write it down under the line; but if it contain tens, reserve them to be added to the next column, writing down only the units of each column; and under the last column put the entire sum whatever it may be. If the sum of any column be an exact number of tens, write 0 for the units and carry the tens to the next column.*

15. The results of the partial additions being furnished by the memory, it is desirable to have some plan of testing the accuracy of the final sum, and this may be done in various ways; but we shall only mention the two following methods of the *proof of addition:*

1. Having found the sum in the usual way, begin at the top and

add the numbers together downward; then if the summation is the same by both methods, it is very probable that the sum is correctly obtained.

2. Separate the numbers into two or three parts; find the sum of each separately, and then add all these partial sums together, which will give the whole amount. It is usual to separate the numbers into two parts only, the uppermost number forming one part.

SUBTRACTION.

16. SUBTRACTION is the taking a less number from a greater, and finding their *difference.* The process of subtraction involves two principles:—the one is the equal augmentation or diminution of each of the numbers. In either way the difference of the two numbers will not be altered; for if the greater number be either increased or diminished by 7, for example, and the less be likewise increased or diminished by 7, the numbers themselves will be altered but not their difference. The other principle is this: since 12 exceeds 7 by 5, and 8 exceeds 6 by 2, then 12 and 8 together, or 20, exceed 7 and 6 together, or 13, by 5 and 2 together, or 7. The sign − (*minus*) is used to indicate subtraction, and 9 − 7 signifies that 7 is to be taken from 9.

EXAMPLES.

1. Let it be required to take 231 from 574.

Write the numbers as in the margin, units under units, tens under tens, and hundreds under hundreds. Then

	Hund.	Tens.	Units
4 units exceed 1 unit by 3 units.	5	7	4
7 tens exceed 3 tens by 4 tens.	2	3	1
5 hundreds exceed 2 hundreds by 3 hundreds.	3	4	3

Therefore by the second principle all the first column together exceeds all the second column together by all the third column together, that is, by 3 hundreds 4 tens and 3 units, or 343, which is the difference between 574 and 231.

2. Let it be required to subtract 23957 from 802126.

Write the numbers at length; thus

		Hund. Thous.	Ten Thous.	Thous.	Hund.	Tens.	Units.
802126	=	8	0	2	1	2	6
23957	=		2	3	9	5	7
778169		–	–	–	–	–	–

Now here a difficulty immediately arises, since 7 is greater than 6, and cannot be taken from it, neither can 5 be taken from 2; 9 from 1; 3 from 2; nor 2 from 0. To obviate this we must have recourse to the first principle, and add the same number to both of these numbers, which will not alter their difference. Add ten to the first number, making 16 units; and add ten also to the second number, but instead of adding ten to the number of units, add one to the number of tens, making 6 tens. Again add ten tens to the first number and one hundred to the second; then add ten hundreds to the first and one thousand to the second, and so on, adding equal numbers to each. In this way the numbers will be changed into the following:—

Hund. Thous.	Ten Thous.	Thous.	Hund.	Tens.	Units.
8	10	12	11	12	16
1	3	4	10	6	7
7	7	8	1	6	9

and the difference 778169 is obtained in the usual manner. Hence, *when the upper figure is the less, we must augment it by ten, and retain one to be added to the lower figure immediately to the left.*

The *proof of subtraction* is deduced from the simple fact, that the difference added to the smaller number is equal to the greater number.

MULTIPLICATION.

17. Multiplication is the finding the amount of a number repeated any number of times. The number which is repeated is called the *multiplicand*, the number denoting the repetitions is called the *multiplier*, and the amount the *product*. The multiplicand and multiplier are termed the *factors* of the product, and the sign $\times$ (*into*) denotes multiplication. Thus 12×3 signifies that 12 is to be repeated three times, and added together; thus $12 + 12 + 12 = 36$.

18. When the multiplicand and multiplier are large numbers, as 1269 and 423, we should have to write 1269, the multiplicand, 423 times, and then to make an addition of enormous length. This operation can be abridged, however, by reducing it into a certain number of partial multiplications which may be easily effected mentally; but previous to explaining a shorter method, the following table must be committed to memory.

Multiplication Table.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

To form this table, write the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, in the first horizontal line; then add each of these numbers to itself to form the second line, which is composed of the products of

each of these numbers multiplied by 2. To each number in the second line add the corresponding one in the first, and the third line is formed, containing the several products of the numbers in the first line multiplied by 3. Again adding the numbers in the third line to the corresponding ones in the first, a fourth line is formed, containing the products of each number of the first line by 4; and so on to the last line. The table may be extended if required. If we take any of the numbers in the first line, as 8, and proceed downwards, we shall find the same succession of numbers as if we had taken 8 at the side and proceeded to the right; hence $8 \times 5 = 40 = 5 \times 8$. This may be shown in the following manner. Place 8 counters in a line, and repeat that line 5 times; then the number of counters in the whole is 5 times 8 if they are counted by rows from the top to the bottom; but if they are counted by vertical columns, we shall find eight rows with five in each row, the whole number of which is 8 times 5. Hence we see that

$$8 + 8 + 8 + 8 + 8 = 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5,$$
$$\text{or } 8 \times 5 = 5 \times 8.$$

This method of proof may be applied to any numbers beyond the range of the table, and hence *in any multiplication the order of the factors may be changed, that is, either of them may be taken as the multiplier.*

19. Let it be required to multiply 739 by the single figure 8.

Since the product of 739 by 8 is evidently equal to the sum of the products of all its parts, we have the following operation:—

Thous.	Hund.	Tens.	Units.		
0	7	3	9	739	
			8	8	
		7	2	72	= product of 9 by 8
	2	4		240	= product of 30 by 8
5	6			5600	= product of 700 by 8
5	9	1	2	5912	= product of 739 by 8.

In practice the partial products 72, 240, and 5600, are not written down, but combined mentally into one sum: thus we say 8 times 9 are 72, write down 2 and reserve the 7 tens; then 8 times 3 are 24, and the reserved 7 added thereto gives 31, write down 1 and carry the 3 to the product of 8 by 7 or to 56 hundreds, and the entire number of hundreds is 59, the whole product being 5912.

$$\begin{array}{r} 739 \\ 8 \\ \hline 5912 \end{array}$$

20. Find the products of 2376 multiplied by 10 and by 100.

To multiply any number by 10, we have only to remove each of the figures of the multiplicand one place to the left and their value will be increased ten times; hence $2376 \times 10 = 23760$. In like manner $2376 \times 100 = 237600$, where the value of each figure is increased a hundred times.

Hence any number will be multiplied by 10, 100, 1000, etc., by writing on the right of the multiplicand as many ciphers as there are in the multiplier.

21. When the significant figure of the multiplier is not a unit, as for example 30, 400, or 7000. Since these multipliers are the same as 10 times 3, 100 times 4, or 1000 times 7; the multiplicand is first multiplied by the significant figure 3, 4, or 7, by Art. 19, and afterwards the

product is multiplied by 10, 100, or 1000, as in Art. 20, by writing one, two, or three ciphers on the right of the product. Thus to multiply 468 by 700, we have the operation in the margin.

$$\begin{array}{r} 468 \\ 700 \\ \hline 327600 \end{array}$$

22. Let it be required to multiply 3729 by 563, where both factors consist of several figures. This is merely to repeat 3729 thrice, 60 times and 500 times, and then to add the whole together. We first multiply 3729 by 3; then by 6 annexing a cipher to the right of the product (20), and lastly, by 5 annexing two ciphers (20). In practice the ciphers on the right may be omitted, provided the first significant figure be made to occupy its proper place in the partial products.

$$\begin{array}{rcl} 3729 & & \\ 563 & & \\ \hline 11187 & = & 3 \text{ times } 3729 \\ 223740 & = & 60 \text{ times } 3729 \\ 1864500 & = & 500 \text{ times } 3729 \\ \hline 2099427 & = & 563 \text{ times } 3729 \end{array}$$

If one or more of the figures of the multiplier be 0, the corresponding partial product or products will be 0, and the lines may be entirely omitted, recollecting to give its proper value to the product arising from multiplying by the next figure. Also if the multiplier be the product of two or more numbers, as 32, which is the product of 8 and 4, we may multiply first by 8 and the product thence arising by 4; or first by 4 and then by 8. This principle is evident, since 4 times any number, repeated 8 times, is the same as repeating that number 32 times. We have, in all cases of multiplication, the following rule:—

RULE. Place the multiplier under the multiplicand, so that the units of the former may be under those of the latter; multiply the whole multiplicand by each figure of the multiplier (19), and place the unit of each line in the column under the figure of the multiplier from which it came; then add all these partial products together, and their sum will be the entire product of the two factors. If the multiplicand contain a cipher, treat it as if it were a number, recollecting that $0 \times 1 = 0$, $0 \times 2 = 0$, and so on.

23. *To prove Multiplication.* Cast the nines out of the sum of the digits of the multiplicand, multiplier, and product separately, and set down each remainder at the side of the number from which it came. Multiply the first two remainders together and cast out the nines from this product, if the sum of the digits exceed nine; then if the remainder which thus arises is the same as that from the product of the two factors, the operation is very likely to be correct, unless there be some compensation of errors, or some figures misplaced.* Thus in the annexed

* This method of proof depends on a property of the number 9, which belongs to no other digit, except 3. It is this: *any number divided by 9 will leave the same remainder as the sum of its figures or digits divided by 9.*

For take the number 563, for instance, which is equal to $500 + 60 + 3$. Now

$$500 = 5 \times 100 = 5 \times (99 + 1) = 5 \times 99 + 5,$$
$$60 = 6 \times 10 = 6 \times (9 + 1) = 6 \times 9 + 6;$$

hence $563 = 5 \times 99 + 6 \times 9 + 5 + 6 + 3$, and $5 \times 99 + 6 \times 9$ contains an exact number of nines; therefore if the number 563 be divided by 9, it will leave the same remainder as $5 + 6 + 3$ divided by 9. The same is true for every other number.

Let now h and k denote the number of nines in the multiplicand and multiplier, and a and b the remainders, then the numbers will be $9h + a$ and $9k + b$, and their product is $(9h + a)$ repeated $9k$ times, $9h$ repeated b times, and the product of a by b. Now these products are each an exact number of nines; except the last $a \times b$; but a and b are the remainders after the nines are cast out of the two factors, and hencet he remainder, after casting the nines out of the product of these factors, must be the same as the remainder after the nines are cast out of the product $a \times b$.

example, we say (omitting the 9) 3 and 7 are 10; then 1 and 6 are 7, which write down. Again, 2 and 8 are 10, then 1 and 3 are 4, which is also put down near the multiplier. Lastly, the product of 4 and 7 is 28, and 2 and 8 are 10, which is 1 above 9. Write then 1 near the product, and cast the nines out of it thus: 1 and 8 are 9; 8 and 2 are 10; 1 and 5 are 6 and 3 are 9; 2 and 8 are 10, which being 1 above 9, shows that the operation most probably is correct.

$$\begin{array}{lr}
\text{Multiply} & 90376\ldots7 \\
\text{By} & 2083\ldots4 \\
\hline
 & 271128 \\
 & 723008 \\
 & 180752 \\
\hline
 & 188253208\ldots1
\end{array}$$

DIVISION.

24. Division is the finding how many times a less number is contained in a greater, or how many times it may be taken out of the greater.

The number to be divided is called the *dividend;* the number to divide by is called the *divisor*, and the number of times the less can be taken out of the greater is called the *quotient.* If a number is left after the division is finished, it is called the *remainder.* The sign ÷ (*divided by*) indicates division.

25. Let it be required to divide 28 by 7, or to find how many sevens the number 28 contains. This is done by subtracting 7 from 28, and then subtracting 7 from the remainder, and so on as often as it can be done. Then count the number of subtractions, and the quotient is obtained. In this example the quotient is 4, because we have found that 28 contains 7 four times. So long as the quotient is a small number, this process of continued subtraction may be employed, but when the dividend contains the divisor a large number of times, it would be necessary to abridge the operation, by taking away as many times the divisor at once as we please, provided the number of times is marked at each step. For example, to divide 115 by 12 we may take away 8 times 12 at once from 115, and afterwards take away 12; therefore 12 may be subtracted 9 times, and the remainder is 7.

$$\begin{array}{r}
28 \\
7 \\
\hline
21 \\
7 \\
\hline
14 \\
7 \\
\hline
7 \\
7 \\
\hline
0
\end{array}$$

$$\begin{array}{rl}
115 & \\
96 & = 8 \text{ times } 12 \\
\hline
19 & \\
12 & = \text{ once } \ 12 \\
\hline
7 & = \text{remainder.}
\end{array}$$

26. Let it be required to divide 3168 by 27. Here the quotient will consist of three digits, and therefore there will be at least 3 separate subtractions. Now the figure in the hundred's place cannot be more than 1, and if the partial product 27 hundreds, or 2700, be subtracted from the total product 3168, the remainder 468 must contain the products of the tens and units of the quotient multiplied by the divisor 27, and thus the question is reduced to another of a similar character, viz., to divide 468 by 27. We now inquire how often 27 is contained ten times in 468, and this is found to be only once ten times; then subtracting the partial product 27 tens or 270 from 468, the remainder is 198. Lastly, we have to divide 198 by 27, which gives 7 for a quotient, and a remainder 9; and, therefore, 3168 contains 27, 100 + 10 + 7, or 117 times, leaving 9 for remainder. It will be readily seen that as often as 27 is contained in 31, so many

$$\begin{array}{rcr}
3168 & & \\
2700 & = & 100 \text{ times } 27 \\
\hline
468 & & \\
270 & = & 10 \text{ times } 27 \\
\hline
198 & & \\
189 & = & 7 \text{ times } 27 \\
\hline
9 & & 117 \text{ times } 27
\end{array}$$

hundred times it will be contained in 3100 or in 3168; and as often as 27 is contained in 46 so many ten times it will be contained in 460 or 468, and in this manner any quotient figure is just as readily obtained as the last or units' figure of it.

27. The preceding articles contain the principles of division, and all that remains is to apply them in the most economical way. Suppose we have to divide 2987618 by 3605.

Operation with ciphers in full.	Operation without annexing ciphers.
3605)2987618(800 + 20 + 8,	3605)2987618(828
2884000 or 828.	28840
103618	10361
72100	7210
31518	31518
28840	28840
2678	2678

Hence we may deduce the following rule:—

Rule.—Write the divisor and dividend in one line, and place parentheses on each side of the dividend. Take off from the left hand of the dividend the least number of figures which make a number not less than the divisor; find what number of times the divisor is contained in these, and write this number as the first figure of the quotient. Multiply the divisor by this figure, and subtract the product from the number which was taken off at the left of the dividend. On the right of the remainder place the next figure of the dividend, and if the remainder thus increased be greater than the divisor, find how many times the divisor is contained in it; put this number at the right of the first figure of the quotient, and proceed as before; but if not, on the right place the next figure of the dividend, or more, until it is greater; recollecting to place a cipher in the quotient for every figure of the dividend so taken, except the first. Find how often the divisor is contained in this number, and proceed in this way until all the figures of the dividend are exhausted.

28. When the divisor is not greater than 12, the subtraction is performed mentally, and the figures of the quotient are written successively under those of the dividend. Also, if the divisor be the exact product of two or more numbers, each of which is not greater than 12, the dividend may be divided by one of these numbers, the quotient thus obtained by the next, and so on, as in the following example:—

Divide 8327965 by 72 and also by 99.

72	9	8327965	99	9	8327965
	8	925329....4		11	925329...4
		115666....1			84120...9

To deduce the remainders which would have been left, had the divisions been performed by 72 and 99 in the usual way, we may observe that the first partial remainder 4 must be *units;* but the second dividend being so many collections of 9 units each, the *second* remainder must be regarded as so many collections of 9 units each; hence the true remainders in these examples are respectively

$$1 \times 9 + 4 = 13, \text{ and } 9 \times 9 + 4 = 85.$$

Division may also be abridged when the divisor is terminated by a cipher or ciphers. Thus cut off as many figures from the right of the dividend as there are ciphers on the right of the divisor; proceed with the remaining figures in the usual manner, and to the right of the remainder annex these figures which were cut off from the dividend.

29. *To prove Division.* Multiply the quotient by the divisor, or the divisor by the quotient, and to the product add the remainder, if there be one. The result ought to be the same as the dividend; because we are only adding the divisor the same number of times as it was subtracted in the operation of division.

30. As the principles of arithmetic can only appear in their full extent when they can be adapted to any scale whatever, it will be useful to show how numbers expressed in one scale may be represented in any other scale, and how the fundamental operations may be performed with numbers expressed in any scale. When more characters than ten are required, as in the duodenary scale, we shall represent 10 by π and 11 by ε. Now all numbers can be expressed in the *binary* scale by two characters 0, 1; in the *ternary* scale by three, 0, 1, 2; in the *quaternary* by four, 0, 1, 2, 3; in the *duodenary* or *duodecimal* scale by 12, viz., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, π, ε, and so on. Let it be required to transform the number 10011101 from the binary to the *denary* or decimal scale.

Since in the binary scale each figure has a value two times greater for each remove to the left, it is obvious that the figure 1 on the left is 7 removes from the units, and expresses $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ or 128 units. The next 1 is 4 places from the units, and it therefore expresses $1 \times 2 \times 2 \times 2 \times 2$ or 16 units; the next 1 will express 8 units; the next 4 units, and consequently the entire number of units is equal to $128 + 16 + 8 + 4 + 1 = 157$. In a similar manner the number represented by $3\,\pi\,8$ in the duodenary scale is expressed by $3 \times 12 \times 12 + \pi \times 12 + 8 = 432 + 120 + 8 = 560$ in the denary scale. These calculations being attended to, it will be easy to see that the same thing may be effected in a more simple manner as in the margin. Thus multiply the left hand figure by 2 and add the figure on the right; multiply the sum 2 by 2 and add the next figure on the right; multiply the sum 4 by 2 and add the third figure on the right, which gives 9; continue this process till all the figures have been taken in, and the last sum will be the equivalent number in the common or denary scale. In this manner a number expressed in any scale may be transformed into an equivalent number in the denary scale. If it be required to convert a number from the common to any other system, we must reverse the preceding process, and *divide* successively by the base of the system in which the number is to be expressed, and the successive remainders will be the figures in the different places, reckoning from the right or units' place. Thus, to convert the number 560 into the duodecimal scale, we divide 560 by 12, which gives 46 and 8 over; hence 8 must be the figure on the right hand of the number; then $46 \div 12$ gives 3 and π over; hence $3\,\pi\,8$

$$\begin{array}{r} 10011101 \\ 2 \\ \hline 2 \\ 2 \\ \hline 4 \\ 2 \\ \hline 9 \\ 2 \\ \hline 19 \\ 2 \\ \hline 39 \\ 2 \\ \hline 78 \\ 2 \\ \hline 157 \\ \hline \end{array}$$

is the number expressed in the duodecimal scale. If a number is to be transformed from one scale to another, neither of which is the decimal one, we must first transform it into the decimal scale and then into the required scale, by the process just adverted to. Thus 226 in the septenary scale is equal to 118 in the denary scale; and this again is equal to $9\,\pi$ in the duodenary scale; consequently 226 in the septenary scale expresses the same number as $9\,\pi$ in the duodenary one. We shall only add one example in illustration of the operation of multiplication in the system whose base is 12, and the student will feel no difficulty in performing the fundamental operations of arithmetic in any system whatever.

Suppose it required to multiply $56\,\varepsilon\,7$ by $30\,\pi\,6$.

By the decimal scale.	By the duodecimal scale.
9643	$56\,\varepsilon\,7$
5310	$30\,\pi\,6$
9643	295 9 6
28929	$4797\,\pi$
48215	$148\pi9$
51204330	151941 7 6

The student may convert the one of these products into the other, and thus test the accuracy of both multiplications. In a similar manner the fundamental operations may be performed in any system of notation.

31. It may be useful to notice here two or three properties and principles connected with the theory of numbers.

We have already seen that a number is divisible by 9 or 3, when the sum of its digits is divisible by 9 or 3. A number is divisible by 2 when the unit's figure is an even number, or 0; it is divisible by 4 when the last two figures are divisible by 4, for every digit except the last two is a number of hundreds, and 100 is divisible by 4 without remainder. A number is divisible by 8 when the last three figures are divisible by 8, for every digit except the last three is a number of thousands, and 1000 is divisible by 8 without remainder. If a number terminate with 5 or 0, it is divisible by 5. Since 5 is the half of 10, the shortest way to multiply by 5 is to annex a cipher and divide by 2; and to divide by 5 multiply by 2 and cut off the last figure, the half of which is the remainder. And since 25 is the fourth of 100, the shortest way to multiply by 25 is to annex two ciphers and divide by 4; and to divide by 25, multiply by 4, and cut off the last two figures, the fourth of which is the remainder. To multiply a number by 9, annexed cipher and subtract the number; and to multiply by 99, annex two ciphers and subtract the number. The reason is very evident, since

$$9 = 10 - 1, 99 = 100 - 1, 999 = 1000 - 1, \text{ and so on.}$$

CONCRETE NUMBERS.

32. The numbers which have been hitherto introduced into our calculations have been independent of any particular unit, that is, *abstract* numbers; but those numbers can only afford a clear conception of the magnitude of objects when the unit is defined and known. By the number 24 we specify that the quantity to be measured is composed of 24 times the unit, supposed to be known; but when we say

that the day is composed of 24 hours, we mean that the unit of time is the duration of *one hour*, and that 24 of these hours are equal in duration to *one day*. Numbers of this kind, composed of a particular unit, which is repeated as many times as are indicated by an abstract number, are termed *concrete numbers*, and are consequently products of which the multiplicand is the unit, and the multiplier an abstract number. Hence 24 hours means 24 times *one hour*, and 36 miles signifies 36 times *one mile*. Now suppose a distance is to be measured, we may take a *mile* as the unit, and the distance may be represented as nearly as we please, and sufficiently accurate for all practical purposes, either by a certain number of miles or a certain number of parts of a mile, and may therefore be expressed either by a whole number or a fraction. If one distance be represented by the *one hundred and fifty-fourth part* of a mile, and another by the *one hundred and forty-fourth part* of a mile, we have but an imperfect notion as to how much the second distance is longer than the first. It is necessary to have some smaller measure, and if a mile be divided into 1760 equal parts, and each of these parts be called a *yard*; then the first distance will be $1760 \div 154$ or 11 yards and three-sevenths of a yard, and the second distance will be $1760 \div 144$ or 12 yards and two-ninths of a yard. We have now a better notion of these different distances, and if the yard be supposed to be divided into 3 equal parts, and each of these parts be called a *foot*, a still clearer notion of the two distances would be obtained. Hence large measures are convenient for measuring large quantities, while smaller measures are necessary and more convenient for measuring smaller quantities.

33. A *compound quantity* is one consisting of several others, expressed in different units, as 17 miles, 57 yards, 2 feet, or £3. 17*s*. 6*d*.; and the different denominations into which money, time, weight, length, or distance, etc., are divided, constitute so many scales or systems of numeration, by means of which operations on concrete and compound numbers are assimilated to those on abstract and simple numbers. The usual tables of the different divisions of money, weights, and measures will be found in the following article, and the determination of the standard weights and measures will be found in a subsequent part of the Arithmetic.

34. TABLES OF MONEY, WEIGHTS, AND MEASURES.

1. *Money.*

2 farthings	= 1 halfpenny.
4 farthings	= 1 penny.
12 pence	= 1 shilling.
2 shillings	= 1 florin.
20 shillings	= 1 pound.

2. *Measure of Length.*

12 inches.	= 1 foot.
3 feet	= 1 yard.
$5\frac{1}{2}$ yards	= 1 pole or rod.
40 poles	= 1 furlong.
8 furlongs	= 1 mile.
1760 yards	= 1 mile.

Special Measures of Length.

Land Measure.

100 links	= 1 chain.
22 yards	= 1 chain.
80 chains	= 1 mile.

Nautical Measure.

3 miles	= 1 league.
20 leagues	= 1 degree.

3. *Measure of Surface.*

144 square inches	= 1 sq. foot.
9 square feet	= 1 sq. yard.
$30\frac{1}{4}$ square yards	= 1 sq. pole.
40 square poles	= 1 rood.
4 roods or 4840 sq. yds.	= 1 acre.

Special Measure of Surface for Land.

10000 square links	= 1 sq. chain.
10 square chains	= 1 acre.
640 acres	= 1 sq. mile.

4. *Measure of Solidity.*

1728 cubic inches = 1 cubic foot.
27 cubic feet = 1 cubic yard.

5. *Measure of Capacity for Liquids, Grain, Fruit, etc.*

Liquid.

4 gills = 1 pint.
2 pints = 1 quart.
4 quarts = 1 gallon.

Dry.

2 gallons = 1 peck.
4 pecks = 1 bushel.
8 bushels = 1 quarter.
5 quarters = 1 load.

6. *Measures of Weight.*

Troy,—by which Gold, Silver, and Precious Stones are weighed.

24 grains = 1 pennyweight.
20 pennyweights = 1 ounce.
12 ounces = 1 pound.

Avoirdupois, or the general Measure of Weight.

16 drams = 1 ounce.
16 ounces = 1 pound.
14 pounds = 1 stone.
2 stones = 1 quarter.
28 pounds = 1 quarter.
4 quarters = 1 hundredweight (cwt).
112 pounds = 1 ditto.
20 cwt. = 1 ton.

Special Measure of Weight for Medical Prescriptions.

20 grains = 1 scruple.
3 scruples = 1 dram.
8 drams = 1 ounce.
12 ounces = 1 pound.

7. *Measure of Time.*

60 seconds = 1 minute.
60 minutes = 1 hour.
24 hours = 1 day.
7 days = 1 week.
365 days = 1 common year.
366 days = 1 leap year.

REDUCTION.

35. ***Reduction*** is to change a concrete number, consisting of one or more denominations, into another, without altering its value.

Ex. 1. Let it be required to reduce £421. 15*s.* 7½*d.* to farthings. Since there are 20 shillings in 1 pound, it is obvious that in 421 pounds there will be 421 times 20, or, which comes to the same thing, 20 × 421, or 8420 shillings. To this product we must add 15 shillings, which may be done mentally while the multiplication by 20 is being made, and the entire number of shillings in £421. 15*s.* is 8435. Again, since there are 12 pence in 1 shilling, there will be in 8435 shillings 8435 times 12 pence, that is, 12 × 8435, or 101220 pence. Adding to this number 7 pence, we have £421. 15*s.* 7*d.*, equal to 101227 pence. Lastly, since 4 farthings are equal to 1 penny, 101227 pence will be equal to 101227 times 4 farthings, or 101227 × 4 = 404908 farthings; hence, in £421. 15*s.* 7½*d.* there are 404908 + 2, or 404910 farthings.

£.	s.	d.
421	15	7½
20		
8435 shillings.		
12		
101227 pence.		
4		
404910 farthings		

Since there are 960 farthings in 1 pound, 48 in a shilling, and 4 in a penny, the previous example may be performed in the following manner:—

960 × 421 =	404160	farthings in 421 pounds
48 × 15 =	720	farthings in 15 shillings
4 × 7 =	28	farthings in 7 pence
2 × 1 =	2	farthings in 1 halfpenny.
hence there are	404910	farthings in £421. 15*s.* 7½*d.*

Ex. 2. How many pounds, shillings, pence, and farthings are in 337587 farthings?

Here the operation must be the reverse of the former, and dividing the number of farthings by 4, the quotient is 84396 pence, and the remainder is 3 farthings. Dividing the number of pence by 12 gives 7033 shillings, and no remainder, and dividing the number of shillings by 20, gives 351 pounds, and a remainder of 13 shillings; hence, 337587 farthings are equal to 84396 pence and 3 farthings, or equal to 7033 shillings and 3 farthings, or equal to £351. 13*s.* 0¾*d.*

4	337587 farthings.
12	84396¾ pence.
2,0	703,3*s.* 0¾*d.*
	£351. 13*s.* 0¾*d.*

Ex. 3. How many half-crowns are equivalent to £227. 12*s.* 1*d.*?

The given sum is first reduced to pence by multiplying by 20 and 12, adding the shillings and pence in succession; then since there are 30 pence in 1 half-crown, the number of pence is divided by 30, which gives 1820 half-crowns and 25 pence remaining, or 1820 *hf.-cr.* 2*s.* 1*d.* These three examples are sufficient to illustrate the principles of Reduction, as, whatever be the denominations, the operations are performed in a similar manner.

```
      £.   s.   d.
      227  12   1
       20
      ----
      4552 shillings.
        12
      ------
3,0)5462,5 pence.
    ----------
     1820 half-crowns 25 pence.
```

COMPOUND ADDITION.

36. *Compound Addition* is the collecting into one sum two or more numbers expressed in different denominations; and the process is precisely similar to that for the addition of simple numbers, with this difference, that the numbers connecting the different denominations must be employed instead of *ten.*

Ex. Find the sum of £73. 2*s.* 9½*d.*, £25. 8*s.* 4¾*d.*, £68. 3*s.* 11¼*d.*, £28. 11*s.* 7¾*d.*, and £17. 14*s.* 11¾*d.*

£.	s.	d.		Farthings.
73	2	9½	=	70214
25	8	4¾	=	24403
68	3	11¼	=	65469
28	11	7¾	=	27439
17	14	11¾	=	17039
213	1	9	=	204564

4	204564 farthings.
12	51141 pence.
2,0	426,1*s.* 9*d.*
	£213. 1*s.* 9*d.*

Here the numbers are arranged so that those of the same denomination are in the same vertical column; then beginning at the lowest denomination, viz., farthings, the sum is 12 farthings, which are equivalent to 3 pence. Then 3 pence are carried to the next column, and added thereto, making the entire number of pence 45. But 45 pence are 3 shillings and 9 pence, and writing 9 under the column of pence, the 3 shillings are added with the numbers in the column of shillings, making 41 shillings, which are equivalent to 2 pounds 1 shilling; writing 1 under the column of shillings, and carrying the 2 pounds to the left column, the entire number of pounds is found to be 213; consequently, the sum of the whole is £213. 1*s.* 9*d.* The sum may also be obtained in the ordinary manner by reducing each of the numbers to the denomination of farthings, as has been done above, and then reducing the sum, viz., 204564 farthings, to pounds, shillings, and pence, in the usual

manner. In practice, it is more useful to adopt a separate scale of notation for each case, as in the first method, without changing the denominations into the lowest.

COMPOUND SUBTRACTION.

37. *Compound Subtraction* is the taking a less number from a greater, when both numbers are composed of different denominations.

Ex. Find the difference between 35 yards 2 feet 8 inches and 52 yards 1 foot 4 inches.

Yds.	Ft.	In.		Yds.	Ft.	In.		Inches.
52	1	4	or	52	4	16	or	1888
35	2	8		36	3	8		1292
16	1	8		16	1	8		596

12	596 inches.
3	49 feet 8 inches.
	16 yds. 1 ft. 8 in.

Since the difference of two quantities is not altered by adding the same quantity to both, we must first add 12 inches to the upper line, and 12 inches or 1 foot to the lower line, making 16 inches in the one and 3 feet in the other. Again, as 3 feet cannot be subtracted from 1 foot, we must add 3 feet to the upper line, and 3 feet or 1 yard to the lower; making 4 feet in the former, and 36 yards in the latter. The subtraction can now be effected, the difference being 16 yds. 1 ft. 8 in. Reducing both numbers to inches, the difference (596 inches) is obtained in the ordinary manner, and then reduced to yards, feet, and inches, as in the example above.

COMPOUND MULTIPLICATION.

38. *Compound Multiplication* is the finding the amount of a number consisting of different denominations, repeated any number of times. If a quantity consists of several parts, and each of these parts be multiplied by a number, and the products be added, the result is the same as would arise from multiplying the quantity by that number.

Ex. 1. Multiply £24. 17*s*. 8¾*d*. by 23.

		£.	*s*.	*d*.
3 farthings × 23 =	69 farthings =	0	1	5¼
8 pence × 23 =	184 pence =	0	15	4
17 shillings × 23 =	391 shillings =	19	11	0
24 pounds × 23 =	552 pounds =	552	0	0
	The sum of all these is	£572	7	9¼

This product may be obtained in another manner; for since $23 = 4 \times 6 - 1$, we may multiply £24. 17*s*. 8¾*d*. first by 4, then the product by 6, and from this last product subtract £24. 17*s*. 8¾*d*. Or, since $23 = 2 \times 11 + 1$, we may first multiply by 2, then the product by 11, and add to this last product £24. 17*s*. 8¾*d*.

	£.	*s*.	*d*.		£.	*s*.	*d*.	
	24	17	8¾		24	17	8¾	
			4				2	
	99	10	11		49	15	5½	
			6				11	
	597	5	6		547	10	0½	
subtract	24	17	8¾		24	17	8¾	add.
	572	7	9¼		572	7	9¼	

39. When the multiplier is a large number, we may reduce the

multiplicand to the lowest demonination included, and proceed in the ordinary way. Or we may multiply by the separate units, tens, hundreds, etc., of the multiplier.

Ex. 2. Multiply £3. 15*s*. $6\frac{1}{2}$*d*. by 327.

£.	*s*.	*d*.	Farthings.
3	15	$6\frac{1}{2}$	= 3626
			327
			25382
			7252
			10878
		4	1185702 far.
		12	$296425\frac{1}{2}d.$
		2,0	2470,2*s*. $1\frac{1}{2}d.$
			£1235 2*s*. $1\frac{1}{2}d.$

£.	*s*.	*d*.		£.	*s*.	*d*.	
3	15	$6\frac{1}{2}$	$\times 7 =$	26	8	$9\frac{1}{2}$	= 7 times.
		10					
37	15	5	$\times 2 =$	75	10	10	= 20 times.
		10					
377	14	2	$\times 3 =$	1133	2	6	= 300 times.
				1235	2	$1\frac{1}{2}$	= 327 times.

COMPOUND DIVISION.

40. *Compound Division* is the dividing a number consisting of several denominations into as many equal parts as there are units in the divisor; or it is the finding how many times one compound number is contained in another consisting of like denominations.

Ex. 1. Divide £172. 11*s*. $5\frac{1}{2}$*d*. by 5.

	£.	*s*.	*d*.
5	172	11	$5\frac{1}{2}$
	34	10	$3\frac{1}{2}$

Dividing 172 pounds by 5, gives a quotient of 34 pounds, and 2 pounds remain to be divided by 5. In 2 pounds there are 40 shillings, and 40 + 11, or 51 shillings, divided by 5, gives 10 shillings, and 1 shilling remains. But in 1 shilling there are 12 pence, and 12 + 5, or 17 pence, divided by 5, gives 3 pence, and 2 pence remain; then 2 pence are 8 farthings, and 8 + 2, or 10 farthings, divided by 5, gives 2 farthings, and £34. 10*s*. $3\frac{1}{2}$*d*. is the quotient required.

Ex. 2. Divide £629. 16*s*. $5\frac{1}{4}$*d*. by 91.

This may be effected either by reducing the dividend to the lowest denomination included, and then dividing in the ordinary way, or by dividing as in Division.

```
     £.    s.   d.
     629   16   5¼
      20
   12596
      12
  151157
       4
91)604629(6644 25/91 farthings,
   546     or £6. 18s. 5 0/4 d. 25/91.
    586
    546
     402
     364
     389
     364
      25
      91
```

```
     £.    s.   d.  £.   s.   d.
  91)629   16   5¼(6   18   5 0/4 25/91.
     546
      83
       20
  91)1676(18
     91
      766
      728
       38
        12
   91)461(5
      455
        6
        4
       25
```

Ex. 3. How many times does £263. 8*s.* 11¼*d.* contain £37. 12*s.* 8½*d.*?

Here both numbers must be reduced to the lowest denomination, which is farthings. The divisor £37. 12*s.* 8½*d.* is equivalent to 36130 farthings, and the dividend £263. 8*s.* 11¼*d.* is equivalent to 252910 farthings; then dividing the latter by the former, we get for a quotient the abstract number 7; hence the former sum contains the latter 7 times.

$$\begin{array}{r} 36130)252910(7 \\ \underline{252910} \\ \hline \end{array}$$

41. It is worthy of remark, that when a concrete number is divided by an abstract number, the quotient is a concrete number of the same kind as the dividend; but when one concrete number is divided by another, the quotient is an abstract number. Thus 24 days divided by 6 gives 4 days for quotient, while 24 days divided by 6 days give the abstract number 4 for quotient.

GREATEST COMMON MEASURE.

42. When one number divides another without remainder, or is contained an exact number of times in it, the former is said to *measure* the latter. Thus 4 is a measure of 28, but it is not a measure of 29. When one number is a measure of two others, it is called a *common measure* of these others; and the greatest of all the common measures of two numbers is called their *greatest common measure.* Thus 2, 3, 6 are common measures of 18 and 30, but 6 is their *greatest* common measure.

43. *If one number measures two others, it measures their sum and difference.* Thus 9 measures 18 and 45, and hence it must necessarily measure both 45 + 18 and 45 − 18, or 63 and 27.

44. *If one number measures a second, it measures every number which the second measures.* Thus 9 measures 18; and 18 measures 36, 54, 72, etc., all of which are evidently measured by 9.

45. *Every number which measures both the dividend and divisor, measures the remainder also; and every common measure of the divisor and remainder is also a common measure of the dividend and divisor.*

Dividing 168 by 63, we get 2 for a quotient, and the remainder is 42; that is, $168 = 63 \times 2 + 42$, and therefore 42 is the difference between 168 and twice 63, or $42 = 168 - 63 \times 2$. Take now any number which measures both 168 and 63 as 3; then, since 3 measures 63, it measures $63 + 63$, or 63×2 by Art. 44, hence (43) it measures $168 - 63 \times 2$, and therefore it measures the remainder 42. The same holds for all other measures of 168 and 63; hence it follows, that every *common measure* of a divisor and dividend is also a *common measure* of the divisor and remainder. Again, since $168 = 63 \times 2 + 42$, let us take any common measure of the divisor 63, and the remainder 42 as 7; then 7 measures 63×2 by Art. 44, and hence (43) it measures $63 \times 2 + 42$, that is, it measures the dividend 168. From this it follows that there is no common measure of the remainder and divisor which is not also a common measure of the divisor and dividend. Hence the *greatest* common measure of the remainder and divisor is also the *greatest* common measure of the divisor and dividend; that is,

the greatest common measure of 42 and 63 is also the greatest common measure of 63 and 168. In a similar manner, by dividing 63 by 42, we find that the greatest common measure of the remainder 21, and the divisor 42 is also the greatest common measure of 42 and 63, and therefore also of 63 and 168.

63)168(2
126
42)63(1
42
21)42(2
42

This process may sometimes be shortened by taking the quotient figure, so that when the divisor is multiplied by it, the product shall be *greater* than the dividend; because it is only the *difference* between 168 and 126 which we have to deal with in the preceding reasoning. Thus taking the first quotient figure 3 instead of 2, the product 189 differs from 168 only by 21; whereas, in the former division, the difference or remainder is 42. The last divisor, 21, is hence the greatest common measure of 63 and 168.

63)168(3
189
21)63(3
63

Otherwise. Since $63 = 7 \times 9 = 7 \times 3 \times 3$, and $168 = 7 \times 24 = 7 \times 3 \times 8$; therefore $7 \times 3 = 21$, is the greatest common measure of 63 and 168.

Hence, *to find the greatest common measure of two numbers, divide the greater by the less, and then the divisor by the remainder; repeat this operation till an exact divisor is obtained; this will be the greatest common measure sought.*

46. A *prime* number is one which can only be measured by unity, and a *composite* number is one which can be measured by some number greater than unity, or it is the product of two or more numbers. Also, two numbers are *prime to each other* when they have no common measure greater than unity.

47. To obtain the greatest common measure of three numbers, as 63, 168, and 189, we must first find that of 63 and 168, which is 21, and since it is manifest that the number sought is either 21 or some measure of it, we have only to find the greatest common measure of 21 and 189, which is 21, because it is an exact divisor of 189. This process is applicable to four or more numbers.

LEAST COMMON MULTIPLE.

48. A *multiple* of any number is one which contains it an exact number of times; a *common multiple* of two or more numbers is one which contains each of them an exact number of times; and the *least common multiple* of two or more numbers is the least number which contains each of them an exact number of times. Thus 15 is a multiple of 3 or 5; 24 is a common multiple of 3 and 4; and 12 is their least common multiple.

49. One method of finding the least common multiple of two numbers is to divide their product by their greatest common measure. For take any two numbers, as 72 and 30, and resolve them into their prime factors; then, since $72 = 2 \times 2 \times 2 \times 3 \times 3$, and $30 = 2 \times 3 \times 5$, we see at once that their greatest common measure is 2×3 or 6, and that if $(2 \times 2 \times 2 \times 3 \times 3) \times (2 \times 3 \times 5)$ be divided by 2×3 or 6, the quotient will be equal to $2 \times 2 \times 3 \times 2 \times 3 \times 5$. This

product must be the least common multiple of 72 and 30, since no smaller number is exactly divisible by $2 \times 2 \times 2 \times 3 \times 3$ and $2 \times 3 \times 5$, or by 72 and 30; hence $2 \times 2 \times 3 \times 2 \times 3 \times 5 = 360$ = least common multiple, and $360 = 72 \times 5 = 30 \times 12 = \frac{72 \times 30}{6}$
$= \frac{2 \times 2 \times 2 \times 3 \times 3 \times 2 \times 3 \times 5}{2 \times 3}$.

50. Hence it follows, that if two numbers be divided by all the numbers which will divide them exactly, the products of the divisors and quotients will be the least common multiple. Thus, to find the least common multiple of 576 and 312, we first divide by 12, and then the quotients by 2, and consequently $12 \times 2 \times 24 \times 13 = 7488$ = the least common multiple required. It is also evident that 12×2, or 24, is the greatest common measure of 576 and 312.

12	576	312
2	48	26
	24	13

51. From these examples we perceive that *the product of two numbers is a common multiple of each, and that if two numbers have a common measure, they also have a common multiple less than their product. Also, when the greatest common measure is unity, the least common multiple of the two numbers is their product. The rule then is:—to find the least common multiple of two numbers, find their greatest common measure, and divide their product by it; or divide either of the numbers by their greatest common measure, and multiply the quotient by the other.*

52. The least common multiple of three or more numbers may be found very simply by the following process. Write the numbers in a line, and divide by any number that will divide them all, or two or more of them, and set down the quotients and undivided numbers (if any) in a line below. Divide the numbers in the second line in a similar manner, and continue the operation until every two of the numbers are prime to each other. Then the continued product of all the divisors, and the numbers in the last line, will be the least common multiple of all the numbers.

Ex. 1. Find the least common multiple of 126, 168, 210, and 294.

Beginning with the least prime divisor, 2, we find the quotients to be as in the second line. As 84 is the only number in this line divisible by 2, we try 3, the next prime number, and thus the third line is obtained; and as no two of these numbers are divisible by 2, 3, or 5, we take the next prime number 7, and this gives the quotients in the last line, every two of which are prime to each other; hence, $2 \times 3 \times 7 \times 3 \times 4 \times 5 \times 7 = 17640$ = the least common multiple sought.

2	126,	168,	210,	294
3	63,	84,	105,	147
7	21,	28,	35,	49
	3,	4,	5,	7

The greatest common measure of all these numbers is evidently the product of all the divisors 2, 3, 7, or 42; and this being the greatest common factor of all the numbers, it is evident that if the product of all the other factors, 3, 4, 5, 7, be multiplied by this number 42, the least common multiple of all will be obtained.

Ex. 2. Find the least common multiple of all the nine digits.

$$\begin{array}{r|l} 2 & 5,\ 6,\ 7,\ 8,\ 9 \\ \hline 3 & 5,\ 3,\ 7,\ 4,\ 9 \\ \hline & 5,\ 1,\ 7,\ 4,\ 3 \\ \hline \end{array} \qquad \begin{array}{r|l} 2 & 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 \\ \hline 2 & 1,\ 3,\ 2,\ 5,\ 3,\ 7,\ 4,\ 9 \\ \hline 3 & 1,\ 3,\ 1,\ 5,\ 3,\ 7,\ 2,\ 9 \\ \hline & 1,\ 1,\ 1,\ 5,\ 1,\ 7,\ 2,\ 3 \\ \hline \end{array}$$

Here we may either take all the digits, or omit 2, 3, 4; because, whatever number is measured by 9 and 8, will be measured by 2, 3, 4. In either way we have the least common multiple

$$= 2 \times 3 \times 5 \times 7 \times 4 \times 3 = 2520.$$

53. The least common multiple of several numbers may be found by first finding the least common multiple of any two of them; then the least common multiple of that multiple, and a third number; and so on.

FRACTIONS.

54. The nature of a fraction will be understood by supposing that a unit of any kind is divided into several equal parts. One or more of these parts is called a *fraction* of that unit, and is represented by one number above a line, and another under it. The number under the line denotes the number of parts into which the unit is divided, and is called the *denominator* of the fraction, and the number above the line shows how many of these parts are represented, and is called the *numerator* of the fraction. Thus if the unit be a yard, the fraction $\frac{3}{8}$ signifies that the yard is divided into 8 equal parts, and three of these parts are taken. It is read *three-eighths*. If the unit be divided into 2, 3, 4, 5, or 6 equal parts, the corresponding fractions, with their names, are as follow: —

$\frac{1}{2}$, *one-half;* $\frac{1}{3}$, *one-third;* $\frac{1}{4}$, *one-fourth;* $\frac{1}{5}$, *one-fifth;* $\frac{1}{6}$, *one-sixth.*

55. Hence it follows that if the numerator of a fraction be less, equal to, or greater than the denominator, its value is less, equal to, or greater than unity.

56. A *proper fraction* is one whose numerator is less than the denominator; and an *improper fraction* is one whose numerator is either equal to, or greater than the denominator. Thus $\frac{3}{5}$, $\frac{9}{10}$ are proper fractions, and $\frac{9}{5}$, $\frac{4}{4}$, $\frac{10}{9}$, are improper fractions. In the latter case we may suppose each of two or more units to be divided into the number of parts indicated by the denominator, and from these units so divided, take as many parts as there are units in the numerator.

57. A *simple fraction* has only one numerator and one denominator, which are called the *terms* of the fraction, as $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{9}{8}$. A *compound fraction* is two or more fractions with the word *of* between them; thus $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{5}{6}$ is a compound fraction. A *mixed number* is composed of

an integer and a fraction, as $1\frac{1}{2}$, $6\frac{3}{4}$, and $15\frac{1}{4}$; and a *complex fraction* has a fraction or a mixed number in one or both of its terms, as $\frac{\frac{1}{2}}{5}$, $\frac{12\frac{1}{2}}{\frac{1}{3}}$.

58. From the notion attached to the words numerator and denominator, it is evident that

1. *A fraction is increased by increasing the numerator.*
2. *A fraction is increased by diminishing the denominator.*
3. *A fraction is diminished by diminishing the numerator.*
4. *A fraction is diminished by increasing the denominator.*

For by increasing or diminishing the numerator, we take more or fewer of the parts of the unit; by diminishing the denominator, the magnitude of the parts is increased, while the same number of parts are taken. Also if the denominator be increased, the magnitude of the parts is diminished, and the fraction is diminished. Thus the fraction

$\frac{3}{5}$ is three times greater than $\frac{1}{5}$; $\frac{4}{7}$ is the double of $\frac{2}{7}$;

$\frac{7}{4}$ is three times greater than $\frac{7}{12}$; $\frac{3}{5}$ is the double of $\frac{3}{10}$;

$\frac{1}{5}$ is the third part of $\frac{3}{5}$; $\frac{5}{21}$ are the half of $\frac{10}{21}$;

$\frac{2}{5}$ are less than $\frac{2}{3}$; $\frac{4}{13}$ are less than $\frac{4}{9}$.

59. Hence *a fraction is multiplied or divided by multiplying or dividing the numerator; and a fraction is divided or multiplied by multiplying or dividing the denominator.* Also *if the numerator and denominator of a fraction be multiplied or divided by the same number, the value of the fraction is not altered.*

Thus $\frac{1}{7} \times 3 = \frac{3}{7}$; $\frac{6}{5} \div 3 = \frac{2}{5}$; $\frac{3}{4} \div 2 = \frac{3}{8}$; $\frac{5}{24} \times 4 = \frac{5}{6}$; and

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14} = \frac{10}{20}, \text{ and so on.}$$

It is worthy of remark that, by suppressing the denominator of a fraction, it becomes multiplied by this number. For example, by suppressing the denominator 8 in the fraction $\frac{3}{8}$, it becomes 3 whole numbers, or is multiplied by 8; that is $\frac{3}{8} \times 8 = 3$. Also since the *integer* 3 is 8 times greater than the fraction $\frac{3}{8}$, it is evident that 3 may be expressed in a fractional form by writing 1 in the denominator; thus $3 = \frac{3}{1}$, and so on.

60. *To reduce a fraction to its lowest terms.*

By Art. 59, the value of a fraction is not altered if both numerator and denominator be divided by the same number; therefore divide the

numerator and denominator by their greatest common measure; or first divide by any common measure of the terms of the fraction, and repeat the operation on the reduced fraction until the terms have no common measure but 1.

Reduce the fraction $\frac{891}{3429}$ to its simplest terms.

```
891)3429(4
    3564
    ----
     135)891(7
         945
         ---
          54)135(2
             108
             ---
              27)54(2
                 54
                 --
```

Hence $\frac{891}{3429} = \frac{891 \div 27}{3429 \div 27} = \frac{33}{127}$.

In finding the greatest common measure, the work has been shortened by taking the quotient figures 4 and 7, because the products 3564 and 945 differ less from the dividends 3429 and 891 than if 3 and 6 had been taken.

Or thus, $\frac{891}{3429} = \frac{297}{1143} = \frac{99}{381} = \frac{33}{127}$, by dividing successively by 3.

In many instances it is unnecessary to find the greatest common measure, the fractions being reducible to lower terms by successive divisions of the numerators and denominators by common factors discovered by *inspection*.

Thus $\frac{4356}{9504} = \frac{1089}{2376} = \frac{363}{792} = \frac{121}{264} = \frac{11}{24}$, by dividing by 4, 3, 3, and 11.

61. *To reduce a compound fraction to the form of a simple fraction.*

Let it be required to reduce $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{7}{8}$ to a simple fraction. By Art. 59 we have, $\frac{1}{6}$ of $\frac{7}{8} = \frac{7}{8} \div 6 = \frac{7}{48}$; $\therefore \frac{5}{6}$ of $\frac{7}{8} = \frac{7}{48} \times 5 = \frac{35}{48}$.

Again, $\frac{1}{3}$ of $\frac{35}{48} = \frac{35}{48} \div 3 = \frac{35}{144}$; $\therefore \frac{2}{3}$ of $\frac{35}{48} = \frac{35}{144} \times 2 = \frac{70}{144}$,

which is a simple fraction, and from the several steps of the process, we have

$$\frac{2}{3} \text{ of } \frac{5}{6} \text{ of } \frac{7}{8} = \frac{2 \times 5 \times 7}{3 \times 6 \times 8} = \frac{70}{144} = \frac{35}{72}, \text{ by dividing by 2.}$$

Hence a compound fraction is reduced to a simple one, by multiplying together all the numerators for the numerator, and all the denominators for the denominator of the simple fraction. The resulting simple fraction may then be reduced to its lowest terms, or the common factors may be suppressed before the multiplication is effected.

62. *To transform fractions having different denominators into others having a common denominator.*

The principle employed in this transformation is the multiplication of the terms of a fraction by the same number; hence we have only to find the least common multiple of the denominators of the fractions (52), and then multiply the numerator and denominator of each fraction separately, by such a number as will raise its denominator to the least common multiple. If the denominators of the several fractions are prime numbers, or every two of them prime to each other, then multiply each numerator by all the denominators except its own for a new numerator,

and multiply together all the denominators for a common denominator.

Reduce $\frac{1}{2}, \frac{2}{3}, \frac{5}{12}$, and $\frac{7}{18}$ to equivalent fractions having a common denominator.

$$\begin{array}{r|l} 2 & 2,\ 3,\ 12,\ 18 \\ \hline 3 & 1,\ 3,\ 6,\ 9, \\ \hline & 1,\ 1,\ 2,\ 3, \end{array}$$

$\therefore 2 \times 3 \times 2 \times 3 = 36 =$ least common multiple of all the denominators.

Hence
$$\left.\begin{aligned} \frac{1}{2} &= \frac{1 \times 18}{2 \times 18} = \frac{18}{36} \\ \frac{2}{3} &= \frac{2 \times 12}{3 \times 12} = \frac{24}{36} \\ \frac{5}{12} &= \frac{5 \times 3}{12 \times 3} = \frac{15}{36} \\ \frac{7}{18} &= \frac{7 \times 2}{18 \times 2} = \frac{14}{36} \end{aligned}\right\} \text{The equivalent fractions.}$$

ADDITION OF FRACTIONS.

63. If the fractions have a common denominator, add the numerators together and place their sum over the common denominator; but if the fractions have different denominators, reduce them to other equivalent fractions (62) having a common denominator; and then add the numerators as before.

64. A mixed number is transformed into an improper fraction by addition. For $7\frac{5}{8} = \frac{7}{1} + \frac{5}{8} = \frac{7 \times 8}{1 \times 8} + \frac{5}{8} = \frac{56}{8} + \frac{5}{8} = \frac{56 + 5}{8} = \frac{61}{8}$. Hence to reduce a mixed number to an improper fraction, we have only to multiply the integer by the denominator of the fraction, and add the numerator to the product. Place this sum over the denominator, and the improper fraction will be obtained.

65. An improper fraction may always be expressed by a mixed number. For $\frac{61}{8} = \frac{56 + 5}{8} = \frac{56}{8} + \frac{5}{8} = 7 + \frac{5}{8} = 7\frac{5}{8}$. This process is evidently the same thing as dividing both numerator and denominator by the *denominator*, noticing the remainder of the former and suppressing the *unit* in the latter; thus $\frac{61}{8} = \frac{61 \div 8}{8 \div 8} = \frac{7\frac{5}{8}}{1} = 7\frac{5}{8}$; which affords the following rule:—

Divide the numerator by the denominator, and the quotient will be the integral part; then place the remainder over the denominator or divisor for the fractional part.

Examples.

1. Find the sum of $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}$, and $\frac{7}{8}$.

The least common multiple of the denominators is 48;

hence $\frac{2}{3} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} = \frac{32}{48} + \frac{36}{48} + \frac{40}{48} + \frac{42}{48} = \frac{150}{48} = \frac{25}{8} = 3\frac{1}{8}$.

2. Find the sum of $24\frac{1}{2}$, $36\frac{3}{4}$, $28\frac{1}{4}$, and $112\frac{7}{8}$.

Here $\frac{1}{2}+\frac{3}{4}+\frac{1}{4}+\frac{7}{8}=\frac{4}{8}+\frac{6}{8}+\frac{2}{8}+\frac{7}{8}=\frac{19}{8}=2\frac{3}{8}$, to which add the sum of the integers, viz., 200, and the entire sum is $202\frac{3}{8}$.

SUBTRACTION OF FRACTIONS.

66. If the fractions have not a common denominator, they must be reduced to other equivalent ones having a common denominator; then the parts into which the unit is supposed to be divided are the same in both fractions, and the difference of the numerators placed over the common denominator will give the difference of the fractions.

Thus $\frac{5}{8}-\frac{3}{8}=\frac{2}{8}=\frac{1}{4}$; $14\frac{7}{8}-10\frac{3}{8}=4\frac{4}{8}=4\frac{1}{2}$;

$$26\tfrac{1}{8}-12\tfrac{5}{8}=25+1\tfrac{1}{8}-12\tfrac{5}{8}=25\tfrac{9}{8}-12\tfrac{5}{8}=13\tfrac{4}{8}=13\tfrac{1}{2};$$

$$\frac{9}{14}-\frac{10}{21}=\frac{9\times 3}{14\times 3}-\frac{10\times 2}{21\times 2}=\frac{27}{42}-\frac{20}{42}=\frac{7}{42}=\frac{1}{6}.$$

MULTIPLICATION OF FRACTIONS.

67. Suppose it required to multiply $\frac{2}{9}$ by $\frac{5}{8}$; then if $\frac{2}{9}$ be multiplied by 5, the product will be $\frac{10}{9}$ by Art. 59; but 5 being 8 times as great as $\frac{5}{8}$, the multiplier used above is 8 times too large, and therefore the product $\frac{10}{9}$ is 8 times too large; hence this product must be diminished 8 times, and (59)

$$\frac{10}{9}\div 8=\frac{10}{72}=\frac{5}{36};\ \text{that is, } \frac{2}{9}\times\frac{5}{8}=\frac{2\times 5}{9\times 8}=\frac{10}{72}=\frac{5}{36}.$$

Hence to multiply two or more fractions together, multiply all the numerators together for the numerator of the product, and all the denominators together for its denominator.

Ex. Find the product of $2\frac{1}{3}$, $\frac{1}{3}$ of $\frac{3}{4}$, and $5\frac{1}{7}$.

Here $2\frac{1}{3}=\frac{7}{3}$, $\frac{1}{3}$ of $\frac{3}{4}=\frac{1\times 3}{3\times 4}=\frac{1}{4}$ and $5\frac{1}{7}=\frac{36}{7}$;

then $\frac{7}{3}\times\frac{1}{4}\times\frac{36}{7}=\frac{7\times 1\times 36}{3\times 4\times 7}=\frac{36}{3\times 4}=\frac{12}{4}=3.$

DIVISION OF FRACTIONS.

68. Suppose it required to divide $\frac{2}{3}$ by $\frac{5}{7}$; then it is evident by Art. 59, that $\frac{2}{3}\div 5=\frac{2}{15}$, which is 7 times too *small*, because the divisor has been taken 7 times too *great*, viz., 5 instead of the *seventh part* of 5;

hence the quotient $\frac{2}{15}$ must be increased 7 times, and $\frac{2}{15} \times 7 = \frac{14}{15}$, is the quotient required. Hence $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{2 \times 7}{3 \times 5} = \frac{14}{15}$; and therefore to divide one fraction by another, invert the terms of the divisor, and multiply the dividend by it.

Ex. Divide $5\frac{3}{5}$ by $\frac{2}{3}$ of $\frac{3}{4}$ of $7\frac{1}{2}$.

Here $\quad 5\frac{3}{5} = \frac{28}{5}$, and $\frac{2}{3}$ of $\frac{3}{4}$ of $7\frac{1}{2} = \frac{2 \times 3 \times 15}{3 \times 4 \times 2} = \frac{15}{4}$;

and then $\quad \frac{28}{5} \div \frac{15}{4} = \frac{28}{5} \times \frac{4}{15} = \frac{28 \times 4}{5 \times 15} = \frac{112}{75} = 1\frac{47}{75}$.

69. A complex fraction may be reduced to a simple fraction by division. For if $\frac{7\frac{1}{3}}{5\frac{1}{2}}$ be the complex fraction, then its value will be the same as that of $7\frac{1}{3} \div 5\frac{1}{2}$. But $7\frac{1}{3} = \frac{22}{3}$ and $5\frac{1}{2} = \frac{11}{2}$, therefore

$$\frac{22}{3} \div \frac{11}{2} = \frac{22}{3} \times \frac{2}{11} = \frac{22 \times 2}{3 \times 11} = \frac{2 \times 2}{3} = \frac{4}{3}.$$

REDUCTION OF FRACTIONS OF CONCRETE NUMBERS.

70. A fraction of a concrete number may be reduced to the fraction of another concrete number of a higher or lower denomination, by means of the principle employed in the reduction of integers from one denomination to another (35).

EXAMPLES.

1. Reduce $\frac{2}{9}$ of a pound to the fraction of a penny.

Since any integer number of pounds is reduced to pence by multiplying the number of pounds by 20, and the product by 12, so any fraction of a pound is reduced to the fraction of a penny by multiplying the numerator by 20×12 or 240.

Thus $\frac{2}{9}$ of a £ $= \frac{2}{9} \times \frac{20}{1} \times \frac{12}{1}$ of a penny $= \frac{480}{9}$ or $\frac{160}{3}$ of a penny.

2. Reduce $\frac{2}{7}$ of a pound to the fraction of a guinea.

Here $\frac{2}{7}$ £ $= \frac{2}{7} \times \frac{20}{1}$ or $\frac{40}{7}$ of a shilling $= \frac{40}{7} \times \frac{1}{21}$ or $\frac{40}{147}$ of a guinea.

3. Reduce 3 cwt. 14 lb. to the fraction of a ton; or what fraction of a ton is 3 cwt. 14 lb.?

. Here 3 cwt. $= \frac{3}{20}$ of a ton, and 14 lb. $= \frac{14}{112}$ or $\frac{1}{8}$ of a cwt. $= \frac{1}{8} \times \frac{1}{20}$ or $\frac{1}{160}$ of a ton; therefore $\frac{3}{20} + \frac{1}{160} = \frac{24}{160} + \frac{1}{160} = \frac{25}{160} = \frac{5}{32}$, the required fraction of a ton.

Otherwise. Since 3 cwt. 14 lb. $= 3 \times 112 + 14$ or 350 lb. and 1 ton $=$ 2240 lb; therefore 3 cwt. 14 lb. $= \frac{350}{2240}$ or $\frac{5}{32}$ of a ton, as before.

We might shorten the operation in this manner:—Since 14 lb. $= \frac{14}{112}$ or $\frac{1}{8}$ of a cwt.; therefore 3 cwt. 14 lb. $= 3\frac{1}{8}$ cwt. $= \frac{25}{8} \times \frac{1}{20}$ or $\frac{5}{32}$ of a ton.

4. What part or fraction of half-a-crown is $11\frac{1}{4}d.$

Here $11\frac{1}{4}d. = 45$ farthings, and half-a-crown is $= 30d.$ or 120 farthings; therefore $11\frac{1}{4}d. = \frac{45}{120}$ or $\frac{5}{8}$ of half-a-crown.

71. The value of a fraction of a concrete number is easily determined in terms of the same or lower denominations. For since $\frac{2}{3}$ of a pound may be obtained either by dividing one pound into 3 equal parts and taking 2 of those parts, or by taking the concrete unit twice, viz., 2 pounds, and dividing it into 3 equal parts; therefore we have

$$\frac{2}{3} \text{ of a £} = \frac{1}{3} \text{ of £2} = \frac{1}{3} \text{ of 40 shillings} = 13s.\ 4d.$$

Similarly $\frac{4}{17}$ of a ton $= \frac{1}{17}$ of 4 tons; and $\frac{3}{5}$ of a mile $= \frac{1}{5}$ of 3 miles.

In every case, then, we have only to multiply the given concrete number by the numerator of the fraction, and divide the product by the denominator.

Examples.

1. Find the value of $\frac{4}{15}$ of a pound; that is, how many shillings and pence are in $\frac{4}{15}$ of a pound?

Here $\frac{4}{15}$ of a £ $= \frac{4}{15} \times \frac{20}{1}$, or $\frac{16}{3}$ or $5\frac{1}{3}$ shillings; but $\frac{1}{3}$ of a shilling $= \frac{1}{3} \times \frac{12}{1}$ or 4 pence; therefore $\frac{4}{15}$ of a £ $= 5$ shillings and 4 pence.

$$\begin{array}{r|rrr} & £. & s. & d. \\ 15\begin{cases}3 \\ 5\end{cases} \begin{array}{c} \\ \\ \end{array} & 4 & 0 & 0 \\ & 1 & 6 & 8 \\ \hline & 0 & 5 & 4 \end{array}$$

2. Find the value of $\frac{3}{7}$ of £24. 12s. 4d.; and also of $\frac{3}{5}$ of 17 cwt. 3 qrs. 7 lb.

$$\begin{array}{rrrr} & £. & s. & d. \\ & 24 & 12 & 4 \\ & & & 3 \\ \hline 7) & 73 & 17 & 0 \\ \hline & 10 & 11 & 0 \end{array} \qquad \begin{array}{rrrr} & \text{Cwts.} & \text{qrs.} & \text{lbs.} \\ & 17 & 3 & 7 \\ & & & 3 \\ \hline 5) & 53 & 1 & 21 \\ \hline & 10 & 2 & 21 \end{array}$$

Hence $\frac{3}{7}$ of £24. 12s. 4d. $=$ £10. 11s.; and $\frac{3}{5}$ of 17 cwt. 3 qrs. 7 lb. $=$ 10 cwt. 2 qrs. 21 lb.

72. By means of these reductions, we may find the sum and difference of fractions of concrete numbers consisting of one or more denominations. Thus to find the sum of $\frac{3}{4}$ of a pound, $\frac{5}{6}$ of a shilling, and $\frac{3}{4}$ of a crown, we may either reduce any two of them to the fraction of the denomination of the other, and then find the sum of these fractions, or we may find the value of each, and then take their sum.

ALIQUOT FRACTIONS.

73. The value of any number of articles may always be found by Compound Multiplication when the price of one of them is given; but when the number is large, and the price of each article is only a small sum, their value may be obtained by the method of *Aliquot Fractions* (*i. e.*, fractions having unity for their numerators), in a very convenient manner. The process is frequently denominated *Practice*, and the following examples will sufficiently illustrate the mode of calculation to be adopted in all cases.

Examples.

1. Suppose it required to find the value of 115 tons, if each ton cost £3. 12*s.* 7½*d.*

The value of 115 tons may be found either by multiplying £3. 12*s.* 7½*d.* by 115, or by separating the price of 1 ton into different parts, as £3, 10*s.*, 2*s.* 6*d.*, and 1½*d.*, and finding the cost of 115 tons at each of these prices. These different sums together will evidently give the value of 115 tons, at the whole price, £3. 12*s.* 7½*d.* The reason for the above separation of the price per ton into different parts will be manifest from the following process, in which £115 is the price of 115 tons at £1 per ton:—

s.	*d.*	£.	£.	*s.*	*d.*				£.	*s.*	*d.*
10	0	½	115	0	0	=	cost of 115 tons at		1	0	0
					3						
			345	0	0	=	,,	,,	3	0	0
2	6	¼	57	10	0	=	,,	,,	0	10	0
0	1½	$\frac{1}{20}$	14	7	6	=	,,	,,	0	2	6
			0	14	4½	=	,,	,,	0	0	1½
			£417	11	10½	=	cost of 115 tons at		3	12	7½

74. The separation of the price into the different parts may be made in a great variety of ways; but in all of them each of the parts must be an *aliquot part* of some one which precedes it, and the first is usually an aliquot part of the unit of the highest denomination in the price.

Thus 10*s.* is an aliquot part of £1, for it is ½ of £1; 2*s.* 6*d.* is an aliquot part of 10*s.*, since it is ¼ of 10*s.*; 14 lb. is an aliquot part of 1 cwt., for it is $\frac{1}{8}$ of 1 cwt.; and so on. Consequently *one number is an aliquot part of another when the former number measures the latter;* that is, when the less number is a fraction of the greater, the numerator of the fraction being unity.

The preceding example may be solved in a different manner, as in

the margin, where the division by 20 is avoided by first taking 2*s.* as an aliquot part of 10*s.*, instead of 2*s.* 6*d.* No rule can be given for the separation of the price, but a little practice will enable the student to effect it in the simplest manner, and he has only to take care that the sum of all the parts be equal to the whole price.

s.	*d.*	£.	£.		
10	0	½	115		
			3		
			345		
2	0	⅕	57	10	
0	6	¼	11	10	
0	1½	¼	2	17	6
			0	14	4½
			£417	11	10½

2. Find the value of 293 quarters, at 18*s.* 10*d.* per quarter.

s.	*d.*	£.	£.		
10	0	½	293		
5	0	½	146	10	
2	6	½	73	5	
0	10	⅓	36	12	6
0	5	½	12	4	2
0	1	⅕	6	2	1
			1	4	5
			£275	18	2

3. What will 358 yards of silk come to, at 5*s.* 9*d.* per yard?

d.	*s.*	*s.*	
6	½	358	
		5	
		1790	
3	½	179	
		89	6
	2,0	205,8	6
		£102 18	6

DECIMAL FRACTIONS.

75. We have seen that the fundamental operations of arithmetic can be applied to fractions with considerable facility, and that they are much more readily performed upon fractions having the same than upon those which have different denominators. Hence in all those parts of mathematics where fractions are often required, it has been customary to use only those which have a common denominator, or such as can be easily transformed to others having the same denominators. As the *decimal numbers* 10, 100, 1000, etc., can be operated upon with the utmost facility, so, by employing those fractions which have decimal numbers for denominators, the highest degree of simplicity will be attained in all calculations involving fractions.

A *decimal fraction* is one whose denominator is any of the decimal numbers 10, 100, 1000, etc. Thus $\frac{5}{10}$, $\frac{27}{100}$, $\frac{4375}{10000}$, are decimal fractions.

76. Instead, then, of dividing the unit into so many different parts, corresponding to all the different denominators which are met with in fractions, it has been found convenient to consider the unit as divided into ten parts, each of which shall be a *tenth;* and each of these tenths into ten parts, of which each is a *hundredth* of a unit; the hundredths into ten parts, each of which is a *thousandth* of a unit, and so on. By continuing this division, the smallest parts may be formed, by means of which all numbers whatever may be measured to any degree of accuracy. These decimal fractions, being composed of parts of a unit which become successively ten times less, are changed one into another, in the same manner as the tens, hundreds, thousands, etc., are changed into units.

For as the unit is 10 tenths, the tenth is 10 hundredths, the hundredth is 10 thousandths, etc.; therefore the tenth is ten times 10 thousandths, or 100 thousandths. Thus 2 tenths, 4 hundredths, and 9 thousandths, are equivalent to 249 thousandths, in the same way as 2 hundreds, 4 tens, and 9 units make 249 units.

77. The decimal fractions may, therefore, be written by means of figures, in the same manner as whole numbers, the *tenths* taking their place, of course, to the right of the units, then the *hundredths* to the right of the tenths, and so on, as in the following table, which may be regarded as an extension of the Numeration Table.

Millions.	Hundreds of thousands.	Tens of thousands.	Thousands.	Hundreds.	Tens.	Units.	Tenths.	Hundredths.	Thousandths.	Ten thousandths.	Hundred thousandths.	Millionths.	Ten millionths.
7	6	5	4	3	2	1 ·	2	3	4	5	6	7	8

To distinguish the figures which express the decimal parts from those which express entire units, a period is placed on the right of the units. Thus 21·234 will represent 21 units and 234 hundredths.

78. Suppose it required to represent $\frac{2}{3}$ and $\frac{1}{5}$ as decimal fractions; then by taking the series of decimal fractions $\frac{1}{1000}, \frac{2}{1000}, \frac{3}{1000}, \ldots \frac{999}{1000}, \frac{1000}{1000}$, we see that if $\frac{2}{3}$ is not exactly equal to one of these decimal fractions, it must fall between some two of them, as $\frac{666}{1000}$ and $\frac{667}{1000}$; and consequently it cannot differ from either by a thousandth part of the unit. With respect to the other fraction, $\frac{1}{5}$, we have $\frac{1}{5} = \frac{2}{10}$, which is a decimal fraction; hence we perceive that though the exact value of every fraction may not be assignable by means of decimals, still we can obtain its approximate value to any degree of accuracy that may be required.

79. The value of the decimal figures depending entirely on the place they occupy with respect to the point which separates the units from the tenths, any number of ciphers on their right may either be annexed or effaced, without altering their value. For instance, 0·7 is the same as 0·70, because the number that expresses the decimal fraction becomes ten times greater, while its parts become hundredths, and are therefore diminished ten times; thus $\frac{7}{10} = \frac{70}{100} = \frac{700}{1000}$, and hence it is evident that annexing ciphers to the right-hand of decimals does not change their value; but if ciphers be prefixed to a decimal, its value will be diminished ten times for each cipher that is prefixed; thus—

$$\cdot 7 = \frac{7}{10},\ \cdot 07 = \frac{7}{100},\ \cdot 007 = \frac{7}{1000}, \text{ and so on.}$$

A decimal may either be considered as the sum of as many fractions as it contains digits, or as a single fraction; thus—

$$\cdot 567 = \frac{5}{10} + \frac{6}{100} + \frac{7}{1000} = \frac{567}{1000},$$

$$\cdot 03057 = \frac{0}{10} + \frac{3}{100} + \frac{0}{1000} + \frac{5}{10000} + \frac{7}{100000} = \frac{3057}{100000},$$

$$18\cdot 204 = 18 + \frac{2}{10} + \frac{0}{100} + \frac{4}{1000} = \frac{18204}{1000}.$$

Hence a decimal is always equivalent to the vulgar fraction whose numerator is the decimal considered integral, and whose denominator is 1, with as many ciphers annexed as there are *decimal places* in it. In all these instances above, the reduction of fractions to a common denominator is entirely dispensed with, and their immediate comparison is one of the great advantages of the decimal notation.

ADDITION AND SUBTRACTION OF DECIMALS.

80. Place the numbers so that all the decimal parts may be in the same vertical line, tens under tens, tenths under tenths, and so on; then add and subtract in the usual manner, as though there were no decimal point, and in the result place the point in the continuation of the same vertical line in which the points in the proposed numbers are situated; that is, between the tens and tenths.

EXAMPLES.

1. Find the sum of 25·6, 4·831, 237·009, and 1850·3074.

```
  25·6
   4·831
 237·009
1850·3074
---------
2117·7474 = sum.
```

2. Find the sum of 4852·791, 4·00745, 3·7, and ·0856.

```
4852·791
   4·00745
   3·7
    ·0856
----------
4860·58405 = sum.
```

3. Find the difference between ·71836 and ·912.

```
·912
·71836
------
·19364 = difference.
```

4. Find the difference between 17·30185 and ·851476.

```
17·30185
  ·851476
---------
16·450374 = difference.
```

MULTIPLICATION OF DECIMALS.

81. Suppose it required to multiply 43·7 by 3·29; then, since 43·7 $= \frac{437}{10}$ and 3·29 $= \frac{329}{100}$, we have

$$43\cdot 7 \times 3\cdot 29 = \frac{437}{10} \times \frac{329}{100} = \frac{437 \times 329}{10 \times 100} = \frac{143773}{1000} = 143\cdot 773.$$

In a similar manner, $\cdot4 \times \cdot03 = \frac{4}{10} \times \frac{3}{100} = \frac{4\times3}{10\times100} = \frac{12}{1000} = \cdot012.$

Hence, *multiply the two numbers as if they were integers, and then point off on the right of the product, as many figures for decimals as there are decimals in both factors; but if there should be a deficiency of figures, prefix as many ciphers on the left as will supply the defect.*

82. If the decimal point be removed one place to the right-hand, the number will be multiplied by 10, for by the principles of notation every figure has ten times its former value; and if the decimal point be removed two places to the right, the number will be multiplied by 100, and so on. Thus,

$$34\cdot5 = 3\cdot45 \times 10, \text{ and } 2\cdot768 = \cdot02768 \times 100.$$

Multiply 170·867 by 4·12, and 21·32 by ·100406.

170·867	21·32
4·12	·100406
341734	12792
170867	8528
683468	2132
703·97204	2·14065592

83. The multiplication might be effected by commencing with the figure of the highest value in the multiplier, and then each of the partial products would have to be carried forward one place to the right. Thus the first line would be that which is generally written last, the last but one would become the second, and so on; and in this manner the figures of the highest value would be found first. We may also terminate each multiplication at any place we choose, and consequently obtain for the product as many figures as may be required. But in order that no mistake may arise in the beginning of each partial multiplication, the figures of the multiplier are written under those of the multiplicand, in a reverse order, in such a manner that each figure of the multiplier is placed under that figure of the multiplicand at which the multiplication ought to be commenced.

EXAMPLES.

1. Let it be required to multiply 348·2617 by 49·20863, reserving only five decimals in the product.

Add a cipher to the right of the decimals in the multiplicand, to make up the requisite number, and mark it by putting a point over it; then reverse the multiplier, and place the unit's figure of it under this cipher, or fifth place of decimals. This is the *ruling* figure of the operation, because the multiplicand 348·26170 being multiplied by the number 9 alone would give a product containing five decimals. Then begin each figure of the multiplier with the figure of the multiplicand which stands above it, taking no account of those to the right, unless it be to ascertain how many are to be carried to the first figure which is written down. The first figures of all the lines must be placed directly below the unit's figure of the multiplier; then add as usual, and mark off five places from the right for decimals.

$348\cdot2617\dot{0}$
3680294
1393046800
313435530
6965234
278609
20896
1045
17137·48114

In this example, a cipher is supposed to be placed above the first figure 4 of the multiplier.

2. Multiply 3·84615 by ·065, reserving four decimals in the product.

Put a point above the fourth figure in the decimals, and place the unit's figure of the multiplier, which is a cipher in this example, under 1, writing the other figures of the multiplier in a reverse order. Then multiply, and place the lines as already directed.

```
3·84615
   5600
-------
   2308
    192
-------
  ·2500
```

DIVISION OF DECIMALS.

84. Divide exactly as in integers, supplying the dividend with ciphers to the right hand, if required; then make as many decimals in the quotient as the number of decimals in the dividend exceeds the number in the divisor; because the dividend must comprise as many decimals as both the divisor and quotient, by the rule of multiplication. Prefix a cipher or ciphers to the quotient, if required to make up the number. If the divisor and dividend have the same number of decimal places, the quotient will be an integer, since there is no excess; and if there be more places in the divisor than in the dividend, ciphers must be supplied so as to make the number in the dividend not less than that in the divisor.

Suppose it required to divide ·1875 by 7·5. These decimals are equal to $\frac{1875}{10000}$ and $\frac{75}{10}$; hence we have

$$\cdot 1875 \div 7\cdot 5 = \frac{1875}{10000} \div \frac{75}{10} = \frac{1875}{10000} \times \frac{10}{75} = \frac{1875}{75} \times \frac{10}{10000} = \frac{25}{1000} = \cdot 025.$$

In this manner the truth of the rule is made evident in every case.

85. If the decimal point be removed one place to the left, the number is divided by 10, for the value of every figure is thus diminished ten times; and if the decimal point be removed two places to the left, the number is divided by 100, and so on.

Thus $\quad 21\cdot 68 \div 10 = 2\cdot 168$, and $3\cdot 65 \div 100 = \cdot 0365$.

Ex. Divide 142·025 by ·0437, and 3·85 by 112.

```
·0437 ) 142·0250 ( 3250
        1311
        ----
         1092
          874
          ----
          2185
          2185
          ----
             0
```

A cipher is added to the decimals in the dividend, because the divisor has four decimals. The quotient, 3250, is therefore an integer.

```
112) 3·850000 ( ·034375
     336
     ---
      490
      448
      ---
       420
       336
       ---
        840
        784
        ---
         560
         560
```

86. When the operation of division does not terminate, the quotient may be limited to a certain number of decimals, and the work may be contracted so as to obtain the specified number. Thus let it be required

to divide 732·10856 by 21·368594, reserving only 4 decimals in the quotient.

Proceed one step in the ordinary manner, and then determine (84) the place which this first figure occupies in the quotient. In this example we ought to add one cipher to the decimals in the dividend to make their number equal to that in the divisor; therefore there will be 2 integers and 4 decimals, making in all 6 figures in the quotient. Now retain 6 figures in the divisor and dividend, and instead of annexing a figure or cipher to the dividend at each division, cut off a figure from the divisor, and work with the curtailed divisor, recollecting to carry the *nearest ten* arising from the product of the figure which was cut off by the quotient figure. Proceed in this manner till all the specified figures are obtained.

```
21·,3,6,8,5,94 ) 732·10856 ( 34·2609
                 64105782
                 --------
                  910507,4
                  854744
                  ------
                   55763
                   42737
                   -----
                   13026
                   12821
                   -----
                     205
                     192
                     ---
                      13
```

REDUCTION OF DECIMALS.

87. By means of the rule of division of decimals, any fraction may be converted into a decimal.

Take $\frac{5}{8}$, for example; then $\frac{5}{8} = \frac{5000}{8000} = \frac{5000 \div 8}{1000} = \frac{625}{1000} = \cdot 625.$

Hence to convert a fraction into a decimal, *add ciphers to the numerator for decimals and divide by the denominator*, as in division of decimals.

Ex. Convert $\frac{5}{32}$, $\frac{25}{36}$, and $\frac{3}{14}$ into decimals.

$$32\begin{cases}4\\8\end{cases}\begin{array}{|l}5\cdot 00000\\1\cdot 25000\\\cdot 15625\end{array}\qquad 36\begin{cases}4\\9\end{cases}\begin{array}{|l}25\cdot 00000\\6\cdot 25000\\\cdot 69444\text{, etc.}\end{array}\qquad 14\begin{cases}2\\7\end{cases}\begin{array}{|l}3\cdot 0000000\\1\cdot 5000000\\\cdot 2142857\text{, etc.}\end{array}$$

Hence $\frac{5}{32} = \cdot 15625$, which is a *finite* decimal, since the division terminates; $\frac{25}{36} = \cdot 69\dot{4}$, which is called a *mixed recurring* or *circulating* decimal, consisting of a *non-recurring* part 69, and the recurring part ·444, etc., usually written with a point or dot above the figure which is repeated; and $\frac{3}{14} = \cdot 2\dot{1}4285\dot{7}$, the recurring part 142857, having a point above its first and last figures, being called its *period*. A *pure* circulating decimal wants the non-recurring part, as ·5555, etc., or ·148148, etc.

Since a number which terminates with a cipher is always divisible by 2 and 5, it is evident that when the factors of the denominator of a fraction are all either 2 or 5, the division will terminate; but if one or more of the factors be 3, 7, 9, etc., the division will never terminate, and the

result will be a circulating decimal, whose period of recurring figures is always less than the denominator of the fraction. This is obvious, since any remainder is less than the divisor, and there cannot be more distinct remainders than the number which is one less than the divisor; hence if there be n units in the divisor, the period will consist of some one of the number of figures $(n-1)$, $\frac{1}{2}(n-1)$, $\frac{1}{3}(n-1)$, etc.

88. Suppose it required to find the fractions equivalent to the decimals $\cdot 0\dot{7}$ and $\cdot 21\dot{2}\dot{3}$,

Let $x = \cdot 0\dot{7}$	Let $y = \cdot 21\dot{2}\dot{3}$
then $10x = \cdot\dot{7}$	$100y = 21\cdot\dot{2}\dot{3}$
and $100x = 7\cdot\dot{7}$	$10000y = 2123\cdot\dot{2}\dot{3}$

Then subtracting the second line from the last in each example, we get

$90x = 7$	$9900y = 2123 - 21 = 2102$
therefore $x = \dfrac{7}{90}$.	$y = \dfrac{2102}{9900} = \dfrac{1051}{4950}$.

In a similar manner may any decimal whatever be converted into a fraction, and in any operation recurring decimals may always be replaced by their equivalent fractions. Finite decimals can always be expressed fractionally by the principles of their notation; thus

$$\cdot 0325 = \frac{325}{10000} = \frac{13}{40}, \text{ and } \cdot 75 = \frac{75}{100} = \frac{3}{4}, \text{ and so on.}$$

89. A compound number of several denominations may be reduced to an equivalent decimal of a higher denomination; and conversely, we can find the value of a decimal in terms of the lower denominations.

EXAMPLES.

1. Reduce 15*s*. 7½*d*. to the decimal of a pound.

Writing the different denominations vertically, beginning with the lowest, we first divide 2 farthings by 4, which gives ·5 of a penny; then dividing 7·5 of a penny by 12, gives ·625 of a shilling; and, lastly, 15·625 shillings divided by 20, gives ·78125 of a pound: hence 15*s*. 7½*d*. = ·78125 of a pound.

$$\begin{array}{r|l} 4 & 2\cdot 00000 \\ 12 & 7\cdot 50000 \\ 20 & 15\cdot 62500 \\ & \cdot 78125 \end{array}$$

2. Find the value of ·625 of a pound.

Multiplying by 20, the product is 12 shillings and ·5 of a shilling; then multiplying ·5 of a shilling by 12, the product is 6 pence: hence £·625 = 12*s*. 6*d*.

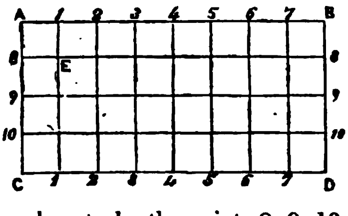
$$\begin{array}{lr} £ & \cdot 625 \\ & 20 \\ \hline s. & 12\cdot 500 \\ & 12 \\ \hline d. & 6\cdot 000 \end{array}$$

DUODECIMALS.

90. When a compound number is multiplied by an abstract number, that is, taken as many times as there are units in the multiplier, the product is a compound number consisting of denominations similar to those of the multiplicand; but in the application of arithmetic to geometry, we meet with examples in which both factors are concrete numbers, as feet and inches, and the denominations of the product are entirely different from those of either of the factors. The connexion between the measures of length and the measures of surface constitutes

the foundation of the application of arithmetic to geometry. This connexion will be understood from the following illustration.

Let A B C D be a figure which geometers term a *rectangle*, having the side A B 8 inches in length, and the side A C 4 inches; then divide A B into 8 equal parts, and also C D, which is equal to A B, into the same number of parts, the points of division in both being 1, 2, 3, 4, 5, 6, 7. In like manner divide A C and B D, which are likewise equal, into four equal parts by the points 8, 9, 10. Join the points 1, 1; 2, 2; 3, 3; . . . . 10, 10; then the figure A B C D is evidently divided into 8 times 4, or 32 smaller rectangles, any one of which, as A 1 E 8, is called a *square*, because it is a rectangle whose sides are all equal, A 1 being equal to A 8. Now each of these 32 squares has its sides one inch in length, and is termed a *square inch;* consequently the figure A B C D contains 8 times 4, or 32 *square inches.* In a similar manner, if A B = 8 feet, and A C = 4 feet, then the figure will contain 32 *square feet*, and so on.



91. Suppose now that A B is 67 inches, or 5 feet 7 inches in length, and A C is 38 inches, or 3 feet 2 inches in length; then, as we have seen, the rectangle A B C D will contain 67 times 38, or 2546 square inches. But as 1 square foot contains 144 square inches, if we divide 2546 by 144, we get 17 square feet and 98 square inches for the content of the rectangle. If, now, 5 feet 7 inches be multiplied by 3 feet 2 inches, the result ought to be the same as we have here obtained. To effect this multiplication, place the factors so that feet may be under feet, inches under inches, and proceed as in the multiplication of decimals, recollecting to carry 1 for every 12, because it is this number which connects the different denominations in the *duodecimal system.* In order to observe the similarity of the processes in the multiplication of decimals and duodecimals, we shall give the mode of multiplying 5·7 by 3·2, in connexion with the other.

In Decimals.

5·7	$5+\frac{7}{10}$
3·2	$3+\frac{2}{10}$
171	$17+\frac{1}{10}$
114	$1+\frac{1}{10}+\frac{4}{100}$
18·24	$18+\frac{2}{10}+\frac{4}{100}$

In Duodecimals.

	Ft.	In.	
	5	7	
	3	2	
	16	9	
		11	2
Sq. ft.	17	8	2
		12	
		98 sq. in.	

Feet.

$$5+\frac{7}{12}$$
$$3+\frac{2}{12}$$
$$16+\frac{9}{12}$$
$$\frac{11}{12}+\frac{2}{144}$$
$$\text{Sq. ft. } 17+\frac{8}{12}+\frac{2}{144}.$$

Thus the processes are precisely similar, and the fractional ones serve to illustrate the other, because if the denominators 10 and 12 were omitted, the corresponding multiplications would have precisely the same

form. In the duodecimal multiplication it is important to notice that while the product of feet by feet is *square feet*, and the product of inches by inches is *square inches*, the product of feet by inches is neither square feet nor square inches, but *rectangles, each one foot in length and one inch in breadth, that is, twelfths of a square foot, or* 12 *square inches.*

92. As another example, let it be required to multiply 23 feet 7 inches 9 parts by 19 feet 3 inches 6 parts.

Let 10 be expressed by π and 11 by ϵ; then 23 transformed to the duodecimal scale will be expressed by *one-twelve* + 11 or 1 ϵ, and 19 by *one-twelve* + 7 or 17; consequently the multiplication may be made *entirely* in the duodecimal scale, or partly in that scale and partly in the decimal scale in the following manner:—

Ft.	In.	Pts.		
23	7	9		
19	3	6		
449	3	3		
5	10	11	3	
	11	9	10	6
456	2	0	1	6

$$
\begin{array}{r}
1\epsilon\cdot79 \\
17\cdot36 \\
\hline
\epsilon9\pi6 \\
5\pi\epsilon3 \\
11963 \\
1\epsilon79 \\
\hline
320\cdot2016 \\
12 \\
\hline
38 \\
12 \\
\hline
456 \text{ sq. feet.}
\end{array}
$$

In this way the multiplication of 23 by 19 is made in the decimal scale, while all the others are effected by the duodecimal scale, and the result is obtained in square feet, twelfths of square feet, hundred and forty-fourths of square feet or square inches, and so on. But in the other process, the result 320·2016 is expressed *entirely* in the duodecimal scale; therefore 320 must be transformed from the duodecimal to the decimal scale, and then the results are precisely the same by both methods. If the multiplication be effected by taking aliquot parts of the multiplicand, the process would be as in the margin, where the multiplicand is multiplied by 19, as in Compound Multiplication, and then the aliquot parts are taken.

		Ft.	In.	Pts.		
		23	7	9		
				2×9+1		
		47	3	6		
				9		
In.	Ft.	425	7	6		
2	$\frac{1}{6}$	23	7	9		
1	$\frac{1}{2}$	3	11	3	6	
Pt.	In.	1	11	7	9	
6	$\frac{1}{2}$	0	11	9	10	6
		456	2	0	1	6

RATIO, PROPORTION, AND PROGRESSIONS.

93. The *ratio* of one number to another is the relation which the former has to the latter, the comparison being made by considering what multiple, part, or parts, or, which is the same thing, what *fraction* the first is of the second. Thus the ratio of 8 to 12 is the same as that of 2 to 3, or of $\frac{2}{3}$ to 1; because in each pair of numbers the first is $\frac{2}{3}$ of the second. Hence if the ratio of any two numbers be represented by a fraction, there must be a tacit reference to unity. In this manner the ratio of 20 to 4

is *equal to* $\frac{20}{4} = \frac{5}{1} = 5$, and the ratio of 15 to 30 is equal to $\frac{15}{30} = \frac{1}{2}$. The ratio of one number to another is expressed by two points placed between them, as 3 : 4. The numbers are called the *terms* of the ratio, the former being called the *antecedent* and the latter the *consequent*.

94. If the first of four numbers has to the second the same ratio which the third has to the fourth, the four numbers are said to form a *proportion*. Thus the ratio of 2 : 3 is $\frac{2}{3}$, and the ratio of 10 to 15 is $\frac{10}{15} = \frac{2}{3}$; hence, these ratios being equal, the four numbers 2, 3, 10, 15, are proportional. This proportion is expressed in the following manner; 2 : 3 :: 10 : 15, and is read as 2 is to 3, so is 10 to 15. In a proportion thus expressed, the numbers 2 and 15 are called the *extreme terms*, or simply the *extremes*, and the numbers 3 and 10 the *means*.

From the equality of the ratios $\frac{2}{3}$ and $\frac{10}{15}$, we have $\frac{2}{3} = \frac{10}{15}$, and multiplying both by 3 times 15 or 45, we get

$$\frac{2}{3} \times 45 = \frac{10}{15} \times 45\text{; that is, } 2 \times 15 = 10 \times 3\text{;}$$

hence, *if four numbers are in proportion, the product of the extremes is equal to the product of the means.*

95. The order of the terms of a proportion may be changed, provided that in the new arrangement the product of the extremes is equal to that of the means. Thus in the proportion 2 : 3 :: 10 : 15, the following arrangements may be made:—

2 : 3 :: 10 : 15	3 : 2 :: 15 : 10
2 : 10 :: 3 : 15	3 : 15 :: 2 : 10
15 : 10 :: 3 : 2	10 : 2 :: 15 : 3
15 : 3 :: 10 : 2	10 : 15 :: 2 : 3

96. Also if the corresponding terms of two proportions be multiplied together, the products thence arising will give four numbers, which are proportionals. Thus if 30 : 15 :: 6 : 3 and 2 : 3 :: 4 : 6 be two proportions, then we have

$$\frac{30}{15} = \frac{6}{3} \text{ and } \frac{2}{3} = \frac{4}{6}\text{; consequently } \frac{30}{15} \times \frac{2}{3} = \frac{6}{3} \times \frac{4}{6}, \text{ or } \frac{30 \times 2}{15 \times 3} = \frac{6 \times 4}{3 \times 6}\text{;}$$

hence $30 \times 2 : 15 \times 3 :: 6 \times 4 : 3 \times 6$, or 60 : 45 :: 24 : 18.

In a similar manner it may be shown that if the terms of a proportion be squared or cubed, or if their square roots or cube roots be extracted, the results will still constitute a proportion.

97. Hence, since the product of the extremes is equal to that of the means, *if the product of the means be divided by one of the extremes, the quotient will be the other extreme; or if the product of the extremes be divided by one of the means, the quotient will be the other mean.* The operation by which the fourth term of a proportion is found, any three of them being known, is called the *Rule of Three.*

98. Numbers may be also compared by observing how much the one differs from the other. Thus the difference between 8 and 19 is 11;

and if we take two other numbers connected in the same manner, that is, two numbers whose difference is 11, as 51 and 62, then the four numbers 8, 19, 51, 62, form what is usually termed an *arithmetical proportion*. It will be obvious that $62 + 8 = 51 + 19$; that is, *the sum of the extremes is equal to the sum of the means.*

99. A set or series of numbers is said to be in *continued* arithmetical proportion, or in *arithmetical progression*, when the difference between every two succeeding terms of the series is the *same*. Thus, the two series of numbers,

2, 5, 8, 11, 14, etc., and 100, 90, 80, 70, 60, etc.,

are in arithmetical progression, the *common difference* of the former being 3, and that of the latter 10. It is obvious that if we take any three terms of the first of these series, as 5, 8, 11, *the sum of the extremes will be equal to twice the mean*, that is, $5 + 11 = 2 \times 8$; since 8 is just as much above 5 as it is below 11. Hence *half the sum of any two numbers is the arithmetical mean between them.*

100. A set or series of numbers is said to be in *continued* proportion, or in *geometrical progression*, when the ratio of any term to the preceding term of the series is the *same*. Thus the two series of numbers,

$$1, 2, 4, 8, 16, \text{etc., and } 9, 6, 4, \frac{8}{3}, \frac{16}{9}, \text{etc.,}$$

are in geometrical progression, the *common ratio* of the former being 2, and that of the latter $\frac{2}{3}$. If we take any three terms of the first of these series, as 4, 8, 16, *the product of the extremes will be equal to the square of the mean*, since $4 \times 16 = 8^2$; hence *the square root of the product of any two numbers is the geometrical mean between them.*

The properties of a series of quantities, either in arithmetical or geometrical progression, are investigated and applied in the Algebra, and require no further notice here.

RULE OF THREE.

101. This is the most extensive and useful rule in Arithmetic, and it is termed the *The Rule of Three*, because in it three quantities are given and a fourth is to be found, to which that one of the three quantities which is of the same kind with it shall have the *same* ratio as one of the remaining two has to the other of these.

Suppose it required to find what 17 yards will cost, if 52 yards cost £10. 8*s*. Reducing £10. 8*s*. to shillings, we get 208 shillings for the cost of 52 yards; therefore, the cost of 1 yard must be $\frac{208}{52}$, or 4 shillings, and since the cost of 17 yards will be 17 times the cost of 1 yard; therefore 17 yards will cost 17 times 4 shillings, or 68 shillings, which is £3. 8*s*. Hence £10. 8*s*. has the same ratio to £3. 8*s* which 52 yards has to 17 yards, and the proportion is thus written:—

52 yards : 17 yards :: £10. 8*s*. : £3. 8*s*.

Again, were it required to find in how many days 27 men would finish a piece of work which 15 men, working at the same rate, could accomplish in 18 days; it is evident that the more men there are employed

the less time will they require to finish it, and *vice versâ*. In this case there exists a proportion, but the order of it is *inverted*; because, if 15 men require 18 days, 27 men will require, not *more*, but *less* than 18 days; consequently 18 days must have the same ratio to the number of days required which 27 men has to 15 men; and therefore the proportion will be

$$27 \text{ men} : 15 \text{ men} :: 18 \text{ days} : x \text{ days},$$

where x denotes the number of days which 27 men would require; consequently (97) the extreme term of the proportion is

$$x = \frac{15 \times 18}{27} = \frac{15 \times 2}{3} = 5 \times 2 = 10 \text{ days}.$$

102. When it is proposed to resolve a question by means of a Rule of Three, first ascertain whether the solution can depend on proportion, and, if it does, then assign to each term the place which it ought to occupy. But as the placing of the numbers constitutes the chief difficulty, the following remarks will be found useful in assisting the student to write down correctly the first three terms of the proportion.

Place that quantity which is of the same kind with the fourth or required quantity as the third term of the proportion; then consider, from the nature of the question, whether the fourth quantity is to be less or greater than that in the third term. If the fourth quantity is to be less than the third, place the less of the two other quantities in the second term; but if the fourth is to be greater than the third, place the greater of the two others in the second term.

If the quantities consist of several denominations, reduce the first and second to the same denomination, the lowest in either, and reduce the third term to the lowest denomination in it; then (97) the fourth term is obtained by multiplying the second and third terms and dividing their product by the first term. The quotient is the fourth quantity or number required, and it is in the same denomination to which the third term was reduced.

If the first and second terms be divided or multiplied by the same number, their ratio will not be altered; and if all the numbers be regarded as abstract numbers, we may also divide the first and third terms by the same number without destroying the proportion.

Ex. 1. Find the value of 36 cwt. 1 qr., if 2 cwt. 2 qrs. 10 lb. cost £4. 7*s*. 9½*d*.

Here the third term must be money, and as 36 cwt. 1 qr. will cost *more* than 2 cwt. 2 qrs. 10 lb., the greater of these quantities must be placed in the second term, and the first three terms of the proportion must be arranged as follows:—

Cwt. qrs. lb.		Cwt. qrs.		£. s. d.
2 2 10	:	36 1	::	4 7 9½
4		4		2
10 qrs.		145 qrs.		8 15 7
28		28		7
80		1160		61 9 1
21		290		
29)29,0 lb.		29)406,0 lb.		*Ans.* £61. 9*s*. 1*d*.
1		14		

Here the second term is 14 times the first; therefore the fourth must be 14 times the third.

Ex. 2. If a person travel 1800 miles in 7 days of 16 hours each, in how many days of 12 hours each will he travel the same distance?

As the fourth term is to be days, the third must be 7 days; but if it require 7 days of 16 hours each to travel the given, or any other distance, it will require a greater number of days of only 12 hours each to accomplish the same distance; therefore the quantities must be placed in the following manner:—

$$\begin{array}{ccccccccccc} \text{Hrs.} & & \text{Hrs.} & & \text{Days.} & & \text{Hrs.} & & \text{Hrs.} & & \text{Days.} \\ 12 & : & 16 & :: & 7; & \text{or} & 3 & : & 4 & :: & 7 \end{array}$$

$$\begin{array}{r} 4 \\ \hline 3)28 \\ \hline 9 \end{array}$$ days and $\frac{1}{3}$ of a day of 12 hours each;

therefore he would travel the distance in 9 days 4 hours. The distance, 1800 miles, has not been employed, being a superfluous quantity.

Ex. 3. If the provisions of a garrison will serve 2500 men for 73 days, how long will they last if the garrison be reinforced by 750 men?

Here the fourth term is evidently less than 73 days, because the garrison is increased from 2500 to 3250 men; hence, 3250 men is the first term, 2500 men is the second, and 73 days is the third term of the proportion.

$$\begin{array}{lccccl} & 3250 \text{ men} & : & 2500 \text{ men} & :: & 73 \text{ days} \\ \text{or} & 65 & : & 50 & & \underline{\quad 10} \\ \text{or} & 13 & : & 10 & & 13)730(56\tfrac{2}{13} \text{ days.} \\ & & & & & \underline{\ 65} \\ & & & & & \quad 80 \\ & & & & & \underline{\quad 78} \\ & & & & & \quad\ \ 2 \end{array}$$

COMPOUND PROPORTION.

103. Questions frequently arise in which five quantities are given to find a sixth, or seven quantities to find an eighth, and so on. In such cases it becomes necessary to repeat the process already adverted to in the Rule of Three, or to combine two or more proportions so as to reduce them all to a single proportion, and then if any three of the terms of the reduced proportion be known, the fourth can be found as before.

Ex. 1. If the expenses of 7 persons for 3 months be 70 guineas, what will be the expenses of 10 persons for 12 months at the same rate?

Here we must find what the expenses of 10 persons for 3 months would be, and this is done by the Rule of Three in the following manner:—

7 persons : 10 persons :: 70 guineas : 100 guineas .... (A).

Now, since 10 persons expend 100 guineas in 3 months, we must next inquire how much they will expend in 12 months; and, by the same rule, we get

3 months : 12 months :: 100 guineas : 400 guineas .... (B).

Instead of repeating the operation in the Rule of Three, the terms of

the proportion (A) may be multiplied by the corresponding terms of the proportion (B), and the products will still be proportional (96). Hence

$$7 \times 3 : 10 \times 12 :: 70 \times 100 : 100 \times 400;$$

but as the ratio of the third term to the fourth will not be altered by dividing each of its terms by 100, it is obvious that the operation for finding the fourth term (100 guineas) in (A) is superfluous, and may be dispensed with entirely. The statement of the terms and the operation will then be as follows:—

7 persons : 10 persons :: 70 guineas;
3 months : 12 months,

then $\frac{70 \times 10 \times 12}{7 \times 3} = 10 \times 10 \times 4 = 400$ guineas, the expense required.

Ex. 2. If the carriage of 30 tons, through 36 miles, cost £12. 10*s*., what weight ought to be carried 48 miles for £6. 13*s*. 4*d*.

Here weight is required, and therefore 30 tons will stand in the third term; then it is obvious that the weight which will be carried through 36 miles for £6. 13*s*. 4*d*. will be less than the weight carried through the same distance for £12. 10*s*. On this account £12. 10*s*., or 3000 pence, will be the first term, and 6*l*. 13*s*. 4*d*., or 1600 pence, the second term of the first proportion. Again, whatever be the weight carried through 36 miles, for £6. 13*s*. 4*d*., a less weight will be carried through 48 miles for the same sum; hence, on this account, 48 miles is to be the first term, and 36 miles the second term of the second proportion. Then multiply the corresponding terms of these two ratios, the reduced proportion is $3000 \times 48 : 1600 \times 36 :: 30$ tons : 12 tons.

3000*d*. : 1600*d*. :: 30 tons
48 miles : 36 miles,

hence $\frac{30 \times 1600 \times 36}{3000 \times 48} = \frac{16 \times 36}{48}$

$= \frac{16 \times 3}{4} = 4 \times 3 = 12$ tons.

104. If one or more of the terms be fractions or decimals, the result will be obtained in the same manner, provided the principles of the multiplication and division of fractions or decimals be carefully observed.

INVOLUTION.

105. By multiplying a number by itself *one*, *two*, *three*, or $(n-1)$ times successively, we obtain the *second*, *third*, *fourth*, or n^{th} powers of that number; hence *a power of a number is the number arising from successive multiplications by itself.* Thus $3 \times 3 = 9$, is the square or second power of 3; and $5 \times 5 \times 5 = 125$, the cube, or third power of 5. In a similar manner the square, the cube, and generally any power of a fraction or a decimal is found by multiplication. So $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, the square of $\frac{2}{3}$, and $\cdot025 \times \cdot025 = \cdot000625$, the square of $\cdot025$, and so on.

These operations are denoted by means of *Indices*, or small figures placed on the right of the numbers a little above the line; thus, $2^2 = 2 \times 2 = 4$, $3^3 = 3 \times 3 \times 3 = 27$, and $2^5 = 32$, where the *index*

or *exponent* denotes the number of factors employed. This method of notation furnishes some important conclusions; for, since $5^2 = 5 \times 5$ and $5^4 = 5 \times 5 \times 5 \times 5$; therefore we have,

$$5^4 \times 5^2 = (5 \times 5 \times 5 \times 5) \times (5 \times 5) = 5^6 = 5^{4+2};$$

$$5^4 \div 5^2 = (5 \times 5 \times 5 \times 5) \div (5 \times 5) = \frac{5 \times 5 \times 5 \times 5}{5 \times 5} = 5^2 = 5^{4-2};$$

consequently *powers of the same number are multiplied by adding their indices and divided by subtracting the index of the divisor from that of the dividend.*

Thus, since $11 = 4 + 4 + 3 = 4 + 4 + 4 - 1$; therefore,

$$5^{11} = 5^4 \times 5^4 \times 5^3 = 5^4 \times 5^4 \times 5^4 \div 5 = 625 \times 625 \times 125 = 48828125.$$

The 12^{th} power of 5 is $5^4 \times 5^4 \times 5^4$ = the cube of 5^4 or $(5^4)^3$, or it is $5^3 \times 5^3 \times 5^3 \times 5^3$ = the fourth power of 5^3 or $(5^3)^4$; hence $(5^3)^4 = (5^4)^3 = 5^{12}$; and, therefore, *the power of a power of a number is expressed by multiplying the index or exponent by the degree of the power.*

Hence, $9^6 = 9^{1 \times 2 \times 3} = 9 \times 9^2 \times 9^3 = 9 \times 81 \times 729 = 531441$, and the same result will be obtained if either of the following indicated operations be performed:—

$$9^6 = 9^2 \times 9^4 = 9^3 \times 9^3 = 9^5 \times 9 = 81^3.$$

EVOLUTION.

106. A *root* of a number is such a number as, being multiplied by itself *one*, *two*, *three*, or $(n-1)$ times, produces the given number; and the operation by which the root is obtained is termed *Evolution.* Thus the second or square root of 25 is 5, since $5 \times 5 = 25 = 5^2$; the third, or cube root of 27 is 3, for $3^3 = 3 \times 3 \times 3 = 27$, and the square root of $\cdot 0625$ is $\cdot 25$, since $\cdot 25 \times \cdot 25 = \cdot 0625$. Also the cube root of $\frac{8}{27}$ is $\frac{2}{3}$, because $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$. The operation of evolution is indicated by the sign $\sqrt{}$, accompanied with a small figure in the opening. The sign $\sqrt{}$ is called the *radical sign,* and the figure in its opening indicates the particular root to be determined. In the case of the square root the figure 2 in the radical sign is always omitted; hence the square root of 25 is expressed by $\sqrt{25}$, the cube, or third root of 27, by $\sqrt[3]{27}$, and so on. Again, by involution $(8^2)^3 = 8^{2 \times 3} = 8^6$; therefore, conversely, the cube, or third root of 8^6 is 8^2, where the index 2 is found by dividing 6, the index of the power, by 3 the index of the root. Hence generally *the n^{th} root of a power of a number is that power of the number whose index is the n^{th} part of the index of the proposed power.* Thus the square root of 2^6 is 2^3; the cube root of 2^6 is 2^2 and the n^{th} root of 2^n is 2.

Suppose, now, we take a number with a *fractional* index, as $8^{\frac{1}{2}}$, and treat the fraction exactly as it were an integer; then since $8^3 \times 8^3 = 8^{3+3} = 8^6$, we have $8^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 8^{\frac{1}{2}+\frac{1}{2}} = 8^1 = 8$; but $\sqrt{8} \times \sqrt{8}$ must necessarily produce 8; hence $\sqrt{8} = 8^{\frac{1}{2}}$, and the fractional index $\frac{1}{2}$ will represent the *second* or square root. Again, since $8^2 \times 8^2 \times 8^2$

$= 8^{3+3+3} = 8^9$, we have in like manner $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$ $= 8^1 = 8$; but $\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = 8$; hence $\sqrt[3]{8} = 8^{\frac{1}{3}}$, and the index $\frac{1}{3}$ characterizes the *third* or cube root; and so on. Hence

$$\sqrt{64} = \sqrt{(16 \times 4)} = \sqrt{16} \times \sqrt{4} = 4 \times 2 = 8;\ \sqrt{35} = \sqrt{5} \times \sqrt{7};$$

$$\sqrt{\frac{49}{36}} = \frac{\sqrt{49}}{\sqrt{39}} = \frac{7}{6};\ \text{for } \frac{7}{6} \times \frac{7}{6} = \frac{49}{36};\ \text{and also}$$

$$(11)^{\frac{1}{6}} = 11^{\frac{1}{2}\times\frac{1}{3}} = (11^{\frac{1}{2}})^{\frac{1}{3}} \text{ or } (11^{\frac{1}{3}})^{\frac{1}{2}};\ \text{hence } \sqrt[6]{11} = \sqrt[3]{\sqrt{11}} = \sqrt{\sqrt[3]{11}};$$

$$(11)^{\frac{3}{2}} = 11^{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}} = 11^{\frac{1}{2}\times 3} = 11^{3\times\frac{1}{2}} = 1331^{\frac{1}{2}} = \sqrt{1331}.$$

EXTRACTION OF THE SQUARE ROOT.

107. The square root of a number is that number which multiplied by itself produces the proposed number. There are many numbers whose square roots cannot be determined exactly, as 5, 7, 10, etc., but they may always be found to any degree of accuracy by means of decimals. And since $10^2 = 100$, $100^2 = 10000$, etc., it follows that the squares of all numbers between 1 and 10 must consist of 1 or 2 figures; the squares of all numbers between 10 and 100 must consist of more than 2, but not more than 4 figures; and the squares of all numbers between 100 and 1000 must consist of more than 4, but not more than 6 figures, etc.; hence, if a number be proposed to find its square root, we must place a point over every second figure, beginning at the *unit's* place; thus dividing the number into periods of two figures each, excepting the last period, which may consist of either one or two figures as the case may be. There will be just as many figures in the root as there are periods. With respect to a number composed of integers and decimals, the points will necessarily fall above the *second*, *fourth*, *sixth*, etc. decimals, counting from the unit's place, which we must do in all cases, and when there is no unit, as in the case of decimals, its place must be supplied with a cipher, and a point placed over it; hence the number of decimals must be one of the even numbers, 2, 4, 6, 8, etc. A cipher must be added to the right of an odd number of decimals to complete the period.

108. Let the several parts of a number be denoted by a, b, c; then $(a + b + c)^2$ is the same as $a + b + c$ repeated a times, b times, and c times. But $a + b + c$ repeated a times is $= a^2 + ab + ac$, b times $= ab + b^2 + bc$, and c times $= ac + bc + c^2$; hence by adding all together we get the square of $a + b + c$ expressed in either of the following forms:—

$$(a+b+c)^2 = a^2 + (2a+b)b + (2a+2b+c)c \qquad \text{or } (a+b+c)^2 = c^2 + 2c(a+b) + b^2 + 2ab + a^2$$

109. From the latter of these forms the rule for squaring will be, *square each part, and multiply all that precede by twice that part;* and a reverse rule for extracting the square root immediately presents itself. Let n denote the given number, and take a number a whose square does not exceed n. Find the remainder; take a second number

b, such that the remainder will bear the subtraction of the square of b and twice b multiplied by the preceding part a. If there be a remainder, take a third number c and find whether the second remainder will allow of the subtraction of the square of c and twice c multiplied by $a + b$. Proceed in this manner till the process terminate, or until the root be obtained to the nearest unit. The first form, however, affords an easy process for forming the numbers to be subtracted. For we have only to *double the sum of all the parts which have been obtained, add the new part, and multiply the sum by the new part.* Thus after a^2 is subtracted; $(2a + b) \times b$ is the next subtrahend; $\{2(a + b) + c\} \times c$ is the third, and so on.

Ex. Let it be required to extract the square root of 273529. Begin at the unit's place and point off the periods as already directed; then since there are 3 periods, the first figure of the root will be in the place of hundreds. This first figure will be 5, because 500^2 is less and 600^2 is greater than 270000. Subtracting 250000 from the number leaves the remainder 23529. Let 2 tens or 20 be the next part; then by the processes given above, the number to be subtracted will be $20^2 + 2 \times 20 \times 500$, or $(2 \times 500 + 20) \times 20$, viz., 20400. Lastly, take 3 as the next part of the root; then, as before, $3^2 + 2 \times 3 \times 520$, or $(2 \times 520 + 3) \times 3$, gives 3129, which, being exactly equal to the second remainder, shows that the root is $= 523$.

$$\begin{array}{rl}
2\dot{7}3\dot{5}2\dot{9} & (500 + 20 + 3 \\
\underline{250000} & \quad \text{or } 523 \\
23529 & \\
\underline{20400} & \\
3129 & \\
\underline{3129} &
\end{array}$$

It will be covenient to arrange the operation of forming the numbers to be subtracted on the left of the successive remainders, as in the following example:—

Ex. Find the square root of 293764.

$$\begin{array}{r|l}
 & 2\dot{9}3\dot{7}6\dot{4}\ (500 + 40 + 2 \\
 & \underline{250000} \quad \text{or } 542 \\
1000 & 43764 \\
40 & \underline{41600} \\
1000 & \ 2164 \\
80 & \ \underline{2164} \\
2 &
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & 2\dot{9}3\dot{7}6\dot{4}\ (542 \\
 & \underline{25} \\
104 & 437 \\
\underline{4} & \underline{416} \\
1082 & \ 2164 \\
 & \ \underline{2164}
\end{array}$$

In the first of these, the numbers are written at length, but in the second the ciphers on the right are omitted, and each period is annexed to the remainder as it is wanted. The parts of the root a, b, c may be any whatever; but when they are confined to the nearest number of hundreds, tens, and units, the parts $2a$, and $2a + 2b$ which form so large a portion of the factors $2a + b$, $2a + 2b + c$ soon become available for the determination of the succeeding parts. Thus 1000 is contained in 43764 more than 40 but less than 50 times. In this way the successive figures of the root can be written in the places which they occupy in the decimal scale, as in the second mode of operation.

110. When the proposed number has not an exact square root, it may be obtained to any degree of accuracy by means of decimals, and

when one more than half the number of figures in the root have been obtained, the remaining figures may be found by dividing the last remainder by its corresponding divisor as in contracted division of decimals.

Ex. Find the square root of 10, and also of 17·108 each to 6 places of decimals.

```
            10(3·162277               17·1080(4·136182
             9                        16
   61   |100                   81   |110
    1   | 61                    1   | 81
   626  |3900                  823  |2980
     6  |3756                    3  |2469
   6322 |14400                 8266 |51100
      2 |12644                    6 |49596
   6,3,2,4  1756               8,2,7,2  1504
            1265                         827
             491                         677
             443                         661
              48                          16
              44                          16
```

111. The square root of a fraction is equal to the square root of the numerator divided by the square root of the denominator. If the terms of the fraction have not exact square roots, the fraction may be reduced to a decimal, and its root extracted, or if the denominator has no exact root, the terms of the fractions may be multiplied by such a number as will render the denominator a complete square. Thus :—

$$\sqrt{\frac{3}{10}} = \sqrt{\frac{30}{100}} = \frac{\sqrt{30}}{\sqrt{100}} = \frac{\sqrt{30}}{10} = \frac{5\cdot4772256}{10} = \cdot54772256;$$

$$\sqrt{\frac{7}{8}} = \sqrt{\frac{14}{16}} = \frac{\sqrt{14}}{\sqrt{16}} = \frac{\sqrt{14}}{4} = \frac{3\cdot7416574}{4} = \cdot9354143.$$

EXTRACTION OF THE CUBE ROOT.

112. The cube root of a number is that number which multiplied twice by itself produces the proposed number. Since $10^3 = 1000$, $100^3 = 1000000$, etc., the cubes of all numbers between 1 and 10 must consist of 1, 2, or 3 figures ; the cubes of all numbers between 10 and 100 must consist of more than 3 but not more than 6 figures, and so on ; hence the number whose cube root is to be found is to be divided into periods of three figures each, beginning at the *unit's* place, and putting a point over it, and a point over every third figure towards the left in integers and the right in decimals. In decimals supply one or two ciphers, if necessary, to complete the period of 3 figures.

113. Let a and b represent the tens and units of the root ; then $(a + b)^2$ is the same as $a + b$ repeated a times and b times. But $a + b$ repeated a times is $a^2 + a\,b$, and repeated b times is $a\,b + b^2$; hence $a + b$ repeated $a + b$ times, or $(a + b)^2 = a^2 + 2\,a\,b + b^2$. Again

$(a+b)^3$ is the same as $a^2+2\,ab+b^2$ repeated a times and b times; that is $(a+b)^3=(a^3+2a^2b+ab^2)+(a^2b+2\,ab^2+b^3)=a^3+3\,a^2b+3\,ab^2+b^3$. This may be put in either of the forms

$$a^3+(3\,a^2+3\,ab+b^2)\,b \text{ or } a^3+3\,ab\,(a+b)+b^3.$$

From this result the following simple rule for finding the cube root of a number is deduced. Let n be the number, and take a number a whose cube does not exceed n. Find the remainder, take a second number b, such that the remainder may bear the subtraction of the cube of b, and the continued product of thrice a, the second number b, and the sum of a and b. If there be a remainder, consider $a+b$ as the first number, and proceed as before.

The following mode of forming the successive numbers to be subtracted is the most convenient in practice. Write down 3 times the first number, and three times its square separately, the former one line lower than the other, and to the left of it as in the margin. To $3\,a$ add the second number b, and multiply the sum by b, placing the product below $3\,a^2$ and adding it thereto. The sum $3\,a^2+3\,ab+b^2$ being multiplied by b, produces the entire number to be subtracted. As the part $3\,a^2$ forms a large part of the factor $3\,a^2+3\,ab+b^2$, it soon becomes available for the determination of the next figure, by using it as a trial divisor. To show how the process may be continued, change a into $a+b$, and b into c in the arrangement in the margin; then $3\,a$ becomes $3\,a+3\,b$ and $3\,a^2$ becomes $3\,a^2+6\,ab+3\,b^2$; now to obtain these, we have only to add $2\,b$ to the one column, and b^2 to the sum of the last two lines of the other column. Then to $3\,a+3\,b$ add the next number c; multiply the sum by c, placing the product below $3\,a^2+6\,ab+3\,b^2$; then adding and multiplying the sum by c, the next entire number to be subtracted will be obtained. In this way the cube root may be extracted with the greatest facility, and when the root cannot be accurately obtained, it may be approximated to, and the work contracted as in the following example.

$$\begin{array}{rr}
 & 3\,a^2 \\
3a+b & \underline{3\,ab+b^2} \\
\underline{2b} & 3\,a^2+3\,ab+b^2 \\
3a+3b & \underline{b^2} \\
 & 3\,a^2+6\,ab+3\,b^2
\end{array}$$

Ex. 1. Find the cube root of 46656. Here the number of tens is evidently 3, for 30^3 is less, and 40^3 greater than 46656, or 3^3 is ess and 4^3 greater than the second period 46. Writing 3 times 30, and 3 times the square of 30 on the left, we ask how many times is 2700 contained in the remainder 19656? The quotient is 7, which will be found a unit too much; taking 6 then it is added to 90, and the sum 96 is multiplied by 6. The product 576 is written below 2700 and added thereto; then the sum 3276 is multiplied by 6, and the product being equal to the remainder, the process terminates. In the next example we shall omit the ciphers, and place the figures as they arise in their proper places.

$$\begin{array}{rrl}
 & & \dot{4}66\dot{5}6\,(30+6=36 \\
90 & 2700 & 27000 \\
\underline{6} & \underline{576} & \overline{19656} \\
\underline{96} & \underline{3276} & \underline{19656}
\end{array}$$

Ex. 2. Find the cube root of 21035·8 to ten places of decimals.

```
                                     21035·800(27·60491055944
               12                    8
67             469 }                 13035
14            1669 }....             11683
                49 }                 1352800
              2187                   1341576
816           4896 }                   11224000000
            223596 }                    9142444864
 12             36 }                    2081555136
          22852800                      2057415281
82804       331216 }                      24139855
        2285611216 }                      22860923
     8          16 }                       1278932
         228594244|8                       1143046
|·8|28|12     7453|1 }                      135886
         228601697|9 }                      114305
          22860915|1                         21581
                 8|3 }                       20575
          22860923|4 }                        1006
      22,8,6,0,9,3|2                           914
                                                92
```

In this manner we have found the cube root to ten or eleven places of decimals with comparatively little trouble. When the contraction is commenced it is only necessary to cut off *one* figure from the right of the middle column, and *two* from the right of the left column; because in this way three figures or a period is struck off from each column, the period on the right-hand column not being annexed to the right of it.

114. The cube root of a fraction may be found by reducing it to a decimal, or by multiplying both numerator and denominator by such a number as will render the denominator a cube number.

$$\text{Thus, } \sqrt[3]{\frac{5}{7}} = \sqrt[3]{\frac{5 \times 49}{7 \times 49}} = \frac{\sqrt[3]{245}}{\sqrt[3]{343}} = \frac{6\cdot257325}{7} = \cdot893904.$$

$$\sqrt[3]{\frac{3}{16}} = \sqrt[3]{\frac{12}{64}} = \frac{\sqrt[3]{12}}{4} = \frac{2\cdot289429}{4} = \cdot572357.$$

EXTRACTION OF ANY ROOT.

(*Dr Hutton's Method.*)

115. Let N be the number whose root is to be extracted, and n the index of the root; then assume a root a whose n^{th} power, a^n, is as near to the given number as convenient, and let R represent the true root of the number. Then will—

$$\mathrm{R} = a \cdot \frac{(n-1)\, a^n + (n+1)\, \mathrm{N}}{(n+1)\, a^n + (n-1)\, \mathrm{N}} = \text{root nearly.*}$$

Ex. 1. Find the cube root of 21035·8.

Here $n = 3$, and if we assume $a = 28$, which is rather too great, since the cube of 28 is $21952 = a^3$, we have

$$\mathrm{R} = 28 \cdot \frac{21952 \times 2 + 21035{\cdot}8 \times 4}{21952 \times 4 + 21035{\cdot}8 \times 2} = 28\, \frac{21952 + 21035{\cdot}8 \times 2}{21952 \times 2 + 21035{\cdot}8}$$

$$= 28 \times \frac{64023{\cdot}6}{64939{\cdot}8} = \frac{1792660{\cdot}8}{64939{\cdot}8} = 27{\cdot}6049, \text{ true to the last figure.}$$

Ex. 2. Find the fifth root of 30.

Here $n = 5$, and if we assume $a = 2$, then $a^5 = 32$, which is a little too great; hence,

$$\mathrm{R} = 2 \cdot \frac{4 \times 32 + 6 \times 30}{6 \times 32 + 4 \times 30} = \frac{2 \times 77}{78} = \frac{77}{39} = 1{\cdot}97436 \text{ nearly.}$$

Take $a = 1{\cdot}97$; then $a^5 = 29{\cdot}670928$, etc.; therefore by the formula

$$\mathrm{R} = 1{\cdot}97 \times \frac{298{\cdot}683712}{298{\cdot}025568} = 1{\cdot}9743504 = \text{root very nearly.}$$

LOGARITHMS.

116. The principle of logarithms is essentially arithmetical, depending on the relation subsisting between the corresponding terms of an arithmetical and a geometrical progression, which is this, viz.:—If any number of terms are arranged in a geometrical progression commencing

* Let $\mathrm{N} = a^n + b$, where b may be either additive or subtractive, and let $a + x$ be the true root R; then by the binomial theorem,

$$\mathrm{N} = a^n + b = (a + x)^n = a^n + n a^{n-1} x + \frac{n(n-1)}{1 \cdot 2} a^{n-2} x^2 + \ldots\ldots;$$

$$\text{hence,} \quad b = \left(n a^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^{n-2} x + \ldots\ldots\right) x;$$

$$\therefore x = \frac{b}{n a^{n-1} + \dfrac{n(n-1)}{1 \cdot 2} a^{n-2} x + \ldots\ldots}.$$

Now if x be small, we may neglect all the terms in the denominator except the first; hence, for a first approximation, we get $x = \dfrac{b}{n a^{n-1}}$. Substitute this value for x in the denominator above, and take two terms of it; then will

$$x = \frac{b}{n a^{n-1} + \dfrac{1}{2}(n-1)\dfrac{b}{a}} = \frac{2ab}{2na^n + (n-1)b};$$

$$\text{and} \quad a + x = a + \frac{2ab}{2na^n + (n-1)b} = a\,\frac{2na^n + (n+1)b}{2na^n + (n-1)b}.$$

But since $\mathrm{N} = a^n + b$, we have $b = \mathrm{N} - a^n$, and this being written for b in the last equation, we get,

$$a + x, \text{ or } \mathrm{R} = a \cdot \frac{2na^n + (n+1)\mathrm{N} - (n+1)a^n}{2na^n + (n-1)\mathrm{N} - (n-1)a^n} = a \cdot \frac{(n-1)a^n + (n+1)\mathrm{N}}{(n+1)a^n + (n-1)\mathrm{N}}.$$

This may be put in the form of a proportion, and easily recollected; thus,

$$(n+1)\,a^n + (n-1)\,\mathrm{N} : (n-1)\,a^n + (n+1)\,\mathrm{N} :: a : \mathrm{R}.$$

with 1, and over these terms are placed a corresponding series of terms of an arithmetical progression commencing with 0, it will be found that the sum of any two of the numbers in the upper line will constitute the number in that line which corresponds to the product of the two numbers in the lower line, and that the difference of any two of the upper line will be the number standing over that number in the lower line, which is equal to the quotient arising from dividing the greater number by the less. Thus let the

Arithmetical series be	0,	1,	2,	3,	4,	5,	6,	7,	etc.
Geometrical series	1,	2,	4,	8,	16,	32,	64,	128,	etc.

Here $8 \times 16 = 128$, and if the numbers 3 and 4 placed over 8 and 16 be added together, the sum is 7, which is the number in the upper line standing over the product 128. In like manner, if 64 be divided by 4, the quotient is 16, and if the number 2 placed over 4 be subtracted from 6, the number standing over 64, the difference 4 is the number standing over the quotient 16.

117. It is a further property of two such progressions, that if we double any one of the terms in the upper line, it will give the number standing over the square of the corresponding number in the lower line, and three times that number will give the term standing over the cube of the corresponding number in the lower line, and so on. The same property extends to the extraction of the square, or cube, or any other root. Thus the square root of 64 is 8, and the number standing over 64 is 6, and, dividing this by 2, gives 3, which is the number standing over 8, the square root of 64; or, if divided by 3, we have 2, the number standing over 4, the cube root of 64.

118. We have here given two of the most simple series for the sake of illustration, but with them we can only deal with the numbers belonging to these series, while in the more general form, viz.,

Arithmetical series	0,	x,	$2x$,	$3x$,	$4x$,	$5x$,	$6x$,	$7x$,	etc.
Geometrical series	1,	a^x,	a^{2x},	a^{3x},	a^{4x},	a^{5x},	a^{6x},	a^{7x},	etc.

we can include every number, integral, decimal, or mixed of both, from 0 to any extent required in the upper series, and in the lower every number, integral, decimal, or both, from 1 to any extent. The upper line constitutes a series of numbers which are termed the *logarithms* of the corresponding numbers in the lower line, and we hence obtain our first general idea and definition of a logarithm, viz.:—*The logarithm of a number is that index of the power of a given radix or base which is equal to that number.*

119. It appears, then, that logarithms are strictly of arithmetical origin, but it would be laborious, if possible, to prove by Arithmetic, unassisted by Algebra, that it is possible, by different powers of any given radix a, to express every intervening number and fraction between 0 and one hundred or a thousand millions, and to supply at the same time the means of computing them. For this reason we shall defer the more extended development of the principle and properties of logarithms to its proper place in the Algebra, and the rules for applying them, with the description of the Tables to the Introduction in the Volume of Logarithms.

APPLICATION OF ARITHMETIC TO COMMERCIAL CALCULATIONS.

PARTNERSHIP.

120. Partnership is the method of dividing any quantity into any proposed number of parts, having a given ratio to one another. By it the gains or losses of partners in trade are adjusted, the effects of bankrupts are divided amongst creditors, and contributions are levied.

When two or more partners invest their money together, and gain or lose a certain sum, it is evident that the gain or loss ought not to be divided equally among them all, unless each partner contributed the same sum. Suppose that P contributes 3 times as much as Q, it is evident that P's share of the gain or loss ought to be 3 times as much as the share of Q; hence dividing the gain or loss into 4 equal parts, P must receive or pay 3 of these parts, and Q one of them.

Ex. 1. A ship is to be insured, in which P has ventured £2500; Q, £3500; and R, £4800. The expense of insurance is £495. 10*s.*; how much must each pay of it?

The entire amount of money risked is £10800, and if the expense of insurance be divided into 10800 equal parts, each of them will express the expense of insurance for one pound of capital; consequently the sum that each must pay will be expressed by

$$\frac{£495.\ 10s.}{10800} \times 2500;\ \frac{£495.\ 10s.}{10800} \times 3500;\ \text{and}\ \frac{£495.\ 10s.}{10800} \times 4800.$$

These results are furnished by the following proportions, in which the first term is the sum of the money risked by all the partners, viz. £10800.

£.	£.	£.	*s.*	£.	*s.*	*d.*
10800 :	2500 ::	495	10 :	114	13	$11\frac{3}{4}\ \frac{1}{9}$
10800 :	3500 ::	495	10 :	160	11	$6\frac{3}{4}\ \frac{5}{9}$
10800 :	4800 ::	495	10 :	220	4	$5\frac{1}{4}\ \frac{3}{9}$
		Proof .	.	495	10	0

Ex. 2. Three persons, A, B, C, have a pasture in common, for which they are to pay £30 per annum, into which A put 7 oxen for 3 months, B put 9 oxen for 5 months, and C put 4 oxen for 12 months; how much must each person pay of the rent?

The principle in questions of this kind is, that the same sum should be paid for the keep of one ox for one month or one year by each person. Now since A put in 7 oxen for 3 months, he might have had an equal share of the pasture by putting in 7 oxen × 3, or 21 oxen for 1 month. In like manner, B might have put in 9 oxen × 5, or 45 oxen for 1 month, and C might have put in 4 oxen × 12, or 48 oxen for 1 month. Hence, if we divide £30 into $7 \times 3 + 9 \times 5 + 4 \times 12$, or 114 equal parts, A must pay 7×3 or 21, B must pay 9×5 or 45, and C must pay 4×12 or 48 of those parts. The three persons A, B, C, must therefore pay respectively,

$$£\frac{7 \times 3 \times 30}{114}\ £\ \frac{9 \times 5 \times 30}{114},\ \text{and}\ £\frac{4 \times 12 \times 30}{114},$$

or £5. 10*s.* $6\frac{1}{4}d.\ \frac{5}{19}$, £11. 16*s.* $10\frac{0}{4}d.\ \frac{8}{19}$, and £12. 12*s.* $7\frac{1}{2}d.\ \frac{6}{19}$.

This question may consequently be resolved in the following manner:

			£.	£.	s.	d.
7 × 3 =	21	Then 114 : 21 ::	30 :	5	10	$6\frac{1}{4}\ \frac{5}{19}$
9 × 5 =	45	114 : 45 ::	30 :	11	16	$10\frac{0}{4}\ \frac{8}{19}$
4 × 12 =	48	114 : 48 ::	30 :	12	12	$7\frac{1}{2}\ \frac{6}{19}$
	114	Proof . .		30	0	0

The reasoning employed above may be conducted in a somewhat different manner; thus, suppose one pound were charged for the pasturage of one ox for a month, it is obvious that A would have to pay 7 pounds for having 7 oxen for 1 month, and consequently £7 × 3, or £21, for having 7 oxen in the pasture for 3 months. In like manner B would have to pay £9 × 5, or £45, for having 9 oxen at pasture for 5 months, and C, £4 × 12, or £48, for having 4 oxen at pasture for 12 months. Hence it is evident that the rent must be divided amongst them in such a manner, that if it be divided into 114 equal parts, A must pay 21, B 45, and C 48 of these parts.

INTEREST.

121. **Interest** is the sum of money paid for the use of other money, and is always estimated at so much for £100 during a year. Thus, if £100 are lent at 4 per cent., it must be understood to mean 4 per cent. per annum, that is, that £4 are paid annually for the use of £100.

Principal is the money lent; the *rate per cent.* is the interest of £100 for a year; and the *amount* is the interest and principal together. *Simple* interest is only the interest of the principal for the whole time it is lent, and *compound* interest is not only the interest of the principal for the whole time it is lent, but if the interest is *not* paid at the stated intervals it is considered as principal as soon as it is due, and then the original principal, together with the unpaid interest, forms a new principal, the interest of which becomes due at the next stated time of payment.

Ex. 1. Find the interest of £355. 12*s.* 6*d.* for 4 years at 4 per cent. per annum.

£.	£.	s.	d.	£.		£.	s.	d.	
100 :	355	12	6 ::	4		14	4	6	= interest for 1 year.
			4					4	
100)	14·22	10	0			56	18	0	= interest for 4 years.
	20								
	4·50								
	12								
	6·00								

In practice, it is usual to multiply the principal by the rate per cent., and by the number of years; then to divide by 100, as in the margin. If the interest be required for any number of days, we must find the interest for one year, or 365 days, and then by the Rule of Three find the interest for any given number of days.

£.	s.	d.
355	12	6
		4
1422	10	0
		4
£56·90	0	0
20		
s.18·00		

122. To find the interest of any sum at compound interest, it is necessary to find the amount of the principal and interest at the end of the first year, because it is this amount on which interest must be charged at the end of two years.

Ex. 2. Find the interest and amount of £400 for 3 years at 5 per cent. per annum, compound interest being allowed.

As £5 is $\frac{1}{20}$ of £100, the interest of any sum at 5 per cent. per annum is found by dividing that sum by 20; hence we have,

$$\begin{array}{rrl}
20)400 & 0 & = \text{first principal.} \\
\underline{20} & \underline{0} & = \text{first year's interest.} \\
20)420 & 0 & = \text{amount at the end of 1 year.} \\
\underline{21} & \underline{0} & = \text{second year's interest.} \\
20)441 & 0 & = \text{amount at the end of 2 years.} \\
\underline{22} & \underline{1} & = \text{third year's interest.} \\
463 & 1 & = \text{amount at the end of 3 years.} \\
\underline{400} & \underline{0} & = \text{original principal.} \\
\underline{63} & \underline{1} & = \text{entire interest for 3 years.}
\end{array}$$

123. But the best way of performing calculations in interest is to find the amount of *one* pound for any given number of years at the proposed rate of interest, and then to multiply the result by the number of pounds in the given sum. Thus in the last example, the interest of £1 for 1 year, at 5 per cent., is $\frac{1 \times 5}{100}$, or ·05 of a £., and the amount is £1·05, or £1 + ·05 of a £. Again, the interest of ·05 of a £. is $\frac{\cdot 05 \times 5}{100}$, or ·0025 of a £., and the interest of £1 being ·05 of a pound as before, the interest of £1·05 is = ·05 of a £. + ·0025, or $(\cdot 05)^2$ of a £.; hence the amount at the end of two years in pounds is $1 + \cdot 05 \times 2 + (\cdot 05)^2$, or $(1\cdot 05)^2$. In like manner, the amount in pounds at the end of three years is $(1\cdot 05)^3$, and if this be the amount of £1 in three years, the amount of £400 will be 400 times £$(1\cdot 05)^3$. The operation by this method will be as in the margin; but when the number of years is very great, the labour is so enormous that recourse must be had either to tables of interest already calculated for £1, or to logarithms.

$$\begin{array}{rr}
\text{£.} & \text{£.} \\
105^3 = & 1\cdot 157625 \\
 & \underline{400} \\
 & \text{£}\,463\cdot 050000 \\
 & \underline{20} \\
 & \underline{1\cdot 00s.}
\end{array}$$

Amount is £463. 1*s.*

If a certain sum of money has at simple interest amounted to £750 in 4 years at 5 per cent., the original sum may be found in the following manner. Take £100, and find what that sum would amount to in 4 years at 5 per cent., simple interest. The interest for one year is £5, the interest for 4 years is £20, and the amount is £120. Now £100 will have the same ratio to £120 which the original sum has to £750; hence, inversely,

$$\overset{\text{£.}}{120} : \overset{\text{£.}}{100} :: \overset{\text{£.}}{750} : \frac{750 \times 100}{120} = \text{£}625, \text{ the original sum.}$$

124. *Commission* and *Brokerage* are charges made by persons acting as agents or brokers, as a remuneration for their skill and trouble in executing business confided to their management, and are calculated at a certain rate per cent. on the amount of the transactions.

125. *Insurance* is a per centage paid to those who engage to make good to the payers any loss they may sustain by accidents from fire, or storms, etc., up to a certain amount named in the agreement. The sum of money paid for insurance is called the *premium*, and it is reckoned by a per centage upon the amount insured.

All questions relating to commission, brokerage, and insurance, are solved by the Rule of Three, as examples in simple interest.

PUBLIC FUNDS, OR STOCKS.

126. The *Public Funds*, or, as they are sometimes called, *Stocks*, are the debts contracted by the Government, and are transferable at pleasure from one person to another. The exigencies of a country often compel the governing body to negociate a *loan* with some monied persons, or great capitalists, who contract with the Government to supply the required sum on condition of receiving a certain interest or annuity until the money is repaid. The contractors do not advance the whole of the money themselves, but bring the stock into the market, and if they dispose of it for more or less than the contracting price, they gain or lose accordingly, and in this way some persons amass immense fortunes, whilst others are ruined. The business in the public funds is transacted at the Stock Exchange, and is confided to the agency of stock-brokers, who usually charge $\frac{1}{8}$ or 2*s*. 6*d*. per cent. on the amount of stock bought or sold. Similar contracts are made by large commercial companies, such as the Bank of England, the East India Company, and some Railway Companies, and their Stocks are denominated accordingly Bank Stock, East India Stock, etc., but we shall only allude to one or two of the Government stocks.

Consols is a description of stock bearing 3 per cent. interest, and it is so named from several annuities being consolidated together, and their dividends are now chargeable on the Consolidated Fund, that is, the permanent taxes.

Reduced Annuities are those which have been reduced from a higher to a lower rate of interest. *Long Annuities* are stock which terminate in 1859 or 1860, and are quoted in the price of stocks at so many years' purchase; thus, if the quotation be $8\frac{1}{2}$, then for an annuity of £30, until 1860, there must be paid $8\frac{1}{2} \times 30$, or £255.

Exchequer Bills are a kind of promissory notes issued from the Exchequer, and entitle the holder to receive the sums for which they are drawn, with interest, when they are advertised to be paid off, which is usually about 12 months from the time they are issued. Exchequer Bills are issued for £100, £200, £500, and £1000, which bear an interest of $1\frac{1}{2}d$. per cent. per diem, and sometimes 2*d*. or more, according to the state of the money market. These bills are much sought after by bankers, as a steady investment, being generally quoted at a *premium*. Thus, if the premium be 17 shillings, we must pay £100. 17*s*. for £100 Exchequer Bills.

Ex. Bought in the 3 per Cent. Consols, £540 at 91⅛; how much was paid for it, and what was the broker's charge at ⅛ per cent.?

$$\begin{array}{rrrrrrl}
& & & & \pounds. & & \\
\pounds.\ \ \pounds.\ \ \pounds. & & & & & & \\
10,0 : 54,0 :: 91\tfrac{1}{8},\ \text{or}\ 91\cdot125 & & & & & & \\
54 & & & & & & \\
\hline
364\,500 & & & & & & \\
4556\,25 & & & & & & \\
\hline
10)4920\cdot750 & & \pounds. & s. & d. & & \\
\hline
492\cdot075 & = & 492 & 1 & 6, & \text{the sum paid.} \\
\text{Brokerage, at } \tfrac{1}{8} \text{ per cent. on } \pounds540 & = & 0 & 13 & 6, & \text{the brokerage.} \\
\cline{3-5}
& & 492 & 15 & 0, & \text{the whole sum paid.}
\end{array}$$

DISCOUNT.

127. Discount is an allowance made for advancing money on securities before they are due, at a certain rate per cent.; and when the discount is subtracted from any proposed sum, the remainder is termed the *present worth*. Suppose that A owes to B £625, to be paid at the end of 4 years, but that A is desirous of paying the debt immediately. Now if A paid £625 to B, he would lose, and B would gain, 4 years' interest. We must therefore find what sum A must pay to B so that, with 4 years' interest, it may amount exactly to £625. Now if interest be reckoned at 5 per cent., then (123),

$$\overset{\pounds.}{120} : \overset{\pounds.}{625} :: \overset{\pounds.}{100} : \frac{625 \times 100}{120},\ \text{or } \pounds520.\ 16s.\ 8d.$$

Hence, £104. 3*s.* 4*d.*, the difference between £625 and £520. 16*s.* 8*d.*, must be allowed in consideration of present payment, and this is the *discount* of £625 payable at the end of 4 years at 5 per cent.; while £520. 16*s.* 8*d.* is the present worth or value of £625 due 4 years hence, discount being at 5 per cent. Since £20 is the discount of £120 due 4 years hence, at 5 per cent., it is evident that the discount of £625 will be found by the following proportion—

$$\overset{\pounds.}{120} : \overset{\pounds.}{625} :: \overset{\pounds.}{20} : \frac{625 \times 20}{120},\ \text{or } \pounds104.\ 3s.\ 4d.$$

As bills or promissory notes are usually made payable in a few months, it is customary for bankers, and those who discount bills, to consider the discount the same as the interest of the money for the short time specified. Thus the discount of £250 for 3 months at 4 per cent. would be considered the interest of the same sum for the given time and rate, viz., £2. 10*s.* The true discount would be found thus—

$$\overset{\pounds.}{101} : \overset{\pounds.}{250} :: \overset{\pounds.}{1} : \frac{250}{101},\ \text{or } \pounds2.\ 9s.\ 6\tfrac{0}{4}d.\ \tfrac{24}{101};$$

and therefore the difference is in favour of the party who discounts the bill.

EXCHANGE.

128. EXCHANGE is a term employed to designate those mercantile transactions by which the debts of individuals residing at a distance from each other are either partially or wholly liquidated without the intervention of money, by means of bills of exchange. A *bill of exchange* is an " order addressed to some person residing at a distance, directing him to pay a certain specified sum to the person in whose favour the bill is drawn." In mercantile phraseology, the person who draws a bill is termed the *drawer;* the person on whom it is drawn, and to whom it is addressed, is called the *drawee*, who is also called the *acceptor* when he engages to pay it, by writing the word *accepted*, together with his name, across the face of the bill; the person to whom it is made payable is called the *payee;* and those persons into whose hands the bill may pass previously to its being paid are, from their writing their names on the back, termed *indorsers*. The person in whose possession the bill is at any given period is termed the *holder* or *possessor*.

Suppose that A in London sells goods to B at Edinburgh, amounting to £500, and B at Edinburgh sells goods to C in London amounting to £300, and to D in London amounting to £200; these several debts may be mutually discharged in this manner:—B, instead of remitting the amount he is indebted to A, draws on C at a specified time, in favour of A, for £300, and on D, in favour of A, for £200. These bills are transmitted to A, who, on their receipt, or as soon after as convenient, presents them to C and D respectively for their acceptance. If C and D accept these bills, each becomes liable to A for the amount named in his bill, and when these bills become due, they are presented by A to C and D for payment. If they are regularly taken up, A writes a receipt on the back, and the several transactions are thus settled in the most simple and convenient manner.

The price of bills of exchange fluctuates according to the abundance or scarcity of them compared with the demand. Thus if the debts reciprocally due by London and Edinburgh were equal, they may all be discharged without the agency of money, and the price of bills of exchange would be at *par;* that is, a sum of £100 or £500 in Edinburgh will purchase a bill for £100 or £500 payable in London, and *vice versâ*. But if these two cities are not mutually indebted in equal sums, then the price of bills of exchange will be increased in the city which has the greater amount of payments to make, and reduced in the other. If Edinburgh owe London £250,000, whilst the debts due by London to Edinburgh amount only to £200,000, it is evident that the price of bills on London would rise in Edinburgh, because of the increased competition, and that the price of bills on Edinburgh would fall in London, on account of the proportionally diminished competition. The exchange would be in *favour* of London and *against* Edinburgh, and bills on London would sell in Edinburgh at a *premium*, whilst bills on Edinburgh would sell in London at a *discount*. An increased demand for bills must always enhance their price, as it would that of any other saleable article. This is the plain principle of exchange being constantly exemplified in the premium paid for bills on London, which has generally a large *balance of trade* in its favour. The premium on bills can never exceed the expense of transmitting money or bullion from one

place to another, which forms the natural limit to fluctuations in exchange.

129. The object of exchange calculations is to ascertain what a sum of money payable in one place is worth in another, either by the ratio of the two sums, considered as equivalent in the coin of those places, or by two or more ratios, formed of equivalent sums, expressed in the coin of three or more places communicating with each other, as in the example in Art. 137.

130. The *par of exchange* between two countries is that sum of the currency of either of the two countries, which, with respect to intrinsic worth, is precisely equal to a given sum of the other. In this definition of the phrase, "par of exchange," it is assumed that a given quantity of gold or silver always possesses the same intrinsic value; or that, bullion being everywhere recognized as the standard currency of the commercial world, the comparative value of the currencies of particular countries depends on the *quantity* of bullion contained in their coins, or for which their paper money, or other circulating medium, may be exchanged. In estimating the quantity of bullion contained in the currencies of different countries, a particular coin of one country is taken as the standard of comparison, and the proportion between it and the coins of other countries of mint standard weight and fineness, is ascertained. A par of exchange is thus established, by ascertaining the amount of the standard currency of any particular country which contains precisely as much gold or silver as is contained in the coin or integer with which it has been compared. Thus, for instance, the franc, which is the principal unit of value in the currency of France, weighs, when of full weight, $77\frac{1}{6}$ troy grains, and is of a fineness of $\frac{9}{10}$ths, that is, $\frac{9}{10}$ths of the pieces are pure silver; and the dollar, which is the principal unit of value in the currency of the United States of America, weighs $412\frac{1}{2}$ troy grains, and is also of a fineness of $\frac{9}{10}$ths; $5 \cdot 346$ francs is evidently, then, the par of exchange for the dollar of the United States, when no loss of time, nor, consequently, of interest, takes place in effecting such exchange; but in the sum quoted by merchants as the par of exchange between two places, an allowance for interest is usually made, depending on the period allowed for the payment of the bills of exchange passing between those places.

131. The par of exchange between different countries is, then, determined by the relative proportion of fine gold or silver which the coins compared respectively contain; but the sums which at any particular time are considered as equivalent in the money of those places are continually varying according to the momentary preponderance of their mercantile transactions. The nominal value thus acquired is termed the *rate of exchange*, and is dependent on the relation between the supply of the bills of exchange of those places and the demand for them.

132. Between two places employing the same description of money as London and Edinburgh, the rate of exchange is easily expressed by saying that the bills are at a certain premium or discount, as the case may be; between foreign countries employing different descriptions of money, the relative value of their respective coins requires to be

considered; and for countries whose standards of value are of different metals, as England and France, the relative value of the metals adopted as their respective standards is a necessary element in a calculation of their par of exchange; and consequently, in determining whether the rate of exchange is either favourable or unfavourable to one of those countries.

The principal unit of value in England is the pound sterling, and this may be defined to be 123·274 troy grains of gold of the fineness of $\frac{11}{12}$ths, while, as already stated, the principal unit of value in France is the franc, containing $77\frac{1}{6}$ troy grains of silver, of a fineness of $\frac{9}{10}$ths; the silver coins of England represent sums somewhat greater than their intrinsic value, being coined only for circulation within the country as convenient subdivisions of the pound sterling, and being only a legal tender in payment of a debt to a small amount; and the gold employed in France is subject to a premium or discount, almost invariably a premium, calculated upon its nominal value, and varying from day to day. To determine the par of exchange at any time between England and France, it is then necessary either to know the market-price of silver at that time in England, or the premium on gold in France. Let it be supposed that the price of fine silver in London is 5*s*. 5*d*. per ounce troy, and consequently, that of an ounce troy, or 480 grains of silver, equivalent in fineness to the franc, 4*s*. $10\frac{1}{2}$*d*., then the intrinsic value of the franc in English money will be 9·4*d*.; and the par of exchange for the pound sterling, 25·53 francs. Or, if it be assumed that the premium on gold in Paris be 15 per mille ($1\frac{1}{2}$ per cent.), the ten-franc gold piece would be worth 10·15 francs; and, as its weight is 49·783 troy grains, and its fineness $\frac{9}{10}$ths, 44·805 grains of pure gold would be worth 10·15 francs; but 123·274 grains of gold of $\frac{11}{12}$ths fine, or 113·001 grains of pure gold being equivalent to the pound sterling, $\frac{113\cdot001}{44\cdot805} \times 10\cdot15 = 25\cdot59$ francs would be the par of exchange for the pound sterling.

If, when the circumstances are in accordance with the above assumptions, the rate of exchange at short, or for bills payable at sight, were 25·75, the exchange would be said to be favourable to England and unfavourable to France.

134. The foregoing calculation is founded on the assumption made in Article 130, that a given quantity of gold or silver has the same intrinsic value in all places and in all forms; but as the transmission of bullion from one country to another is regulated by the same principles as the export and import of other commodities, it is evident that a difference in the value of bullion in two countries may exist equal to the expense of its transmission, and that the difference in value of bullion in two countries, like the rate of exchange between those countries, as stated at the end of Article 128, is limited by the cost of transmission. The regulations, also, under which the mints of different countries execute the coinage, establish generally a certain difference between the value of coin and the value of bullion.

135. Under the laws by which the currency of this country is regulated the price of gold may be considered invariable. Since the Bank of England has been by law obliged to purchase all bullion tendered to them at 3*l*. 17*s*. 9*d*. per oz., of $\frac{11}{12}$ths fineness, gold bullion has

been almost invariably quoted at that price in this country, and English gold coin is always worth 3*l.* 17*s.* 10½*d.* per oz., that value being assumed as the basis of our currency. In France a kilogramme = 15,432 English grains, of gold of $\frac{9}{10}$ths fineness, is coined into 3100 francs, of which six are retained to cover the expense of fabrication, and the price of gold bullion of the fineness of $\frac{9}{10}$ths may therefore be considered, when there is no premium, 3094 francs per kilogramme, or 3437·8 francs per kilogramme of fine gold, or the price of a kilogramme of gold bullion of English standard or $\frac{11}{12}$ths fineness, 3151·3 francs; or of one ounce, of which the value in England would be 3*l.* 17*s.* 9*d.*, 98·019 francs: the par of exchange between the two countries deduced from the relative prices for gold bullion would therefore be 25·21 francs to the pound sterling; and if at that time the rate of exchange was 25·24, gold must be dearer in London than at Paris, for ·03 in 25·21 francs, or ·12 per cent. more gold would be given for a bill on London than that bill would exchange for in London, or the same proportion less would be given in London for a bill on Paris than it would afterwards command in that city.

If, as is generally the case, gold in Paris be at a premium, say of 12 per mille, the par of exchange calculated by the price of gold bullion would be 25·51; and if at that time the rate of exchange were 25·61, gold would be about $\frac{4}{10}$ per cent. dearer in London than in Paris.

In a similar manner, if the par of exchange between Hamburgh * and London be 13 marks banco, 11¼ schillings, but if the rate of exchange at Hamburgh on London at short be 13·12½, then it follows that gold is dearer in London than in Hamburgh. To find the per centage we have only to make the following proportion:—

Mks. Sch. Mks. Sch.
As 13 11¼ : 13 12½ :: 100 : 100·51;

hence gold is ·51 per cent. dearer in London than in Hamburgh.

136. In foreign exchange, one place always gives another a fixed sum of money for a variable price, which fluctuates according to the balance of debt; the former is called the *certain* price, and the latter the *uncertain* price. Thus London is said to give to Paris the certain for the uncertain; that is, the pound sterling for a variable number of francs; and to give to Spain the uncertain for the certain; that is, a variable number of pence sterling for the dollar of exchange.

As examples in direct exchange are only applications of the Rule of Three, it will be unnecessary to occupy more space with it, and we shall now advert shortly to the subject of indirect exchanges.

ARBITRATION OF EXCHANGE.

137. Indirect exchanges are those produced through the medium of some other country or countries, and the proportional or mean rate deduced is termed the arbitrated rate of exchange, and the object of

* In France, accounts are kept in francs and centimes, and 100 centimes make 1 franc. In Hamburgh, there are two kinds of money—banco and currency. Accounts are kept in banco, which is a nominal valuation of the Cologne mark. A mark banco is worth about 17½*d.* sterling, and 16 schillings make 1 mark. In Frankfort, accounts are kept in florins and kreutzers, and 60 kreutzers make 1 florin. A rix dollar is equal to 90 kreutzers.

the arbitration of exchanges is to ascertain whether, in remitting or drawing bills, it will be most advantageous to do so *directly* or *indirectly*. *Simple* arbitration comprehends the exchanges of *three* places only, and *compound* arbitration of *more* than three places. A single example will be sufficient to explain the method of calculation in all cases.

Suppose a merchant of London has to receive 4500 marks banco at Hamburgh, whether will it be more advantageous for him to draw on Hamburgh directly at 13 marks 8 schillings per pound sterling, or to direct his agent to remit the sum to Frankfort, at 148 rix dollars per 300 marks banco, with directions to invest the value in a bill on London, which can be effected at 10 florins per pound sterling, allowing 1 per cent. for the charge of commission?

By the direct mode of exchange,

Mks. B.	Sh.		Mks. B.		£.		£.	*s.*	*d.*
13	8	:	4500	::	1	:	333	6	8

Now let x represent the value by the indirect course of exchange, then we have the following equalities:—

$$\begin{aligned} \pounds\, x &= 4500 \text{ marks banco.} \\ 300 \text{ marks banco} &= 148 \text{ rix dollars.} \\ 1 \text{ rix dollar} &= 1\tfrac{1}{2} \text{ florins.} \\ 10 \text{ florins} &= \pounds 1. \\ \pounds 100 &= \pounds 99 \text{ on account of commission.} \end{aligned}$$

Hence the continued product of the numbers in the first column must be equal to the continued product in the second, that is

$$x \times 300 \times 10 \times 100 = 4500 \times 148 \times 1\tfrac{1}{2} \times 99;$$

$$\therefore x = \frac{4500 \times 148 \times 3 \times 99}{300 \times 10 \times 100 \times 2} = \frac{15 \times 37 \times 3 \times 99}{10 \times 25 \times 2}$$

$$= \frac{3 \times 37 \times 3 \times 99}{10 \times 5 \times 2} = \pounds\,\frac{32967}{100} = \pounds 329.\ 13s.\ 4\tfrac{1}{4}d.$$

The direct is preferable to the indirect exchange, the difference being £3. 13*s.* 3¾*d.* in favour of the direct exchange.

STANDARDS OF ENGLISH WEIGHTS AND MEASURES.

I. *Measure of Time.*

138. For a standard measure of time we must look to the daily and yearly revolutions of the earth, which, by the immutable laws of Nature, has performed its revolutions round the sun with unerring regularity for ages past, and will continue to do so for ages to come, unless some great and unknown change should occur in the solar system of which it is a part. The sun, which regulates the operations of man, and determines the periods of labour and rest, is pointed out by nature to fix the standard of time; but as the length of a *solar day* is not always the same, it is an unfit measure of time, and therefore the solar day is superseded by a duration called a *mean solar day*, that is, the mean interval of time which elapses between two passages of the sun across the meridian of any place. This interval of time is supposed to be divided into 24 equal portions, each of which is called an *hour*. Astronomers have determined that a *solar year*, or the time of one tropical revolution of the earth

round the sun, consists of 365·24224 mean solar days. This is very nearly 365¼ days, and a year is made to consist of 365 days, while to every fourth year a whole day is added, and such year is termed a *leap year*. The average year 365¼ days is too long by the difference between ·25 and ·24224 or ·00776 of a day; consequently if 1 day be divided by ·00776, the quotient 128·87 is the number of years in which the error will amount to 1 day, so that the error is 3 days in 3 times 128·87 years, or nearly 3 days 2¼ hours in 400 years. This error is corrected by allowing only one out of four of the years which terminate the centuries to be leap years. Every year whose date is divisible by 4 is a leap year, except that date be exactly a *number of centuries*, in which case that number must be divisible by 4, in order that it may be a leap year: thus 1600, 2000, 2400, etc., are leap years, whilst 1900, 2100, 2200, 2300, etc., are not accounted leap years, for 19, 21, 22, 23, etc., the numbers of the centuries, are not divisible by four. Hence in 400 years there are only 97 leap years and 303 common years.

The *day* is then the first measure that is obtained, and it is divided into 24 parts, each of which is called an *hour*; each hour is divided into 60 parts, called *minutes*, and each of these is divided into 60 parts which are termed *seconds*. Since $24 \times 60 \times 60 = 86400$, it is obvious that 1 second (marked thus 1") is the 86400th part of a day.

A civil month consists of 4 weeks or 28 days. The 12 calendar months, and the number of days in each are as follow:—January 31, February 28 in common years, and 29 in leap years, March 31, April 30, May 31, June 30, July 31, August 31, September 30, October 31, November 30, and December 31. The mean solar year consists of 365·24224 days, or 365 days 5 hours 48 minutes and 49·5 seconds.

II. *Standard of Lineal Measure.*

139. It is an object of the first importance in a civilized state of society that standards of weights and measures shall remain without alteration, and that we should be able to replace them in the event of their being destroyed either by accident or design. With a view to this object the English Government has had reference to the length of the seconds' pendulum. It has been seen that the second of time is an unalterable period of duration, and by the application of very ingenious contrivances, and the known laws of mechanics, the actual length of the seconds' pendulum has been ascertained, in the latitude of London, at the level of the sea in a vacuum, and at a certain degree of the thermometer, so that in case our standard of length should be destroyed, it would always be possible, while the laws of Nature remain unchanged, to restore that standard without alteration. The seconds' pendulum is not the unit of our measure of length. By an *Act of Parliament*, passed on the 17th June, 1824, a certain brass rod is declared to be the *standard yard* of England; and that its length, as compared with that of the seconds' pendulum, determined as above, is in the proportion of 36 to 39·1393; it follows, therefore, that although the seconds' pendulum is not the unit of our measure of length, it furnishes the means of restoration if at any time hereafter the standard yard should be lost or destroyed.* The

* The standard yard was destroyed in the conflagration of the Houses of Parliament, in 1834, and has not yet been restored.

measures of length are given in (34), and for particular purposes some other denominations are employed. Thus, for measuring cloth of all kinds, $2\frac{1}{4}$ inches = 1 nail, 4 nails = 1 quarter, 4 quarters = 1 yard, and 5 quarters = 1 ell. For measuring the height of horses, 4 inches = 1 hand. For measuring depths, 6 feet = 1 fathom. The chain for measuring land is 22 yards, or 66 feet, or 792 inches in length, and consists of 100 links, each of which is consequently 7·92 inches. Mechanics usually divide the inch into halves, quarters, and eighths. A degree of the equator is 69·156 miles, and a degree of the meridian is 69·044 miles.

Since a square inch is a square whose side is 1 inch, and a cubic inch is a cube whose side is 1 inch, it necessarily follows that a lineal inch is the foundation of the unit of square and cubic measure. In the superficial measurement of stone, brick or slate-work, 36 square yards are termed a *rood*, and 100 square feet of flooring a *square*.

III. *Standard of Weight.*

140. By the Act of Parliament already referred to, and which came into operation on the 1st of January, 1826, "a cubic inch of distilled water, weighed in air by brass weights, at the temperature of 62 degrees of Fahrenheit's thermometer, the barometer being at 30 inches, is equal to 252·458 grains." Of the grains thus determined 5760 are a *troy* pound, and 7000 are a pound *avoirdupois*. These pounds are divided, as in the tables of troy and avoirdupois weight, in (34). In *troy* weight, and that usually termed *apothecaries'* weight, the grain, ounce, and pound are the same. The former, or troy weight, is used for the precious metals and for jewels, as also in trying the strength of spirituous liquors, etc.; and the latter is employed in medical prescriptions. The *carat*, used for weighing diamonds, is $3\frac{1}{6}$ grains. The *assay weights*, which are only used to show the fineness of gold, are 4 grains = 1 carat, and 24 carats = 1 pound. *Avoirdupois* weight is the general weight of commerce, and by it all articles are bought and sold, except precious metals and precious stones. The avoirdupois pound is larger than the troy pound, the former being to the latter as 7000 grains to 5760 grains, or as 175 : 144. The troy ounce is to the avoirdupois ounce as 480 grains to 437·5 grains, or as 192 : 175.

IV. *Standard of Capacity.*

141. The standard unit of the measure of capacity is the *imperial gallon*, containing 10 pounds avoirdupois of distilled water at the temperature of 62 degrees Fahrenheit, the barometer 30 inches, and is equal to 277·274 cubic inches. The *imperial bushel*, consisting of 8 gallons, will consequently contain 2218·192 cubic inches, or 80 pounds avoirdupois of distilled water.

V. *Standard of Value.*

142. The usual coins of England are in copper, silver, and gold, of the value stated in the Table of Money, Art. 34. Of these, gold is the only legal tender above 40 shillings. A farthing is the coin of least value, and it is usual to denote farthings as fractions of a penny. From the Latin words *libra*, a pound; *solidus*, a shilling; and *denarius*, a penny; £. *s*. *d*. are made to represent pounds, shillings, and pence re-

spectively; hence one farthing = $\frac{1}{4}d.$, two farthings = $\frac{1}{2}d.$, and three-farthings = $\frac{3}{4}d.$ The English pound is generally called a *pound sterling*, to distinguish it from a pound weight, as well as from stock or foreign coins.

The *standard* gold coin is made of a metal composed of 22 parts of *pure* or *fine gold*, and 2 parts of copper. The *standard* silver coin is made of a metal composed of 37 parts of *pure silver* and 3 parts of copper. These compositions are better fitted for the purposes of a circulating medium than either pure gold or silver, which are too soft and flexible. Of these standard metals, a pound troy of gold is coined into £46. 14*s*. 6*d*., and a pound troy of silver into 66 shillings; hence the mint price of standard gold, or gold of a fineness $\frac{11}{12}$, is £3. 17*s*. $10\frac{1}{2}d.$ per ounce; and of our silver coinage of a fineness $\frac{37}{40}$, or nearly $\frac{12}{13}$, is 5*s*. 6*d*. per ounce.

The weight of a sovereign, or pound sterling, is consequently 5 dwts. 3 $\frac{171}{623}$ grains, and the weight of a shilling is 3 dwts. 15 $\frac{3}{11}$ grains. An avoirdupois pound of copper is coined into 24 pence, each of which weighs $10\frac{2}{3}$ drams avoirdupois, or $291\frac{2}{3}$ grains troy. The weight of the silver florin is double that of the shilling, or 7 dwts. $6\frac{6}{11}$ grains.

COMPARISON OF ENGLISH AND FOREIGN MEASURES, COINS, ETC.

143. It is explained (139) that the standard of English lineal measure is a brass rod preserved in the Exchequer of a certain length called a *yard*, and that in order to the recovery of this measure or standard if it should be accidentally destroyed, a comparison has been made between it and the length of the seconds' pendulum vibrating under certain conditions. The standard lineal measure of France is in like manner a certain fixed length called a *mètre*. This length has been obtained as follows:—About $9\frac{1}{2}$ degrees of the meridian of Paris were carefully measured by some of the most distinguished mathematicians and astronomers of France and Spain, in terms of a certain *assumed mètre*, and by means of the length of the whole of this measured terrestrial arc, and the lengths of separate parts of the same, the figure of the earth, and the entire measure of the quadrant from the Equator to the Pole, was computed in terms of the assumed mètre, and then the ten-millionth part of this arc was adopted as the *standard mètre* of France. Several comparisons have been made by the most carefully-conducted experiments to ascertain the ratio of the length of this mètre to that of the English standard yard, and the result considered the most accurate, makes the metre = 39·370079 English inches. This ratio being thus established, we easily arrive at the standards of lineal measure of the several European nations, which are generally founded on either the English or French system.

It is shown also (139) that the unit of lineal measure becomes necessarily the foundation of the unit of square and cubic measure, as also of the measures of solidity and capacity. And by availing ourselves of the unalterable condition of water at a fixed temperature and barometrical pressure, it furnishes also the means of determining a fixed unit of weight, or rather for fixing a means of comparison of different units of weight. Thus a certain brass weight in the Exchequer is determined to be the English troy pound, and the 5760th part of this is one

grain; and by the means of careful experiments, it has been ascertained that a cubic inch of water under the conditions above referred to weighs 252·458 grains. These grains thus become a sort of universal term of comparison of the weights of different nations. Some discrepancy is found to exist in the results as obtained by different philosophers, but they are very inconsiderable; and we believe the numbers in the following tables are those which are generally considered as the most correct.

144. The metrical system of weights, measures, and coins, adopted in France, both as regards the multiples and submultiples of the unit, proceed according to the decimal scale, the units of the different measures being denominated as follows:—

1. The unit of length is the *mètre.*
2. The unit of surface is the *are*, which is a square whose side is ten mètres.
3. The unit of volume is the *stère*, which is a cube whose side is a mètre.
4. The unit of capacity is the *litre*, which is a cube whose side is a tenth part of a mètre, and
5. The unit of weight is the *gramme*, which is the weight in vacuo of a cubic *centimètre* (the hundredth part of a mètre) of water at the temperature of 4 degrees of the centigrade thermometer (39·2° Fahrenheit), being the temperature of water when its density is a maximum.

The Latin derivatives *déci* to denote the tenth part, *centi*, the hundredth, and *milli*, the thousandth part, being prefixed to any of the preceding units, serve to denominate its decimal submultiples; whilst the Greek prefixes, *deca* to denote ten times, *hecto* a hundred, *kilo* a thousand, and *myria* ten thousand times, will express the decimal multiples. Thus a *décimètre* signifies the tenth of a mètre, and a *decamètre* is 10 mètres. Taking the length of the mètre to be 39·370079 inches, the following tables of comparison of French and English measures will be easily understood.

Comparison of English and French Measures.

Measures of Length.

		Inches.
Millimètre	=	·03937
Centimètre	=	·39370
Décimètre	=	3·93701
Mètre	=	39·37008
Décamètre	=	393·70079
Hectomètre	=	3937·00790
Kilomètre	=	39370·07900
Myriamètre	=	393700·79000

Measures of Capacity.

		Cubic Inches.
Centilitre	=	·6102
Decilitre	=	6·1023
Litre	=	61·0237
Decalitre	=	610·2379
Hectolitre	=	6102·3791
Kilolitre	=	61023·7917
Myrialitre	=	610237·9179

Measures of Surface.

		Square Yards.
Are	=	119·599
Décare	=	1195·990
Hectare	=	11959·906

Measures of Volume.

		Cubic Feet.
Décistère	=	3·53146
Stère	=	35·31469
Décastère	=	353·14694

Measures of Weight.

		Grains, Troy.
Centigramme	=	·1543
Décigramme	=	1.5432
Gramme	=	15·4327
Décagramme	=	154·3272
Hectogramme	=	1543·2720
Kilogramme	=	15432·7200
Myriagramme	=	154327·2000

Comparison of English and French Coinage.

145. In England the only legal tender above 40 shillings is the pound sterling or sovereign, a gold coin of the standard fineness of $\frac{11}{12}$, viz., eleven parts of fine gold with one of alloy. In France either gold or silver may be legally tendered to any amount. The coins of France are of $\frac{9}{10}$ fineness, viz., nine parts of fine metal with one of alloy, and the gold coins are considered to have a variable value with respect to the silver coins; the 20 and 40 franc gold coin being commonly at a premium in respect of 20 or 40 silver francs. The silver franc is the unit of French money, which is subdivided and multiplied, like their weights and measures, according to the decimal scale. The subdivisions are stated in centimes or hundredths of a franc.

The weight of a silver franc is the 200th part of a kilogramme, or of 15432·72 grains = 77·1636 grains, of which 69·447 grains are fine silver and 7·716 grains alloy. A franc in gold is the 3100th part of a kilogramme; its weight in standard gold is therefore 4·9783 grains, of which 4·48047 are fine gold and ·49783 alloy.

Now (135) it is shown that the English pound sterling or sovereign contains 113 grains of fine gold; therefore, estimating the value of the two coins by their respective weights of pure gold, the English sovereign $= 113 \div 4{\cdot}4804 = 25{\cdot}22$ gold francs, whatever may be the price of gold, the exchange being made in the same place, and at the same time; but other conditions must be considered under the general question of exchange between the two countries, as is shown in (135). From the above are obtained the following comparative intrinsic values of the French and English coinage, viz. :—

1 franc (*silver*)	=	77·1636	grains of silver	$\frac{9}{10}$	fine.
10 franc (*gold*)	=	49·783	ditto	gold $\frac{9}{10}$	ditto.
20 franc ditto	=	99·566	ditto	ditto.	
40 franc ditto	=	199·132	ditto	ditto.	
1 franc	=	9·516	English pence, and		
25·22 francs in gold	=	1 sovereign.			

COMPARATIVE TABLES of ENGLISH, FRENCH, and GERMAN MEASURES and WEIGHTS.

I. MEASURES OF LENGTH.				II. MEASURES OF SURFACE.				III. MEASURES OF VOLUME.			
English Foot.	French Mètre.	Paris Old Foot.	Prussian Foot.	English Sq. Foot.	French Sq. Mètre.	Paris Sq. Foot.	Prussian Sq. Foot.	English Cub. Foot.	French Cub. Mètre.	Paris Cub. Foot.	Prussian Cub. Foot.
1	·304794	·938293	·971136	1	·092899	·880393	·943105	1	·028315	·826067	·915884
3·280899	1	3·078444	3·186199	10·76430	1	9·476817	10·15187	35·31658	1	29·17385	32·34587
1·065765	·324839	1	1·035003	1·135856	·105521	1	1·071232	1·210556	0·34277	1	1·108728
1·029722	·313853	·966181	1	1·060327	·098504	·933505	1	1·091842	0·30916	·901935	1

120 Prussian quarts are equal to 30·23 English gallons, nearly; a Prussian scheffel = 1·512 English bushels, nearly; a Prussian lb. = 1·0311 lb. avoirdupois; a Prussian centner or quintal = 110 Prussian lbs.; and a French quintal metric = 100 kilogrammes.

IV. MEASURES OF CAPACITY.						V. WEIGHTS.					
English Gallon.	French Litre.	Prussian Quart.	English Bushel.	French Hectolitre.	Prussian Scheffel.	English lbs. Avoird.	French Kilogramme.	Prussian Pound.	English Cwt.	French Quintal.	Prussian Centner.
1	4·543458	3·967977	1	·363477	·661330	1	·453597	·969824	1	·508029	·987458
·220097	1	·873338	2·751208	1	1·819455	2·204597	1	2·138072	1·968390	1	1·943702
·252018	1·145031	1	1·512105	·549615	1	1·031114	·467711	1	1·012702	·514482	1

ALGEBRA.

DEFINITIONS AND ELEMENTARY PRINCIPLES.

ART. 1. ALGEBRA is the science of computation by general symbols, or it is a general method of reasoning on quantity by means of symbolical characters.

In algebra, quantities are represented by the symbols a, b, c, x, y, z, etc., and by employing this concise notation, the steps and results of algebraic investigations can be expressed not only in simple and intelligible forms, but free from that prolixity of which common language could never divest them. These very convenient symbols are made the arbitrary representatives of the quantities under consideration, and by the fundamental principles and operations of algebra, which are the same as those of common arithmetic, results are deduced exhibiting in what manner the various quantities are combined, and showing their relations to each other, even before particular values have been assigned to these quantities. In arithmetic each question requires a separate investigation, but in algebra all questions of the same class are considered together, and included in the same investigation, and the result of the reasoning with general symbols is expressed in a general form that applies with equal facility to every question of the same kind.

Thus, suppose that a represents the number of days in which a person, A, can perform a certain piece of work, and that b denotes the number of days in which another person, B, can perform the same, or an equal piece of work; then, by the principles of algebra it can easily be shown that the time in which A and B, working in conjunction, can perform the specified piece of work, is expressed in symbols by $\frac{a \times b}{a + b}$ Now, to apply this to an example or two, let us suppose that A can do a piece of work in 10 days, and that B can do it in 15 days, then a will denote 10 and b will denote 15; hence

$\frac{a \times b}{a + b} = \frac{10 \times 15}{10 + 15} = \frac{150}{25} = 6$ days, the time in which A and B jointly can perform the piece of work.

Again, suppose A takes 12 days to perform a piece of work, and B takes 20 days, in what time will A and B jointly perform the piece of work?

In the general expression $\frac{a \times b}{a + b}$, we must now write 12 for a and 20 for b, then we get

$\frac{a \times b}{a + b} = \frac{12 \times 20}{12 + 20} = \frac{240}{32} = \frac{15}{2} = 7\frac{1}{2}$ days, which is the time that A and B jointly would take to perform the work.

Thus, by the use of general symbols, results may be deduced which are true for all numbers, and formulas which apply to all questions of the same character. Other advantages will present themselves as we proceed in the development of the principles of this useful science.

2. Quantities or magnitudes of all kinds are, as we have seen, represented by the letters of the alphabet; so also the operations to be performed with these symbols or letters are indicated by certain other signs, instead of being expressed in words at length.

3. Quantities are divided into classes distinguished by the terms *known* or *given*, and *unknown* or *required*. Known or given quantities are usually represented by the leading letters of the alphabet, as a, b, c, etc., and unknown or required quantities by the final letters x, y, z, etc. Also if a represent 8, then *twice* a, or $2\,a$, would represent 16; or if b represent 5, then *thrice* b, or $3\,b$, would represent 15.

The signs which are employed to indicate the operations to be performed with the symbols of quantity are as follow:—

4. The sign $+$ (*plus*) denotes that the quantity which it precedes is *added;* thus $a + b$ signifies that b is added to a, and it is read *a plus b.*

5. The sign $-$ (*minus*) denotes that the quantity which it precedes is *subtracted;* thus $a - b$ signifies that b is subtracted from a, and read *a minus b.*

6. The sign $\pm$ (*plus* or *minus*) comprehends both the former signs, and denotes that the quantity which it precedes is added when $+$ is used, and subtracted when $-$ is employed.

7. All quantities preceded by $+$ are called *positive* quantities, and those which are preceded by $-$ are called *negative* quantities; and if neither $+$ nor $-$ precedes a quantity, $+$ is understood, and the quantity is a positive one: thus c means $+ c$.

8. The sign $=$ (*equal*) denotes that the quantities between which it is placed are equal to one another: thus $3 + 2 = 2 + 3$ signifies that 2 added to 3 is equal to 3 added to 2; and $a = b + c$, means that a is equal to b plus c, or that a is equal to the sum of b and c.

9. The sign $\times$ (*into*) denotes that the quantities between which it is placed are to be multiplied together. A period, or full point, is often used instead of $\times$, and it is usual in algebra to write the quantities to be multiplied together in succession without the intervention of any sign between them. Thus $a \times b \times c$, $a \, . \, b \, . \, c$, and $a\,b\,c$, all signify that a is multiplied by b, and the product is multiplied by c, or they all denote the continued product of a, b, c.

10. The sign $\div$ (*by*) denotes that the quantity which precedes it is to be divided by that which follows it; but the same operation is usually expressed by placing the dividend over the divisor, with a line between them, in the form of a fraction. Thus $a \div b$ and $\frac{a}{b}$ signify the same operation, *a divided by b.*

11. The sign $<$ (*inequality*) denotes that one of the quantities between which it is placed is greater than the other, the opening of the sign being turned towards the greater quantity. Thus $a > b$ signifies that the quantity denoted by a is *greater* than that represented by b, and $c < d$ signifies that c is *less* than d.

12. The *coefficient* of a quantity is the number, whether positive or

negative, which is prefixed to it, and it expresses the number of times the quantity is taken. Thus in $5a$, 5 is the coefficient of a, and it signifies that the quantity a is taken 5 times.

13. If no numerical coefficient is prefixed, the coefficient is understood to be 1, because a means *once* a, and x means *once* x.

14. A *power* of a quantity is the product arising from the multiplication of that quantity by itself any number of times, and it is usually expressed by writing the quantity with a small figure above it to the right, denoting the number of equal factors to be multiplied together. Thus a^6 means $a \times a \times a \times a \times a \times a$, or $a\,a\,a\,a\,a\,a$, and also $5a^2b^3c^4x$ stands for $5aa\;bbb\;cccc\;x$.

15. A *root* of a quantity is that quantity which, if multiplied by itself a certain number of times, produces the proposed quantity. Thus a is the *second* or *square* root of a^2, for $a \times a = a^2$; $2x$ is the *third* or *cube* root of $8x^3$, for $2x \times 2x \times 2x = 8x^3$, and y is the n^{th} root of y^n for $y \times y \times y \times \ldots$ to n factors $= y^n$.

16. The *exponent* or *index* of a quantity is the small figure employed to denote the power to which it is to be raised.

Thus a (or a^1) is the *first* power of a,
a^2 is the *second* power or *square* of a,
a^3 is the *third* power or *cube* of a, and so on.

The small figures, ^{2}, ^{3}, etc., are the indices of the second, third, etc., powers of a respectively.

17. The symbol $\sqrt{\ }$ is employed to denote a root, and a small figure placed over the sign shows the root to be extracted, except in the case of the square root, where the figure 2 is understood. Thus $\sqrt{a}$ signifies the square root of a, $\sqrt[3]{a}$ the cube root of a, $\sqrt[n]{a}$ the n^{th} root of a, and so on. A different notation for expressing roots of quantities will be explained in a subsequent article.

18. The terms of an expression are those parts of it which are connected by the sign $+$ or $-$, and the expression itself is either simple or compound.

19. A *simple* quantity consists of a single term, as $3ax$.

20. A *compound* quantity consists of two or more terms, such as $a - 2b + 5y$.

21. Brackets ( ), { }, [ ]. are employed when a compound quantity is to be the subject of any operation, as $(a+b-c) \times 5$, or $5(a+b-c)$; $\sqrt{(a+b-c)}$, and $(a+b-c)^3$, which respectively denote that the quantity $(a+b-c)$ is to be multiplied by 5, that its square root is to be taken, that it is to be cubed. Sometimes a line over the quantity is used instead of a bracket; thus the last two expressions might be written $\sqrt{a+b-c}$, and $\overline{a+b-c}^{\,3}$; but the latter of these is very inelegant, and ought not to be employed.

22. Algebraic quantities are called *like* or *unlike*, according as they contain the *same* or *different* letters, or as they contain the *same* or *different* combinations of letters. Thus $2a$ and $5a$; $2x^2y$ and $3x^2y$ are *like* quantities; and $2a$, and $3b$; $3x^2y$ and $5xy^2$ are *unlike* quantities.

23. A *monomial* quantity consists of one term only, as am, cxy, etc.

24. A *binomial* quantity consists of two terms, as $a + b$, $3x - 5y$, etc.

25. A *trinomial* involves three terms, as $a + b - 5x$.

26. A *multinomial* or *polynomial* quantity is composed of several terms, as $a + 2b - 3c + 4d$.

27. The *reciprocal* of a quantity is unity divided by that quantity. Thus the reciprocal of a is $\frac{1}{a}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

28. The signs $: \, :: \, :$ (*proportion*) denote that the four quantities between which they are placed are proportional. Thus $a : b :: c : d$ signifies that a has the same relation or ratio to b that c has to d.

29. The sign $\propto$ (*varies as*) signifies that the quantity which precedes it varies as that which follows it; thus $A \propto B$ signifies that A varies as B.

30. The abbreviation $\therefore$ for *therefore* or *consequently* is now much used instead of either of those words.

31. We may here add a few examples of the substitution of numbers for letters, for the purpose of familiarizing the student with the symbols. The very little power which even many advanced students possess of reducing algebraical into arithmetical results, operates very much to their disadvantage, and seriously checks their progress in the higher branches of analysis, inasmuch as the numerical result alone is the object of almost every practical inquiry.

Numerical Valuation of Algebraic Expressions.

If $a = 6$, $b = 5$, $c = 4$, $d = 1$ and $e = 0$, find the numerical values of the following expressions:—

1. $a + 2b + 3c + 4d + 5e = 6 + 10 + 12 + 4 = 32.$

2. $a^2(a+b) - 2abc = 36 \times (6+5) - 2 \times 6 \times 5 \times 4 = 396 - 240 = 156.$

3. $\frac{2a}{c} + \frac{2b}{d} + \frac{5c}{4d} - \frac{6e}{13a} = \frac{12}{4} + \frac{10}{1} + \frac{20}{4} = 3 + 10 + 5 = 18.$

4. $4abc^2 - 3a^2bd + \frac{3ab^2c}{a+b-2c} = 1920 - 540 + \frac{1800}{3} = 1980.$

5. $\sqrt{\frac{3}{2}a} + \sqrt{5b} - \sqrt{c} + 4\sqrt{d} = 3 + 5 - 2 + 4 = 10.$

6. Suppose that, at the commencement of an investigation, the numbers 5, 3, 8 and 10 had been denoted by a, b, c, d respectively, and that the result of the investigation gave $x = \frac{2}{3}ab + \frac{3}{4}bc + \frac{4}{5}cd$, what would be the numerical value of x?

Here
$$\begin{aligned} x &= \frac{2}{3} \times 5 \times 3 + \frac{3}{4} \times 3 \times 8 + \frac{4}{5} \times 8 \times 10 \\ &= 2 \times 5 + 3 \times 3 \times 2 + 4 \times 8 \times 2 \\ &= 10 + 18 + 64 = 92. \end{aligned}$$

For practice, the student may compute the numerical values of the following expressions, taking $a = 5$, $b = 2$, $c = 4$, $d = 3$, and $n = 4$.

1. $x = \frac{a+b+c}{2} + \frac{a+b-c}{2} + \frac{a-b+c}{2} + \frac{b+c-a}{2}.$

Ans. $x = 11.$

2. $x = \frac{a^2-b^2}{c-d} + \frac{4ab-c^2}{2d} - \frac{a^2+b^2+c^2-d^2}{ab+cd}.$ *Ans.* $x = 23\frac{4}{11}.$

3. $x = a + b - (c-d) + \sqrt{(a^2 - 2bc)}$. *Ans.* $x = 9$.

4. $x = \frac{6}{a} - \frac{3}{b} + \frac{10}{c-d} - \frac{14}{c+d}$. *Ans.* $x = 7\frac{7}{18}$.

5. $x = \sqrt{(a^2 + b^2 - c^2 - d^2)} - \sqrt[3]{(8bc)} + b^a$. *Ans.* $x = 14$.

6. $y = \sqrt{(a^2+b^2)} - \sqrt{(a^2-b^2)} + \sqrt[3]{(a^3-b^3)}$. *Ans.* $y = 5{\cdot}693562$.

7. $y = \left\{\frac{a^2 b}{c} \times d\right\} \div \left\{\frac{ab^2}{c} + d\right\}$. *Ans.* $y = 4{\cdot}6875$.

8. $y = \frac{a^2 b^2 c^2 d^2 + 2abcd + 1}{abcd + 1}$. *Ans.* $y = 121$.

9. $y = \frac{a^2+b^2-d^2}{a+b+d} + \frac{abcd}{2b+c} - \frac{4a^2 - 10bc + 2}{2c+d}$. *Ans.* $y = 15$.

10. $y = \sqrt{\left\{\frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{b+c-a}{2}\right\}}$.
Ans. $y = 3{\cdot}799671$.

11. There is a certain expression consisting of four terms connected by the sign $+$. The first term is the square of a, divided by b; the second is the square of a by b; the third is the square root of the excess of the square of a above the square of b; and the fourth is the reciprocal of the sum of the squares of a, b, c. Write down the expression.

12. An expression consists of three positive terms; the *first* is the square of x, the *second* is x with a coefficient equal to the sum of a and b; and the *third* is the product of a and b. What is the expression?

ADDITION.

32. Addition is the method of connecting quantities together by means of the signs prefixed to them, and *incorporating* such as are like into one sum. Unlike quantities cannot be incorporated, and their sum can only be expressed by writing them in succession, prefixing to each its proper sign.

Thus the sum of $3a$ and $4a$ is $7a$; for the sum of 3 times any quantity a and 4 times the same will be 7 times that quantity, in the same manner as the sum of 3 pounds and 4 pounds is 7 pounds.

Again, the sum of $3a - 2b$ and $7a - 3b$ is $10a - 5b$; for the sum of $3a$ and $7a$ is $10a$; but the first quantity $3a - 2b$ is less than $3a$ by $2b$, and the second quantity $7a - 3b$ is less than $7a$ by $3b$; therefore the sum of $3a$ and $7a$, viz., $10a$, must be diminished by the sum of $2b$ and $3b$, or $5b$, to obtain the correct sum of the two proposed quantities.

33. *When the like quantities have like signs*, their sum is obtained by adding the coefficients together, prefixing the common sign to that sum, and annexing the common quantity.

It is usual to arrange the quantities which are like in the same vertical column, which facilitates the process of summation.

EXAMPLES.

(1)

$$\begin{array}{r}
3a - 3bx + bxy + 3z \\
9a - 5bx + 2bxy + 2z \\
5a - 4bx + 5bxy + 4z \\
12a - 2bx + bxy + z \\
a - 7bx + 3bxy + 5z \\
2a - bx + 6bxy + 6z \\
\hline
32a - 22bx + 18bxy + 21z \\
\hline
\end{array}$$

(2)

$$\begin{array}{r}
3x^2 + 5xy + 2ax - 4y \\
x^2 + xy + ax - y \\
2x^2 + 4xy + 5ax - 3y \\
6x^2 + 2xy + 3ax - 5y \\
4x^2 + 3xy + ax - 2y \\
x^2 + xy + 2ax - 3y \\
\hline
17x^2 + 16xy + 14ax - 18y \\
\hline
\end{array}$$

(3)

$$\begin{array}{r}
4ab - 6\sqrt{x} + 3x^2\sqrt{y} - 4c \\
3ab - 5\sqrt{x} + x^2\sqrt{y} - 5c \\
ab - 2\sqrt{x} + 2x^2\sqrt{y} - c \\
2ab - \sqrt{x} + x^2\sqrt{y} - 7c \\
ab - 15\sqrt{x} + x^2\sqrt{y} - c \\
\hline
\\
\hline
\end{array}$$

(4)

$$\begin{array}{r}
2x^2y - 3xy^2 + 2(a+b) \\
3x^2y - xy^2 + (a+b) \\
x^2y - 2xy^2 + 3(a+b) \\
8x^2y - 5xy^2 + 4(a+b) \\
7x^2y - 9xy^2 + (a+b) \\
\hline
\\
\hline
\end{array}$$

34. *When the like quantities have different signs*, add all the positive coefficients together, and then all the negative coefficients: subtract the less sum from the greater, and prefix the sign of the greater, and annex the common letter or letters.

(5)

$$\begin{array}{r}
4a + 3ax^2 + 8n^3 + 3y \\
-5a + 4ax^2 - 5n^3 + 4y \\
6a - 8ax^2 - 16n^3 + 5y \\
a - 6ax^2 + 3n^3 - 7y \\
-3a + 5ax^2 + 2n^3 - 2y \\
\hline
3a - 2ax^2 - 8n^3 + 3y \\
\hline
\end{array}$$

(6)

$$\begin{array}{r}
-3a^2 + 3b^2y^3 + 4ab + 4 \\
-5a^2 + 9b^2y^3 - 4ab + 12 \\
-10a^2 - 10b^2y^3 + 7ab - 14 \\
10a^2 - 19b^2y^3 + ab + 3 \\
14a^2 - 2b^2y^3 - 5ab - 10 \\
\hline
6a^2 - 19b^2y^3 + 3ab - 5 \\
\hline
\end{array}$$

In the left column of example (5), the sum of the positive coefficients is 11, and the sum of the negative coefficients is 8; subtracting 8 from 11, the remainder 3 is positive, and hence the sum of the quantities in the first column is $3a$, the sign $+$ being understood.

In the second column the sum of the positive coefficients is 12, and the sum of the negative 14; subtracting the former from the latter gives 2, to which the sign $-$ is to be prefixed, and the sum is $-2ax^2$. This is obvious, since the sums of the positive and negative terms are $12ax^2$ and $-14ax^2$ respectively, and the quantity to be subtracted exceeds the quantity to be added by $2ax^2$, and this must therefore be written, not with the sign of addition, but with the sign of subtraction, prefixed.

(7)

$$\begin{array}{r}
10\sqrt{ax} + 4a\sqrt{x} - 3y \\
-3\sqrt{ax} - 5a\sqrt{x} + y \\
4\sqrt{ax} + 3a\sqrt{x} - 2y \\
-12\sqrt{ax} - 2a\sqrt{x} + 6y \\
\sqrt{ax} - a\sqrt{x} - 10y \\
\hline
\\
\hline
\end{array}$$

(8)

$$\begin{array}{r}
12x^2y - 3xy^2 + 2(a+b)z^2 \\
-11x^2y + 4xy^2 - (a+b)z^2 \\
4x^2y + xy^2 + 4(a+b)z^2 \\
-3x^2y - xy^2 + 2(a+b)z^2 \\
x^2y + xy^2 + (a+b)z^2 \\
\hline
\\
\hline
\end{array}$$

Examples for Practice.

1. Find the sum of $a + b + c$, $a + b - c$, $a - b + c$, and $-a + b + c$. *Ans.* $2a + 2b + 2c$, or $2(a + b + c)$.

2. Find the sum of $2ax + 3by$, $3ax + 2by$, $7ax + by$, and $8ax + 7by$. *Ans.* $20ax + 13by$.

3. Find the sum of $2a^2 - 17ab + 3b^2$, $5a^2 + 12ab - 5b^2$, $12a^2 + 6ab - 9b^2$, $3a^2 + 6ab + 3b^2$. *Ans.* $22a^2 + 7ab - 8b^2$.

4. Find the sum of $x^3 - y^3 + 2xy^2 - 3x^2y$, $2x^3 - 3xy^2 - 5x^2y + 2y^3$, $6x^2y + 6xy^2 - x^3 - y^3$, and $5xy^2 - 2y^3 - 4x^3 + 8x^2y$. *Ans.* $-2x^3 + 6x^2y + 10xy^2 - 2y^3$.

5. Find the sum of $2x + 3y - 4z - 10$, $8y - 4x + 7z + 8$, $11z + 5x - 10y - 2$, and $16 + 10x + 12y + 14z$. *Ans.* $13x + 13y + 28z + 12$.

6. Find the sum of $3x^3 + 2y^3 + z^3 + 8xyz$, $y^3 + 2x^3 - 3z^3 - 4xyz$, $z^3 + 3x^3 - 2y^3 - 2xyz$, and $xyz + x^3 + y^3 + z^3$. *Ans.* $9x^3 + 2y^3 + 3xyz$.

7. Find the sum of $x^4 + 3x^3y + x^2z - 2xv$, $30x^4 - 29x^2z + 18xv - 17x^3y$, $22x^3y - 15x^4 - 32x^2z + 16xv$, and $17x^2z - 12x^4 + 6x^3y - 11xv$. *Ans.* $4x^4 + 14x^3y - 43x^2z + 21xv$.

35. If the coefficients be literal instead of numerical, they may be summed, when they are like, by the preceding methods; and, when they are unlike, their sum can only be expressed by writing them in succession, prefixing to each its proper sign. In this case annex the common quantity to the sum of the literal coefficients inclosed in brackets.

Thus the sum of ax, bx, cx, dx may be written either

$$ax + bx + cx + dx, \text{ or } (a + b + c + d)\,x.$$

8. Add together $ax - by$, $x - y$, and $(a - 1)\,x + (b + 1)\,y$. *Ans.* $2ax$.

9. Add together $(a + 2b)\,x + (4b - 3a)\,y$, $(2a + b)\,x + (2a - b)\,y$, and $4ax + 3by$. *Ans.* $(7a + 3b)\,x + (6b - a)\,y$.

10. Add together $px + qy + rz - c$, $2px - 2qy + 2c$, $3qy - px + 4c$, and $7px - 8qy - rz - 3c$. *Ans.* $9px - 6qy + 2c$.

SUBTRACTION.

36. Subtraction is the method of finding the difference of any two quantities of the same kind. When the quantities are unlike, their difference can only be indicated by writing the quantity to be subtracted after the other, interposing the sign $-$, which indicates subtraction.

Let it be required to subtract $2x + 3y$ from $7x + 6y$. Here, if from $7x + 6y$ we subtract $2x$, there remains $5x + 6y$, and if from this we subtract $3y$, the result is $5x + 3y$, which is the difference of the two proposed quantities.

Again, let it be required to subtract $c - d$ from $a - b$. Here, if from $a - b$, c be subtracted, the remainder will be $a - b - c$; but, by subtracting c from $a - b$, we have subtracted too much, since $c - d$ is a quantity less than c by d. Having therefore subtracted too much from $a - b$ by d, it is evident that the remainder, $a - b - c$, must be too little by d, and must hence be increased by d. The correct remainder is therefore $a - b - c + d$, or $a - c - b + d$, and if we inspect the signs of $c - d$,

$$\begin{array}{r} a - b \\ c - d \\ \hline a - b - c + d \\ \text{or } a - c - b + d \end{array}$$

the quantity to be subtracted, it will be seen that the signs are changed, the one from $+$ to $-$, and the other from $-$ to $+$.

Lastly, let it be required to subtract $12a - 13b$ from $16a - 9b$. By the preceding example the remainder would be $16a - 12a - 9b + 13b$, which, by incorporation, gives $4a + 4b$. This is the same result that would be found by changing $12a$ into $-12a$; $-13b$ into $+13b$, and then adding the columns.

$$\begin{array}{r} 16a - 9b \\ 12a - 13b \\ \hline 4a + 4b, \\ \text{or } 4(a+b). \end{array}$$

Hence *to subtract one quantity from another*, conceive the sign, or signs of the quantity to be subtracted to be changed, and proceed as in addition.

In many cases it will be unnecessary to apply this rule of changing the sign, because the difference of the coefficients may be found at once, as in common arithmetic, and the common quantity annexed will complete the difference. Thus, in the preceding example, it is unnecessary to change the sign of $12a$, because 12 taken from 16 leaves 4, and therefore $4a$ is the difference. So also, if $-7y$ be taken from $-10y$, the remainder is $-3y$.

When the coefficients are literal, the operation of subtraction must be performed upon these coefficients, and to the remainder, enclosed in brackets, if necessary, annex the common quantity.

Examples.

(1)

$$\begin{array}{r} 6a^2 - 8b \\ 3a^2 - 5b \\ \hline 3a^2 - 3b \end{array}$$

(2)

$$\begin{array}{r} 9x^2 - 4xy + 8 \\ 3x^2 + 6xy - 2 \\ \hline 6x^2 - 10xy + 10 \end{array}$$

(3)

$$\begin{array}{l} 8xy - 3 + 6x - y \\ 4xy - 7 - 6x - 4y \\ \hline 4xy + 4 \quad * \quad + 3y \end{array}$$

(4)

$$\begin{array}{r} 5ab - 6 \\ -2ab + 6 \\ \hline \end{array}$$

(5)

$$\begin{array}{r} 4y^2 - 3y + 4 \\ 2y^2 + 2y + 4 \\ \hline \end{array}$$

(6)

$$\begin{array}{r} 7x^2 - \sqrt{xy} + \sqrt{z} \\ 3x^2 - 2\sqrt{xy} + 2\sqrt{z} \\ \hline \end{array}$$

7. From $219a^3 - 117a^2b + 218ab^2 + 145b^3$ subtract $193a^3 + 157ab^2 - 121a^2b + 155b^3$. *Ans.* $26a^3 + 4a^2b + 61ab^2 - 10b^3$.

8. Subtract $2x^3 - 3x^2y + xy^2 + xy^3$ from $5x^3 + x^2y - 6xy^2 + y^3$. *Ans.* $3x^3 + 4x^2y - 7xy^2 + y^3 - xy^3$.

9. Subtract $x^4 - 4x^3y + 6x^2y^2 - 5xy^3 + y^4$ from $3x^4 - x^3y + 7x^2y^2 - 4xy^3 + y^4$. *Ans.* $2x^4 + 3x^3y + x^2y^2 + xy^3$.

10. From $2px^2 - 3qxy + ry^2$ subtract $px^2 - 4qxy + 2ry^2$. *Ans.* $px^2 + qxy - ry^2$.

11. Subtract $a^2 - 3ax + 2x^2 - 16a^2x + 12ax^2 - 12ax^3 - 4x^3 + 2a^2x^2$ from $2a^2 + ax + x^2 - 12a^2x + 20ax^2 - 4x^3 + 6a^2x^2 - 10ax^3$. *Ans.* $a^2 + 4ax - x^2 + 4a^2x + 8ax^2 + 4a^2x^2 + 2ax^3$.

37. To indicate the subtraction of a polynomial quantity, without actually performing the operation, enclose the polynomial in brackets, and prefix the sign $-$. Thus,

$$2x^3 - 3x^2y + 2xy^2 - (x^3 + y^3 - xy^2)$$

signifies that the quantity $x^3 + y^3 - xy^2$ is to be subtracted from the preceding quantity. By the principle of subtraction, we must change

all the signs of $x^3 + y^3 - x y^2$ from $+$ to $-$, and *vice versâ*, and proceed as in addition. Hence the preceding expression becomes

$$2x^3 - 3x^2y + 2xy^2 - x^3 - y^3 + xy^2 = x^3 - 3x^2y + 3xy^2 - y^3.$$

Also, $12a - (a + b - c) = 12a - a - b + c = 11a - b + c$,

and $a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)$

$$= a^2 + 2ab + b^2 - a^2 + 2ab - b^2 = 4ab.$$

On this principle we can make polynomials undergo several transformations, which are useful in various calculations.

Thus, $a^2 - 2ab + b^2 = a^2 - (2ab - b^2) = b^2 - (2ab - a^2)$;

$$a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - (3a^2b - 3ab^2 + b^3)$$
$$= a^3 + 3ab^2 - (3a^2b + b^3)$$
$$= a^3 - b^3 - (3a^2b - 3ab^2).$$

12. Reduce $3x - \{(x - 3a) - (2y - a)\}$ to its simplest form.
Ans. $2x + 2y + 2a$.

13. Reduce $a^2 - (b^2 - c^2) - \{b^2 - (c^2 - a^2)\} + c^2 - (b^2 - a^2)$ to its simplest form. *Ans.* $a^2 - 3b^2 + 3c^2$.

14. To what is $x + y + z - (x - y) - (y - z) - (-y)$ equal?
Ans. $2y + 2z$.

15. From $a(x + y) - bxy + c(x - y)$ subtract $4(x + y) + (a + b)xy - 7(x - y)$.
Ans. $(a - 4)(x + y) - (a + 2b)xy + (c + 7)(x - y)$.

38. We may now make a remark on the occurrence of a negative quantity in a detached form, and unconnected with a positive quantity.

If 8 is to be subtracted from 6, it cannot be performed, in the arithmetical sense, since 8 is greater than 6 by 2: subtracting $6 + 2$ from 6 by the principle of algebraic subtraction, we get -2 for remainder, implying that 2 still remains to be subtracted from some other positive quantity. In this way the student will notice that $-7a$ is *greater* than $-11a$, because if a positive quantity, for example $14a$, be added to each: the sum of $14a$ and $-7a$ is $7a$, but the sum of $14a$ and $-11a$ is $3a$; and the former sum is greater than the latter; therefore $-7a$ is greater than $-11a$. In a similar manner it can be shown that 0 *is greater than any negative quantity.* For if we subtract successively 1, 3, 5, 8, 10 from 5, we get the remainders 4, 2, 0, -3 and -5, each of which, except the first, is evidently less than the one preceding it, so that -3 is *less* than nothing, and *greater* than -5.

MULTIPLICATION.

39. MULTIPLICATION is the method of finding the product of two or more quantities, and the quantities themselves are termed *factors*.

Let it be required to multiply $a - b$ by c. Here the product of a and c is ac; but the product of $a - b$ and c must be less than the product of a and c by the product of b and c, that is, by bc; hence the product of $a - b$ and c is $ac - bc$, as in the margin.

$$\begin{array}{r} a - b \\ c \\ \hline ac - bc \end{array}$$

Again, if it be required to multiply $a - b$ by $c - d$, let $c - d$ be called x; then we have,

$$\begin{array}{r} a - b \\ c - d \\ \hline ac - bc \\ -ad + bd \\ \hline ac - bc - ad + bd. \end{array}$$

$$(a - b)(c - d) = (a - b)x = ax - bx$$
$$= a(c - d) - b(c - d) = ac - ad - (bc - bd)$$
$$= ac - ad - bc + bd.$$

Hence we see that the product of $+a$ and $+c$ is $+ac$; that of a and $-d$ is $-ad$; that of $-b$ and c is $-bc$; and that of $-b$ and $-d$ is $+bd$.

The same may be shown in the following manner. The product of $a-b$ and c is $ac-bc$; but $a-b$ is to be multiplied, not by c, but by $c-d$, a quantity less than c by d; hence the product of $a-b$ and c, that is, $ac-bc$ will be too much by the product of $a-b$ and d, that is, by $ad-bd$. Subtracting this from the former, by changing the signs, we get for the correct product of $a-b$ and $c-d$ the expression $ac-bc-ad+bd$, as before.

Hence, if quantities are multiplied by a positive term, their signs are unchanged in the product, but if multiplied by a negative term, their signs are changed. In other words, like signs produce *plus* in the product, and unlike signs *minus*.

Powers of the same quantity are multiplied together by adding their indices for the index of the power of that quantity in the product.

Thus, $a^2 \times a^5 = a^{2+5} = a^7$; for $a^2 = a \times a$ and $a^5 = a \times a \times a \times a \times a$; hence $a^2 \times a^5 = a \times a \times a \times a \times a \times a \times a = a^7$.

Also, $x^m \times x^n = x^{m+n}$; $x \times x^n = x^{n+1}$; $2 \times 2^{n-1} = 2^n$, and $x^{m-n} \times x^n = x^m$.

Terms which have coefficients are multiplied by prefixing the product of the coefficients to that of the other quantities.

Thus the product of $5x$ and $3y$ is $15xy$.

EXAMPLES.

1.

$$\begin{array}{r} 9ax \\ 4x \\ \hline 36ax^2 \end{array} \qquad \begin{array}{r} -2x^2y \\ 2xy^2 \\ \hline -4x^3y^3 \end{array} \qquad \begin{array}{r} -6x \\ -4a \\ \hline 24ax \end{array} \qquad \begin{array}{r} -4xy \\ -\ xy \\ \hline 4x^2y^2 \end{array} \qquad \begin{array}{r} 5ab \\ -3ac \\ \hline -15a^2bc. \end{array}$$

2.

$$\begin{array}{l} 5a-3c \\ 2a \\ \hline 10a^2-6ac \end{array} \qquad \begin{array}{l} 3ac-4b \\ 3a \\ \hline 9a^2c-12ab \end{array} \qquad \begin{array}{l} 4x-b+3ab \\ 2ab \\ \hline 8abx-2ab^2+6a^2b^2 \end{array}$$

3.

$$\begin{array}{l} a+x \\ a+x \\ \hline a^2+ax \\ \quad\ +ax+x^2 \\ \hline a^2+2ax+x^2 \end{array} \qquad \begin{array}{l} a+x \\ a-x \\ \hline a^2+ax \\ \quad\ -ax-x^2 \\ \hline a^2 \quad * \quad -x^2 \end{array} \qquad \begin{array}{l} a^2+ax+x^2 \\ a-x \\ \hline a^3+a^2x+ax^2 \\ \quad\ -a^2x-ax^2-x^3 \\ \hline a^3 \quad * \quad * \quad -x^3 \end{array}$$

4.

$$\begin{array}{l} 2x^2+xy-2y^2 \\ 3x\ -3y \\ \hline 6x^3+3x^2y-6xy^2 \\ \qquad -6x^2y-3xy^2+6y^3 \\ \hline 6x^3-3x^2y-9xy^2+6y^3 \end{array} \qquad \begin{array}{l} x^2-2x+1 \\ x^2+2x+1 \\ \hline x^4-2x^3+x^2 \\ \quad +2x^3-4x^2+2x \\ \qquad\qquad +\ x^2-2x+1 \\ \hline x^4 \quad * \quad -2x^2 \quad * \quad +1 \end{array}$$

5. Multiply together $10ac$, $2a$, $3ac$, and $5c$. *Ans.* $300a^3c^3$.

6. Multiply together a^2bc^3, ab^2c^3, and a^3b^3c. *Ans.* $a^6b^6c^6$.

7. Multiply together $3a^2-2b$, $3b$, and $4a$. *Ans.* $36a^3b-24ab^2$.

8. Multiply $3a + 2b$ by $3a - 2b$. *Ans.* $9a^2 - 4b^2$.

9. Multiply $x^2 - xy + y^2$ by $x + y$. *Ans.* $x^3 + y^3$.

10. Multiply $a^3 + a^2 b + a b^2 + b^3$ by $a - b$. *Ans.* $a^4 - b^4$.

11. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$. *Ans.* $a^4 + a^2 b^2 + b^4$.

12. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 6$.
Ans. $3x^4 + 4x^3 y - 4x^2 y^2 - 13x^2 + 22xy - 30$.

13. Multiply $x^6 - x^5 y + x^4 y^2 - x^3 y^3 + x^2 y^4 - x y^5 + y^6$ by $x + y$.
Ans. $x^7 + y^7$.

14. Multiply $x^4 - 2x^3 y + 4x^2 y^2 - 8xy^3 + 16y^4$ by $x + 2y$.
Ans. $x^5 + 32y^5$.

15. Multiply $27a^2 - 13ab + 5b^2$ by $7a^2 + b^2$.
Ans. $189a^4 - 91a^3 b + 62a^2 b^2 - 13ab^3 + 5b^4$.

16. Multiply together $x + a$, $x + b$, and $x + c$.
Ans. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$.

17. Multiply $a^{(p-1)q} - b^{(q-1)p}$ by $a^q - b^p$.
Ans. $a^{pq} - a^q b^{(q-1)p} - b^p a^{(p-1)q} + b^{pq}$.

18. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.
Ans. $a^3 + b^3 + c^3 - 3abc$.

DIVISION.

40. Division is the method of finding a quantity called the quotient; such, that if it be multiplied by the divisor, the product will be the dividend.

Thus—

$$14ax \div 7a = \frac{14ax}{7a} = 2x, \text{ since } 2x \times 7a = 14ax, \text{ the dividend;}$$

$$14ax \div -2a = -7x, \text{ since } -7x \times -2a = 14ax;$$

$$-24xy \div 3x = -8y, \text{ since } -8y \times 3x = -24xy,$$

$$-24xy \div -3x = 8y, \text{ since } 8y \times -3x = -24xy.$$

Wherefore, as in multiplication, like signs produce $+$ and unlike signs $-$, so the same rule must necessarily hold good in the division of one quantity by another. For since $+a$ multiplied by $-c$ produces $-ac$, and therefore $-ac$ divided by $+a$ gives $-c$, and so in the other three cases.

Again, if a^7 is to be divided by a^3, the quotient is $a^{7-3} = a^4$; for since $a^7 = a \times a \times a \times a \times a \times a \times a$, and $a^3 = a \times a \times a$; therefore

$$\frac{a^7}{a^3} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a} = a \times a \times a \times a = a^4.$$

Hence one power of a quantity is divided by another by subtracting the index of the latter from that of the former, and placing the remainder as the index of the quantity in the quotient.

From this we can show that the symbol a^0 is equal to unity, whatever may be the value of a. For since

$$\frac{a^m}{a^m} = a^{m-m} = a^0, \text{ and } \frac{a^m}{a^m} = 1; \text{ therefore } a^0 = 1.$$

Hence $(a + x)^0 = 1$, $x^0 = 1$, and generally $(\textit{any quantity})^0 = 1$.

EXAMPLES.

1. $6ab \div 2a = \frac{6ab}{2a} = 3b$; $abx \div bxy = \frac{abx}{bxy} = \frac{a}{y}$.

2. $16x^2 \div 8x = \frac{16x^2}{8x} = 2x$; $12a^2x^2 \div -3a^2x = \frac{12a^2x^2}{-3a^2x} = -4x$.

3. $-15ay^2 \div 3ay = \frac{-15ay^2}{3ay} = -5y$.

4. $-26y^3 \div -13y^4 = \frac{-26y^3}{-13y^4} = \frac{2}{y}$.

41. If the dividend consists of several terms, and the divisor of only one term, each term of the dividend must be separately divided by the divisor.

5. Divide $6ab - 8ax + 4a^2y$ by $2a$.

$$\frac{6ab - 8ax + 4a^2y}{2a} = \frac{6ab}{2a} - \frac{8ax}{2a} + \frac{4a^2y}{2a} = 3b - 4x + 2ay.$$

6. Divide $10a^2x - 15x^2 - 25x$ by $5x$.

Here $$\frac{10a^2x - 15x^2 - 25x}{5x} = 2a^2 - 3x - 5,$$

or,
$$\begin{array}{l} 5x)\ 10a^2x - 15x^2 - 25x \\ \hline \quad\ \ 2a^2 \ \ - \ \ 3x \ \ - \ \ 5 = \text{the quotient as before.} \end{array}$$

42. If both the dividend and divisor be compound quantities, the division is effected as in arithmetic, taking care to arrange the terms of both divisor and dividend so that each index may be less, or that each may be greater, than the succeeding one.

7. Divide $12x^4 - 26x^3y - 8x^2y^2 + 10xy^3 - 8y^4$ by $3x^2 - 2xy + y^2$.

$$\begin{array}{rl} 3x^2 - 2xy + y^2) & 12x^4 - 26x^3y - 8x^2y^2 + 10xy^3 - 8y^4 (4x^2 - 6xy - 8y^2. \\ & \underline{12x^4 - \ \ 8x^3y + 4x^2y^2} \\ & \quad -18x^3y - 12x^2y^2 + 10xy^3 \\ & \quad \underline{-18x^3y + 12x^2y^2 - \ \ 6xy^3} \\ & \qquad\qquad -24x^2y^2 + 16xy^3 - 8y^4 \\ & \qquad\qquad \underline{-24x^2y^2 + 16xy^3 - 8y^4} \end{array}$$

In the preceding example, if $12x^4$ is divided by $3x^2$, the quotient is $4x^2$, which is the first term of the quotient; then the product of the divisor and $4x^2$ is $12x^4 - 8x^3y + 4x^2y^2$, which, placed under the first three terms of the dividend, and subtracted therefrom, leaves a remainder $-18x^3y - 12x^2y^2$, to which the next term $+10xy^3$ is added. Then if $-18x^3y$ is divided by $3x^2$, the quotient is $-6xy$, which is placed next in the quotient, and the work proceeds as before.

8. Divide $48x^3 - 96ax^2 - 64a^2x + 150a^3$ by $2x - 3a$.

$$\begin{array}{rl} 2x - 3a) & 48x^3 - 96ax^2 - 64a^2x + 150a^3 (24x^2 - 12ax - 50a^2. \\ & \underline{48x^3 - 72ax^2} \\ & \quad -24ax^2 - 64a^2x \\ & \quad \underline{-24ax^2 + 36a^2x} \\ & \qquad\quad -100a^2x + 150a^3 \\ & \qquad\quad \underline{-100a^2x + 150a^3} \end{array}$$

9. Divide $a^4 - 3x^4$ by $a + x$.

$$\begin{array}{l} a + x)\, a^4 - 3x^4 \,(a^3 - a^2 x + a x^2 - x^3 - \dfrac{2x^4}{a+x}. \\ \quad a^4 + a^3 x \\ \quad -a^3 x - 3x^4 \\ \quad -a^3 x - a^2 x^2 \\ \qquad a^2 x^2 - 3x^4 \\ \qquad a^2 x^2 + a x^3 \\ \qquad\quad -a x^3 - 3x^4 \\ \qquad\quad -a x^3 - x^4 \\ \qquad\qquad -2x^4 \end{array}$$

In this example, the remainder is $-2x^4$, and to complete the quotient it is made the numerator of a fraction whose denominator is the divisor, the negative sign being prefixed to the fraction.

10. Divide $x^3 - (a + p)x^2 + (ap + q)x - aq$ by $x - a$.

$$\begin{array}{l} x - a)\, x^3 - (a+p)\, x^2 + (ap + q)\, x - aq \,(x^2 - px + q. \\ \quad x^3 - \quad a x^2 \\ \qquad -p x^2 + (ap + q)\, x \\ \qquad -p x^2 + apx \\ \qquad\qquad qx - aq \\ \qquad\qquad qx - aq \end{array}$$

11. Divide 1 by $1 - 2x + x^2$.

$$\begin{array}{l} 1 - 2x + x^2)\, 1 \qquad (1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots. \\ \quad 1 - 2x + x^2 \\ \qquad 2x - x^2 \\ \qquad 2x - 4x^2 + 2x^3 \\ \qquad\quad 3x^2 - 2x^3 \\ \qquad\quad 3x^2 - 6x^3 + 3x^4 \\ \qquad\qquad 4x^3 - 3x^4 \\ \qquad\qquad 4x^3 - 8x^4 + 4x^5 \\ \qquad\qquad\quad 5x^4 - 4x^5. \end{array}$$

In this example the division does not terminate, and the operation may be continued at pleasure to any length.

12. Divide $2x^{3n} - 6x^{2n}y^n + 6x^n y^{2n} - 2y^{3n}$ by $x^n - y^n$.

$$\begin{array}{l} x^n - y^n)\, 2x^{3n} - 6x^{2n}y^n + 6x^n y^{2n} - 2y^{3n} \,(2x^{2n} - 4x^n y^n + 2y^{2n}. \\ \quad 2x^{3n} - 2x^{2n}y^n \\ \qquad -4x^{2n}y^n + 6x^n y^{2n} \\ \qquad -4x^{2n}y^n + 4x^n y^{2n} \\ \qquad\qquad 2x^n y^{2n} - 2y^{3n} \\ \qquad\qquad 2x^n y^{2n} - 2y^{3n} \end{array}$$

Examples for Practice.

1. Divide $12\, a^2 b^2 c^2$ by $6\, abc$. *Ans.* $2\, abc$.
2. Divide $-48\, a^3 b^6$ by $8\, ab^2$. *Ans.* $-6\, a^2 b^4$.
3. Divide $36\, a^5 b^6 c d^3$ by $-4\, a^3 b c d^2$. *Ans.* $-9\, a^2 b^5 d$.
4. Divide $-48\, a^3 b^3 c^4 d^3 x^2$ by $-8\, a b^3 c d^2 x$. *Ans.* $6\, a^2 c^3 d x$.
5. Divide $12\, x^m y^{n+1}$ by $-6\, x^{m-1} y^{n-1}$. *Ans.* $-2\, x y^2$.

6. Divide $15\,a^2bc + 25\,ab^2c - 30\,abc^2$ by $-5\,abc$.
Ans. $-3\,a - 5\,b + 6\,c$.

7. Divide $y^{n+1} + 2\,y^{n+2} - 3\,y^{n+3} + y^{n+4}$ by y^n.
Ans. $y + 2\,y^2 - 3\,y^3 + y^4$.

8. Divide $6\,a^2x^4y^2 - 12\,a^3x^3y^2 + 15\,a^4x^2y$ by $3\,a^2x^2y$.
Ans. $2\,x^2y^2 - 4\,axy + 5\,a^2$.

9. Divide $12\,a^4(a+x)^2 - 18\,a^3(a+x)^3 + 24\,a^2(a+x)^4$ by $6\,a^2(a+x)^2$. *Ans.* $2\,a^2 - 3\,a(a+x) + 4\,(a+x)^2$.

10. Divide $a^2 + 4\,ax + 4\,x^2$ by $a + 2\,x$. *Ans.* $a + 2\,x$.

11. Divide $a^3 - 3\,a^2x + 3\,ax^2 - x^3$ by $a - x$. *Ans.* $a^2 - 2\,ax + x^2$.

12. Divide $a^3 + 5\,a^2x + 5\,ax^2 + x^3$ by $a + x$. *Ans.* $a^2 + 4\,ax + x^2$.

13. Divide $a^4 - 4\,a^3y + 6\,a^2y^2 - 4\,ay^3 + y^4$ by $a^2 - 2\,ay + y^2$.
Ans. $a^2 - 2\,ay + y^2$..

14. Divide $a^4 - b^4$ by $a^3 + a^2b + ab^2 + b^3$. *Ans.* $a - b$.

15. Divide $12\,x^4 - 192$ by $3\,x - 6$. *Ans.* $4\,x^3 + 8\,x^2 + 16\,x + 32$.

16. Divide $x^6 - 3\,x^4y^2 + 3\,x^2y^4 - y^6$ by $x^3 - 3\,x^2y + 3\,xy^2 - y^3$.
Ans. $x^3 + 3\,x^2y + 3\,xy^2 + y^3$.

17. Divide $x^{4n} + x^{2n}y^{2n} + y^{4n}$ by $x^{2n} + x^ny^n + y^{2n}$.
Ans. $x^{2n} - x^ny^n + y^{2n}$.

18. Divide $x^3 + ax^2 + bx + c$ by $x - r$.
$$Ans.\ x^2 + (r+a)\,x + (r^2 + ar + b)\,x + \frac{r^3 + ar^2 + br + c}{x - r}.$$

19. Divide $1 + 2\,x$ by $1 - 3\,x$. *Ans.* $1 + 5\,x + 15\,x^2 + 45\,x^3 +$ etc.

20. Divide $1 + 2\,x$ by $1 - x - x^2$. *Ans.* $1 + 3\,x + 4\,x^2 + 7\,x^3 +$ etc.

21. Divide $a^2 - b^2 + 2\,bc - c^2$ by $a - b + c$. *Ans.* $a + b - c$.

Supplementary Processes.

43. There are certain products and quotients of algebraic expressions which are of frequent occurrence in all investigations, and if the *forms* of these products and quotients could be recollected by the student, he would be enabled to obtain results with increased facility.

(1). By multiplication, $(x+a)\ (x+b) = x^2 + (a+b)\,x + ab$: Hence,

$$\begin{aligned}
(x+6)\ (x+7) &= x^2 + (6+7)\,x + 6.7 = x^2 + 13\,x + 42,\\
(x+8)\ (x-5) &= x^2 + (8-5)\,x - 8.5 = x^2 + 3\,x - 40,\\
(x-4)\ (x-7) &= x^2 - (4+7)\,x + 4.7 = x^2 - 11\,x + 28,\\
(3\,x+6)\,(3\,x-5) &= (3x)^2 + (6-5)\,3\,x - 6.5 = 9\,x^2 + 3\,x - 30.
\end{aligned}$$

In a similar manner, since

$$(x+a)\ (x+b)\,(x+c) = x^3 + (a+b+c)\,x^2 + (ab + ac + bc)\,x + abc,$$

$$\therefore\ (x+2)\ (x+3)\ (x+5) = x^3 + (2+3+5)\,x^2 + (2.3 + 2.5 + 3{\cdot}5)\,x + 2.3.5 = x^3 + 10\,x + 31\,x + 30.$$

(2). Since

$$(a+b)\,(c+d) = c\,(a+b) + d\,(a+b) = ac + bc + ad + bd;$$
$$\therefore (x+5)\ (y+6) = y\,(x+5) + 6\,(x+5) = xy + 6\,x + 5\,y + 30.$$
$$(2\,x - 7)\,(3\,y+8) = 2\,x \times 3\,y - 7 \times 3\,y + 2\,x \times 8 - 7 \times 8$$
$$= 6\,xy + 16\,x - 21\,y - 56.$$

(3). Since $(a+b)\ (a+b)$ or $(a+b)^2 = a^2 + 2\,ab + b^2$; therefore

$$(x+6)^2 = x^2 + 2.6\,x + 6^2 = x^2 + 12\,x + 36,$$
$$61^2 = (60+1)^2 = 60^2 + 2.60.1 + 1^2 = 3600 + 120 + 1 = 3721.$$

Also since $(a-b)(a-b)$ or $(a-b)^2 = a^2 - 2ab + b^2$; therefore

$$(x-9)^2 = x^2 - 2.9.x + 9^2 = x^2 - 18x + 81;$$
$$59^2 = (60-1)^2 = 60^2 - 2.60.1 + 1^2 = 3600 - 120 + 1 = 3481.$$
$$(3x-2y)^2 = (3x)^2 - 2(3x)(2y) + (2y)^2 = 9x^2 - 12xy + 4y^2.$$

(4.) Since by multiplication $(x+y)(x-y) = x^2 - y^2$; therefore

$$(x+8)(x-8) = x^2 - 8^2 = x^2 - 64,$$
$$(3x+7)(3x-7) = (3x)^2 - 7^2 = 9x^2 - 49,$$
$$(a+b+c)(a+b-c) = (a+b)^2 - c^2 = a^2 + 2ab + b^2 - c^2,$$
$$(a+b-c)(a-b+c) = \{a+(b-c)\}\{a-(b-c)\}$$
$$= a^2 - (b-c)^2 = a^2 - b^2 + 2bc - c^2.$$
$$(x^2+xy+y^2)(x^2-xy+y^2) = \{(x^2+y^2)+xy\}\{(x^2+y^2)-xy\}$$
$$= (x^2+y^2)^2 - (xy)^2 = x^4 + 2x^2y^2 + y^4 - x^2y^2$$
$$= x^4 + x^2y^2 + y^4.$$

Conversely, since $x^2 - y^2 = (x+y)(x-y)$; therefore

$$x^2 - 81 = x^2 - 9^2 = (x+9)(x-9);$$
$$4y^2 - 9a^2 = (2y)^2 - (3a)^2 = (2y+3a)(2y-3a);$$
$$a^2x^2 - b^2y^2 = (ax)^2 - (by)^2 = (ax+by)(ax-by);$$
$$(x+y)^2 - z^2 = (x+y+z)(x+y-z);$$
$$(a+b)^2 - (c-d)^2 = \{(a+b)+(c-d)\}\{(a+b)-(c-d)\}$$
$$= (a+b+c-d)(a+b-c+d).$$

(5). By the converse of (1) we have

$$x^2 + (a+b)x + ab = (x+a)(x+b);$$
$$\therefore x^2 + 8x + 15 = x^2 + (3+5)x + 3.5 = (x+3)(x+5);$$
$$x^2 + 5x - 14 = x^2 + (7-2)x - 7.2 = (x+7)(x-2);$$
$$x^2 - x - 6 = x^2 - (3-2)x - 3.2 = (x-3)(x+2).$$

In this manner, the student may find the simple factors in each of the following expressions:

1. $x^2 - x - 30.$	5. $3a^2 - 6ab + 3b^2.$
2. $a^2x - x^3.$	6. $x^3 - 2x^2 - 15x.$
3. $3b^3c - 3bc^3.$	7. $a^2 - b^2 + 2bc - c^2.$
4. $x^2 - 7x + 12.$	8. $x^3 + 3x^2 + 2x.$

(6). The following formulas, which occur very frequently, should also be noticed:

$a^2 - b^2 = (a+b)(a-b)$	$(a^2-b^2)\div(a-b) = a+b$
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$	$(a^2-b^2)\div(a+b) = a-b$
$a^3 + b^3 = (a+b)(a^2-ab+b^2)$	$(a^3-b^3)\div(a-b) = a^2+ab+b^2$
$a^4 - b^4 = (a^2+b^2)(a^2-b^2)$	$(a^3+b^3)\div(a+b) = a^2-ab+b^2$
$= (a^2+b^2)(a+b)(a-b)$	$(4x^2-1)\div(2x-1) = 2x+1.$

THE GREATEST COMMON MEASURE OF TWO OR MORE QUANTITIES.

44. A *measure* of a quantity is one which divides it without remainder: thus $4a$ is a measure of $8ax$, and $2, 4, 8, 2a, 2x, 4a, 4x, 2ax$, etc., are all measures of $8ax$.

45. A *common measure* of two or more quantities is one which divides each of them without remainder: thus $2a$, $4x$, $2ax$, etc., are all common measures of $8ax$ and $12a^2x$.

46. The *greatest common measure* of two or more quantities is the greatest of all these quantities which divides each of them without remainder. Thus $4ax$ is the greatest common measure of $8ax$ and $12a^2x$.

47. The greatest common measure of quantities which have a monomial form is readily found by inspection, as in the following examples:

Thus the greatest common measure of $8a^2x^3$ and $12a^4x^5$ is $4a^2x^3$, since $8a^2x^3 = 4a^2x^3 \times 2$, and $12a^4x^5 = 4a^2x^3 \times 3a^2x^2$, and the factors 2 and $3a^2x^2$ have no common measure.

The greatest common measure of $5xy(x-y)^2$ and $15x^2yz(x-y)$ is $5xy(x-y)$, since the factors $x-y$ and $3xz$ have no common measure.

The greatest common measure of $6a^2x^2y^2$, $15a^3xy^3$, $21ax^3y$ and $3ax^3y^3$ is $3axy$, since the quotients, $2axy$, $5a^2y^2$, $7x^2$, and x^2y^2 have no common measure.

48. In order to investigate a method for determining the greatest common measure of two or more multinomial quantities, it will be necessary to show that

If a quantity c is a common measure of two other quantities a and b, it will measure both the sum and difference of any multiples of a and b, as $ma + nb$ and $ma - nb$.

Let c be contained h times in a, and k times in b; then will $a = hc$, $b = kc$, $ma = mhc$, $nb = nkc$, and therefore $ma + nb = mhc + nkc = (mh + nk)c$, and $ma - nb = (mh - nk)c$; hence c is contained $mh + nk$ times in $ma + nb$, and $mh - nk$ times in $ma - nb$; and therefore c measures both $ma + nb$ and $ma - nb$.

Thus, 5 measures 10 and 15; and 5 will measure both $10m + 15n$ and $10m - 15n$, or $15n - 10m$, as is apparent.

Let a and b be any two given polynomial expressions, and suppose that a divided by b gives h for quotient, and c for remainder; that b divided by c gives k for quotient, and d for remainder, and that c divided by d gives l for quotient, without remainder: then will the last divisor d be the greatest common measure of a and b.

$$\begin{array}{l} b\,)\,a\,(\,h \\ \quad hb \\ \hline \quad c\,)\,b\,(\,k \\ \qquad kc \\ \hline \qquad d\,)\,c\,(\,l \\ \qquad\quad ld \\ \hline \end{array}$$

For *all* the common measures of a and b are measures of $a - hb$ or c; therefore *all* the common measures of a and b are also common measures of b and c, and conversely, *all* the common measures of b and c are measures of $hb + c$ or a, and are consequently common measures of a and b. Hence it is obvious that the *greatest common measure* of a and b is also the *greatest common measure* of b and c. In a similar manner it is proved that the greatest common measure of b and c is the same as that of c and d. Now d divides c, without remainder: therefore d is the greatest common measure of c and d, and consequently d is the greatest common measure of a and b.

49. In finding the greatest common measure of numerical quantities, the quotient should be that which, when multiplied by the divisor, gives a product nearer to the dividend than any other, whether that product be in *excess* or *defect* with respect to the dividend, because it is the *difference* to which the above reasoning applies, whether it be $a - hb$ or $hb - a$. Thus in the annexed example, where the greatest common measure of 2433 and 13787 is required, we take 6 for a quotient, because 6 times the divisor (2433) is *nearer* to 13787 than 5 times the divisor, and the operation is completed at two divisions, instead of

$$\begin{array}{l} 2433\,)\,13787\,(\,6 \\ \qquad\quad 14598 \\ \hline \qquad\quad 811\,)\,2433\,(\,3 \\ \qquad\qquad\quad 2433 \\ \hline \end{array}$$

three, if the quotient figure had been taken 5. In this manner a division is avoided each time that the nearest quotient figure is in *excess*, and the general rule is to *take that figure for quotient which will give the remainder less than the half of the divisor.*

50. If each of the terms of either of the quantities contains a factor which is not contained in each term of the other quantity, such factor forms no part of the *common measure*, and ought to be suppressed. This principle should be recollected at every step of the operation; but if the given quantities have a factor *common to both*, and that factor be suppressed in the operation, it must not be neglected altogether, because it forms a part of the common measure, and it must necessarily be introduced finally as a factor of the greatest common measure.

51. Also if the first term of any dividend is not exactly divisible by the first term of the divisor, it may be made so by multiplying the dividend by the least factor which will avoid fractional quotients. This multiplication will not affect the greatest common measure of the two quantities, because the factor thus introduced into the dividend is not found as a factor in the divisor, which has no simple factor, or if it had a simple factor, it has been divested of it.

Examples.

1. Find the greatest common measure of $x^3 - 19x + 30$ and $x^3 - 2x^2 - 7x + 14$.

Since the highest power of x is the same in both quantities we may take either as a divisor, and the work will be nearly the same in both cases.

$$
\begin{array}{rl}
x^3 - 2x^2 - 7x + 14\,) & x^3 - 19x + 30 \quad (1 \\
& \underline{x^3 - 2x^2 - 7x + 14} \\
& 2x^2 - 12x + 16, \\
\text{or} \quad x^2 - 6x + 8\,) & x^3 - 2x^2 - 7x + 14\,(x + 4 \\
& \underline{x^3 - 6x^2 + 8x} \\
& 4x^2 - 15x + 14 \\
& \underline{4x^2 - 24x + 32} \\
& 9x - 18, \\
\text{or} \quad x - 2\,) & x^2 - 6x + 8\,(x - 4 \\
& \underline{x^2 - 2x} \\
& -4x + 8 \\
& \underline{-4x + 8}
\end{array}
$$

Here in the first remainder, the simple factor common to all the terms is 2 which is suppressed as no part of the common measure of $2x^2 - 12x + 16$ and $x^3 - 2x^2 - 7x + 14$. In like manner, the simple factor 9 is rejected as no part of the common measure of $x^2 - 6x + 8$ and $9x - 18$. The last divisor, $x - 2$, is therefore the greatest common measure.

2. Find the greatest common measure of $3x^2 - 2xy - y$ and $2x^3 - 2x^2y + 7xy^2 - 7y^3$.

$$
\begin{array}{rl}
3x^2 - 2xy - y^2\,) & 2x^3 - 2x^2y + 7xy^2 - 7y^3 \\
& \underline{3} \\
& 6x^3 - 6x^2y + 21xy^2 - 21y^3\,(2x \\
& \underline{6x^3 - 4x^2y - 2xy^2} \\
& -2x^2y + 23xy^2 - 21y^3
\end{array}
$$

Multiplying by 3,

$$\begin{array}{r|l} 3x^2 - 2xy - y^2) & -6x^2y + 69xy^2 - 63y^3(-2y \\ & \underline{-6x^2y + 4xy^2 + 2y^3} \\ & \qquad\qquad 65xy^2 - 65y^3 = 65y^2(x-y). \end{array}$$

Rejecting the factor $65y^2$ as no part of the common measure, and dividing by $x - y$,

$$\begin{array}{l} x - y)\,3x^2 - 2xy - y^2\,(3x + y \\ \qquad\ \ \underline{3x^2 - 3xy} \\ \qquad\qquad\quad xy - y^2 \\ \qquad\qquad\quad \underline{xy - y^2} \end{array}$$

Hence $x - y$ is the greatest common measure of the two proposed quantities.

3. Required the greatest common measure of the two polynomials

$$6x^3 - 6x^2y + 2xy^2 - 2y^3 \text{ and } 12x^2 - 15xy + 3y^2.$$

Suppressing the factor 3 in the latter of these, and introducing the factor 2 in the former, to avoid fractions, we have

$$\begin{array}{l} 4x^2 - 5xy + y^2)\,12x^3 - 12x^2y + 4xy^2 - 4y^3\,(3x \\ \qquad\qquad\qquad \underline{12x^3 - 15x^2y + 3xy^2} \\ \qquad\qquad\qquad\qquad\quad 3x^2y + xy^2 - 4y^3 \\ \qquad\qquad\qquad\qquad\quad \underline{4} \\ \qquad\qquad\qquad\qquad\quad 12x^2y + 4xy^2 - 16y^3\,(3y \\ \qquad\qquad\qquad\qquad\quad \underline{12x^2y - 15xy^2 + 3y^3} \\ \qquad\qquad\qquad\qquad\qquad\quad 19xy^2 - 19y^3 = 19y^2(x-y). \end{array}$$

Rejecting the factor $19y^2$ as no part of the common measure, we have

$$\begin{array}{l} x - y)\,4x^2 - 5xy + y^2\,(4x - y \\ \qquad\ \ \underline{4x^2 - 4xy} \\ \qquad\qquad\ -xy + y^2 \\ \qquad\qquad\ \underline{-xy + y^2} \end{array}$$

Hence $x - y$ is the greatest common measure.

4. Find the greatest common measure of the polynomials

$$a^3 - 2a^2x + ax^3 + ax^2 - x^4 \text{ and } a^3 + ax^3 - ax^2 + x^4.$$

Arranging the terms according to the ascending powers of x, we have

$$\begin{array}{l} a^3 - 2a^2x + ax^2 + ax^3 - x^4)\,a^3 + 0 \quad - ax^2 + ax^3 + x^4\,(1 \\ \qquad\qquad\qquad\qquad\qquad\quad \underline{a^3 - 2a^2x + ax^2 + ax^3 - x^4} \\ \qquad\qquad\qquad\qquad\qquad\quad 2x)\,2a^2x - 2ax^2 + 2x^4 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad a^2 - \ ax + x^3 \\ a^2 - ax + x^3)\,a^3 - 2a^2x + ax^2 + ax^3 - x^4\,(a - x \\ \qquad\qquad\qquad \underline{a^3 - \ a^2x \qquad\quad + ax^3} \\ \qquad\qquad\qquad\quad -\ a^2x + ax^2 - x^4 \\ \qquad\qquad\qquad\quad \underline{-\ a^2x + ax^2 - x^4} \end{array}$$

Hence $a^2 - ax + x^3$ is the greatest common measure required.

5. Find the greatest common measure of $a^6 - a^2x^4$ and $a^6 + a^5x - a^4x^2 - a^3x^3$.

Here a^2 is a simple factor of the former quantity, and a^3 is a simple factor of the other, and the greatest common measure of a^2 and a^3 is

obviously a^2, which must be reserved as a simple factor of the greatest common measure. We have then the following operation:

$$
\begin{array}{l}
a^3+a^2x-ax^2-x^3)\ a^4+0\quad+0\quad+0\quad-x^4\ (a-x \\
\qquad\qquad\qquad\quad a^4+a^3x-a^2x^2-ax^3 \\
\hline
\qquad\qquad\qquad\quad -a^3x+a^2x^2+ax^3-x^4 \\
\qquad\qquad\qquad\quad -a^3x-a^2x^2+ax^3+x^4 \\
\hline
\qquad\qquad\qquad\quad 2x^2)\ 2a^2x^2-2x^4 \\
\qquad\qquad\qquad\qquad\quad a^2-x^2)\ a^3+a^2x-ax^2-x^3\ (a+x \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^3\qquad\quad -ax^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^2x\qquad -x^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^2x\qquad -x^3 \\
\hline
\end{array}
$$

Hence $a^2 \times (a^2-x^2)$ or $a^4-a^2x^2$ is the greatest common measure.

52. Let a, b, c denote three algebraic quantities, and let d be the greatest common measure of a and b, and m the greatest common measure of c and d; then because d is a common measure of a and b, every measure of d is also a common measure of a and b; therefore every common measure of c and d is also a common measure of a, b, c; and therefore the *greatest* common measure of c and d, that is, m, is also the greatest common measure of a, b, c. In a similar manner the greatest common measure of four or more quantities may be found.

Examples for Practice.

1. Find the greatest common measure of $3x^2y$ and $12xy^2$; of $6a^2x^3$ and $9ax^5$; of $12x^3y^2z^4$ and $8x^4y^3z^2$. *Ans.* $3xy$, $3ax^3$, and $4x^3y^2z^2$.

2. Find the greatest common measure of $2a^2x^2$, $4x^2y^2$, and $8x^3y$; and also of $3a^nx^{n-1}y^{n+1}$, $6a^{2n}x^{n+1}y^{n-1}$, and $21a^{n-1}x^{2n}y^{2n}$.
Ans. $2x^2$, and $3a^{n-1}x^{n-1}y^{n-1}$.

Find the greatest common measure

3. Of x^2+x-12 and x^3-5x^2+7x-3. *Ans.* $x-3$.
4. Of a^2-4 and a^2+4a+4. *Ans.* $a+2$.
5. Of a^3-ab^2 and $a^2+2ab+b^2$. *Ans.* $a+b$.
6. Of $a^5-a^3b^2$ and a^4-b^4. *Ans.* a^2-b^2.
7. Of $a^3x+2a^2x^2+2ax^3+x^4$ and $5a^5+10a^4x+5a^3x^2$.
Ans. $a+x$.
8. Of $3x^5-10x^3+15x-8$ and x^4-2x^2+1.
Ans. x^2-2x+1.
9. Of $2x^2-xy-6y^2$ and $3x^2-8xy+4y^2$. *Ans.* $x-2y$.
10. Of $7x^3-23x^2y+6xy^2$ and $5x^3-18x^2y+11xy^2-6y^3$.
Ans. $x-3y$.
11. Of $a^2+b^2+c^2+2ab+2ac+2bc$ and $a^2-b^2-c^2-2bc$.
Ans. $a+b+c$.
12. Of x^2+2x+1 and x^3+hx^2+hx+1.
Ans. $x+1$.
13. Of y^5+5y^3+6 and $3y^4+120y+117$. *Ans.* $y+1$.
14. Of $x^4-x^3-3x^2+5x-2$ and $x^5-2x^4-x^3+5x^2-4x+1$.
Ans. x^3-3x^2+3x-1.
15. Of x^3+5x^2+7x+3, x^3+3x^2-x-3, and x^3+x^2-5x+3.
Ans. $x+3$.

THE LEAST COMMON MULTIPLE OF TWO OR MORE QUANTITIES.

53. A *multiple* of a quantity is one which can be divided by it without remainder; thus $8ax$ is a multiple of $2a$, and $16x^3y^3$ is a multiple of $4xy^2$, or of $2xy^3$.

54. A *common multiple* of two or more quantities is one which can be divided by each of them without remainder. Thus $36ax$ is a common multiple of $2a$ and $9x$.

55. The *least common multiple* of two or more quantities is the least quantity which can be divided by each of them without remainder. Thus the least common multiple of $8a^2x^3$ and $12a^4x^5$ is $24a^4x^5$; and the least common multiple of $6a^2x^2y^3$, $15a^3xy^3$, and $21ax^3y$, is $210a^3x^3y^3$.

56. Let a and b be two quantities, m their greatest common measure, and l their least common multiple; then we have,

$$a = hm \text{ and } b = km,$$

where h and k have no common measure, since m is the *greatest* common measure of a and b; hence hk is the least common multiple of h and k; therefore the least common multiple of hm and km is hkm; consequently the least common multiple of a and b is

$$l = hkm = \frac{hkm^2}{m} = \frac{hm \times km}{m} = \frac{a \times b}{m}.$$

Hence the least common multiple of two quantities a and b is found by *dividing their product ab by their greatest common measure;* or, which amounts to the same thing, *divide either of the quantities by their greatest common measure, and multiply the quotient by the other quantity.*

This is evident, since $\dfrac{a \times b}{m} = \dfrac{a}{m} \times b = \dfrac{b}{m} \times a.$

57. Let a, b, c be three quantities, and l the least common multiple of a and b; then the least common multiple of l and c is the least common multiple of a, b, and c. For any common multiple of a and b contains l, their *least* common multiple, and therefore every multiple of l is a common multiple of a and b, and every common multiple of l and c is a common multiple of a, b, and c; consequently the *least* common multiple of l and c is the *least* common multiple of a, b, and c.

To find the least common multiple of three quantities: find the least common multiple of two of them, and then the least common multiple of this last multiple and the third quantity will be the least common multiple of all three, and so on, if there are four or more quantities.

EXAMPLES.

1. Find the least common multiple of $15x^2y^2$, $6x^3y$, and $12xy^3$.

The greatest common measure of $15x^2y^2$ and $6x^3y$ is (47) $3x^2y$;

$\therefore \dfrac{15x^2y^2}{3x^2y} \times 6x^3y = 5y \times 6x^3y = 30x^3y^2 =$ the least common multiple of $15x^2y^2$ and $6x^3y$. Again, the greatest common measure of $30x^3y^2$ and $12xy^3$ is $6xy^2$; hence the least common multiple of all three is $= \dfrac{30x^3y^2}{6xy^2} \times 12xy^3 = 5x^2 \times 12xy^3 = 60x^3y^3$.

Otherwise,

$$\begin{array}{r|l}
3xy & 15x^2y^2,\ 6x^3y,\ 12xy^3 \\ \hline
2 & 5xy\ ,\ 2x^2\ ,\ 4y^2 \\ \hline
x & 5xy\ ,\ x^2\ ,\ 2y^2 \\ \hline
y & 5y\ ,\ x\ ,\ 2y^2 \\ \hline
 & 5\ ,\ x\ ,\ 2y
\end{array}$$

$\therefore\ 3xy \times 2 \times x \times y \times 5 \times x \times 2y = 60x^3y^3$, as before.

This last method is similar to that employed in arithmetic for finding the least common multiple of any number of quantities.

2. Find the least common multiple of $8x^2(x-y)$, $15x^5(x-y)^2$ and $12x^3(x^2-y^2)$.

$$\begin{array}{r|l}
x^2(x-y) & 8x^2(x-y)\ ,\ 15x^5(x-y)^2\ ,\ 12x^3(x^2-y^2) \\ \hline
3x & 8\ ,\ 15x^3(x-y)\ ,\ 12x\ (x+y) \\ \hline
4 & 8\ ,\ 5x^2(x-y)\ ,\ 4\ (x+y) \\ \hline
 & 2\ ,\ 5x^2(x-y)\ ,\ (x+y)
\end{array}$$

$\therefore\ x^2(x-y) \times 3x \times 4 \times 2 \times 5x^2(x-y) \times (x+y)$
$= 120x^5(x^2-y^2)(x-y)$, the least common multiple.

3. Find the least common multiple of $3x^3-2x^2-x$ and $6x^2-x-1$.

Suppressing the simple factor x in the former quantity, we have

$$\begin{array}{r|c|c|l}
x & 3x^2-2x-1 & 6x^2-x-1 & 2 \\
 & 3x^2+x & 6x^2-4x-2 & \\ \hline
-1 & -3x-1 & 3x+1 & \\
 & -3x-1 & & \\ \hline
\end{array}$$

Hence $3x+1$ is the greatest common measure, and therefore

$$\frac{3x^3-2x^2-x}{3x+1} \times (6x^2-x-1) = x(x-1)(6x^2-x-1)$$
$$\therefore = 6x^4 - 7x^3 + x = \text{least common multiple.}$$

Find the least common multiple

4. Of $8x^4$, $10x^3y$, and $12x^2y^2$. *Ans.* $120x^4y^2$.

5. Of $10(x^2+xy)$, $8(xy-y^2)$, and $5(x^2-y^2)$.
Ans. $40xy(x^2-y^2)$.

6. Of $2a^2(a+x)$, $4ax(a-x)$, and $6x^2(a^2-x^2)$.
Ans. $12a^2x^2(a^2-x^2)$.

7. Of x^3-1 and x^2+x-2. *Ans.* x^4+2x^3-x-2.

8. Of $6x^2-x-1$ and $2x^2+3x-2$. *Ans.* $6x^3+11x^2-3x-2$.

9. Of $a-x$, a^2-x^2, and a^3-x^3. *Ans.* $a^4+a^3x-ax^3-x^4$.

10. Of x^2-1, x^2+2x-3, and x^3-7x^2+6x.
Ans. $x^5-3x^4-19x^3+3x^2+18x$.

11. Of $x^3-x^2y-xy^2+y^3$, $x^3-x^2y+xy^2-y^3$, and x^4-y^4.
Ans. $x^5-x^4y-xy^4+y^5$.

12. Of $a^3-2a^2x+ax^3+ax^2-x^4$ and $a^3+ax^3-ax^2+x^4$.
Ans. $a^4-a^3x-a^2x^2+(a^3+a)x^3-x^5$.

FRACTIONS.

58. The principles of algebraic fractions are precisely similar to those of arithmetical ones, and the management of fractions consists either in certain transformations of them to others of the same value, or in the usual processes of addition, subtraction, multiplication, and division.

59. If $\frac{a}{b}$ denote an algebraic fraction, then b denotes the number of equal parts into which the unit is supposed to be divided, and a the number of such parts to be taken; or we may conceive the quantity represented by a to be divided into b equal parts, and *one* of them to be represented by $\frac{a}{b}$.

Hence, if the *numerator* of a fraction be increased any number of times, the fraction itself is increased as many times; therefore

$$\frac{ac}{b} = \frac{a}{b} \times c.$$

Again, if the denominator be diminished any number of times, the magnitude of the equal divisions of the unit will be *increased* as many times, and consequently the same number of the increased divisions being taken, the fraction will still be increased as many times; hence

$$\frac{a}{b \div c} = \frac{a}{b} \times c.$$

We have therefore this principle: *a fraction is multiplied by a quantity, either when its numerator is multiplied, or its denominator is divided by that quantity.*

Again, if the numerator of a fraction be diminished any number of times, the fraction itself is diminished as many times; therefore

$$\frac{a \div c}{b} = \frac{a}{b} \div c.$$

Or, if the denominator of a fraction be increased any number of times, the number of divisions of the unit will be increased as many times, and the magnitude of the equal divisions will be *diminished* as many times, and consequently the same number of the diminished divisions being taken, the fraction will still be diminished as many times; hence

$$\frac{a}{bc} = \frac{a}{b} \div c.$$

We have therefore another principle: *a fraction is divided by a quantity, either when its numerator is divided or its denominator is multiplied by that quantity.*

Lastly, if the numerator of a fraction be increased any number of times, the fraction will be increased as many times, and if the denominator be increased as many times, the fraction will be diminished the same number of times as it was before increased; hence its value will not be altered. In a similar manner, it may be shown that if the terms of a fraction are both divided by the same quantity, its value is not altered.

Thus $$\frac{a}{b} = \frac{ma}{mb}, \text{ or } \frac{a}{b} = \frac{a \div m}{b \div m}.$$

We have then a third principle: *a fraction is neither multiplied nor divided by a quantity, if both its terms are either multiplied or divided by that quantity;* that is, the fraction remains unaltered in value, though the *forms* of its terms are changed by equal multiplication or division.

Any algebraical symbol a may be represented in a fractional form by writing it as the numerator of a fraction whose denominator is 1; thus

$$a = \frac{a}{1},\ 2x = \frac{2x}{1},\ \text{and } 3x^2(a+x) = \frac{3x^2(a+x)}{1}.$$

REDUCTION OR TRANSFORMATION OF FRACTIONS.

60. *To reduce a fraction to its lowest terms, or its simplest form.*

The value of a fraction is not altered, if both its terms are divided by the same quantity (59); hence it is evident that to reduce any fraction to its lowest terms, is to divide its terms by their greatest common measure. If the terms of the fraction have no common measure other than unity, the fraction is already in its simplest form. Thus $\frac{ma}{mb} = \frac{a}{b}$, by dividing both terms by m, their greatest common measure.

EXAMPLES.

1. Reduce $\frac{5a^2xy}{a^2y^2}$ and $\frac{a^2-x^2}{(a-x)^2}$ to their simplest forms.

Here $\frac{5a^2xy}{a^2y^2} = \frac{a^2y \times 5x}{a^2y \times y} = \frac{5x}{y}$, by dividing both terms by their greatest common measure a^2y.

Also $\frac{a^2-x^2}{(a-x)^2} = \frac{(a-x)\times(a+x)}{(a-x)\times(a-x)} = \frac{a+x}{a-x}$, by dividing by $a-x$.

2. Reduce $\frac{12x^2-15xy+3y^2}{6x^3-6x^2y+2xy^2-2y^3}$ to its lowest terms.

Here $12x^2 - 15xy + 3y^2 = 12x^2 - 12xy - (3xy - 3y^2)$
$= 12x(x-y) - 3y(x-y) = (x-y)\times(12x-3y)$;
and $6x^3 - 6x^2y + 2xy^2 - 2y^3 = 6x^2(x-y) + 2y^2(x-y)$
$= (x-y)\times(6x^2+2y^2)$;

therefore $\frac{12x^2-15xy+3y^2}{6x^3-6x^2y+2xy^2-2y^3} = \frac{(x-y)\times(12x-3y)}{(x-y)\times(6x^2+2y^2)}$

$$= \frac{3(4x-y)}{2(3x^2+y^2)}.$$

In this example we have found the greatest common measure, $x-y$, of the terms of the fraction, without having recourse to the usual method, which is seldom required in practice, and is frequently a very tedious operation.

Reduce each of the following fractions to its lowest terms:—

3. $\frac{16abx^2}{24a^2b^2x}$, $\frac{12a^2b^2cd}{16abc^2d^2}$, and $\frac{ab+b^2}{ac^2+bc^2}$. *Ans.* $\frac{2x}{3ab}$, $\frac{3ab}{4cd}$, and $\frac{b}{c^2}$.

4. $\dfrac{a^3 - a b^2}{a^2 + 2ab + b^2}$ and $\dfrac{3a^5 + 6a^4 b + 3a^3 b^2}{a^3 b + 3a^2 b^2 + 3ab^3 + b^4}$.

Ans. $\dfrac{a(a-b)}{a+b}$ and $\dfrac{3a^3}{b(a+b)}$.

5. $\dfrac{a^2 - 2ab + b^2}{a^2 - b^2}$ and $\dfrac{a^4 - a^3 - 3a^2 + 5a - 2}{a^4 + a^3 - 9a^2 + 11a - 4}$.

Ans. $\dfrac{a-b}{a+b}$ and $\dfrac{a+2}{a+4}$.

6. $\dfrac{x^4 - 1}{x^6 - 1}$, $\dfrac{a^3 + x^3}{(a+x)^3}$ and $\dfrac{x^2 + 2x - 3}{x^2 + 5x + 6}$.

Ans. $\dfrac{x^2 + 1}{x^4 + x^2 + 1}$, $\dfrac{a^2 - ax + x^2}{a^2 + 2ax + x^2}$, and $\dfrac{x-1}{x+2}$.

7. $\dfrac{8x^3 - 27y^3}{4x^2 - 9y^2}$ and $\dfrac{16y^4 - 53y^2 + 45y + 6}{8y^4 - 30y^3 + 31y^2 - 12}$.

Ans. $\dfrac{4x^2 + 6xy + 9y^2}{2x + 3y}$ and $\dfrac{4y^2 + 9y + 1}{2y^2 - 3y - 2}$.

8. $\dfrac{x^{3n} + x^{2n} - 2}{x^{2n} + x^n - 2}$ and $\dfrac{a^2 + b^2 - c^2 - 2ab}{a^2 - b^2 - c^2 - 2bc}$.

Ans. $\dfrac{x^{2n} + 2x^n + 2}{x^n + 2}$ and $\dfrac{a - b + c}{a + b + c}$.

9. $\dfrac{3x^2 y + 3xy^2}{3x^2 + 6xy + 3y^2}$ and $\dfrac{xy + ab + bx + ay}{mx + 2an + 2nx + am}$.

Ans. $\dfrac{xy}{x+y}$ and $\dfrac{y+b}{m+2n}$.

61. *To reduce a fraction to the form of a mixed quantity, when the reduction can be effected.*

If the numerator of the fraction $\dfrac{n}{c}$ can be divided by the denominator c, and leave a remainder b, then the form of the numerator will be $ac + b$; hence

$$\frac{ac+b}{c} = \frac{ac}{c} + \frac{b}{c} = a + \frac{b}{c}, \text{ a mixed quantity.}$$

Also $$\frac{ac-b}{c} = \frac{ac}{c} - \frac{b}{c} = a - \frac{b}{c}, \text{ a mixed quantity.}$$

Thus $$\frac{16}{5} = \frac{15}{5} + \frac{1}{5} = 3\,\frac{1}{5}, \text{ or } \frac{16}{5} = \frac{20}{5} - \frac{4}{5} = 4 - \frac{4}{5};$$

$$\frac{9a^3}{4} = 2a^3 + \frac{a^3}{4}, \text{ or } \frac{9a^3}{4} = 3a^3 - \frac{3a^3}{4}.$$

This transformation or reduction is nothing else than dividing the numerator by the denominator, as in common division.

EXAMPLES.

1. Reduce $\dfrac{a^3 + x^3}{a + x}$ to a mixed quantity; that is, divide $a^3 + x^3$ by $a + x$.

$$a + x)\,a^2 + x^2\,(a - x + \frac{2x^2}{a + x} = \text{the mixed quantity.}$$
$$\begin{array}{l} a^2 + ax \\ \hline -ax + x^2 \\ -ax - x^2 \\ \hline \qquad 2x^2 \end{array}$$

2. Reduce $\dfrac{15x^3 + 5x^2}{3x^3 + 2x^2 - 2x - 4}$ to a mixed quantity; that is, divide the numerator by the denominator.

$$3x^3 + 2x^2 - 2x - 4)\,15x^3 + 5x^2 \quad (5 - \frac{5\,(x^2 - 2x - 4)}{3x^3 + 2x^2 - 2x - 4}\ \textit{Ans.}$$
$$\begin{array}{l} 15x^3 + 10x^2 - 10x - 20 \\ \hline -\ 5x^2 + 10x + 20 \end{array}$$

3. Reduce $\dfrac{a^2}{a - x}$ and $\dfrac{10a^2 - 4a + 6}{5a}$ to mixed quantities.

Ans. $a + x + \dfrac{x^2}{a - x}$ and $2a - \dfrac{2\,(2a - 3)}{5a}$.

4. Reduce $\dfrac{3ax + 4x^2}{a + x}$ and $\dfrac{2y^4 + 19y^2 + 35}{y^3 - 3y^2 + 7y - 21}$ to mixed quantities.

Ans. $3x + \dfrac{x^2}{a + x}$, and $2y + 6 + \dfrac{23}{y - 3}$.

5. Reduce $\dfrac{a^3 + x^3}{a^2 + 2ax + x^2}$ and $\dfrac{30 - 11x - 38x^2 + 40x^3}{15 + 17x - 4x^2}$ to mixed quantities.

Ans. $a - 2x + \dfrac{3x^2}{a + x}$ and $2 - 3x + \dfrac{7x^2}{5 - x}$.

62. *To reduce a mixed quantity to a fractional form.*

Let $a + \frac{c}{b}$ and $a - \frac{c}{b}$ be mixed quantities, then since (59, p. 23)

$$a = \frac{a}{1} = \frac{ab}{b}, \text{ we have } a \pm \frac{c}{b} = \frac{a}{1} \pm \frac{c}{b} = \frac{ab}{b} \pm \frac{c}{b} = \frac{ab \pm c}{b}.$$

This is nothing else than a particular case of the addition or subtraction of algebraic fractions, in which the denominator of one quantity is unity, and the common denominator is that of the fractional quantity.

EXAMPLES.

1. Reduce $1 + \dfrac{3a}{x}$, $12 + \dfrac{4x - 18}{5x}$, and $x + y + \dfrac{x^2 + y^2}{x - y}$ to entire fractions.

Here $1 + \dfrac{3a}{x} = \dfrac{1}{1} + \dfrac{3a}{x} = \dfrac{x}{x} + \dfrac{3a}{x} = \dfrac{x + 3a}{x}$;

$$12 + \frac{4x - 18}{5x} = \frac{12}{1} + \frac{4x - 18}{5x} = \frac{60x}{5x} + \frac{4x - 18}{5x} = \frac{64x - 18}{5x};$$

$$x + y + \frac{x^2 + y^2}{x - y} = \frac{x + y}{1} + \frac{x^2 + y^2}{x - y} = \frac{x^2 - y^2}{x - y} + \frac{x^2 + y^2}{x - y} = \frac{2x^2}{x - y}.$$

2. Reduce $4 + 2x - \dfrac{2x^2 - 3a}{5a}$ and $a - \dfrac{a^2 + b^2 - x^2}{2b}$ to entire fractions.

Writing 1 for a denominator, and multiplying by $5a$, we get

$$\frac{5a(4 + 2x)}{5a} - \frac{2x^2 - 3a}{5a} = \frac{20a + 10ax - 2x^2 + 3a}{5a}$$
$$= \frac{23a + 10ax - 2x^2}{5a}.$$

Also $$\frac{a}{1} - \frac{a^2 + b^2 - x^2}{2b} = \frac{2ab - a^2 - b^2 + x^2}{2b} = \frac{x^2 - (a^2 - 2ab + b^2)}{2b}$$
$$= \frac{x^2 - (a-b)^2}{2b} = \frac{\{x + a - b\}\{x - a + b\}}{2b}.$$

3. Reduce $a + \dfrac{ax}{a - x}$ and $\dfrac{ab + ac + bc}{a + b + c} - c$ to entire fractions.

Ans. $\dfrac{a^2}{a - x}$ and $\dfrac{ab - c^2}{a + b + c}$.

4. Reduce $\dfrac{a^2 + b^2}{2ab} + 1$ and $\dfrac{a^2 + b^2}{2ab} - 1$ to entire fractions.

Ans. $\dfrac{(a + b)^2}{2ab}$ and $\dfrac{(a - b)^2}{2ab}$.

5. Reduce $1 + \dfrac{a^2 + b^2 - c^2}{2ab}$ and $1 - \dfrac{a^2 + b^2 - c^2}{2ab}$ to entire fractions.

Ans. $\dfrac{(a + b)^2 - c^2}{2ab}$ and $\dfrac{c^2 - (a - b)^2}{2ab}$;

or $\dfrac{(a + b + c)(a + b - c)}{2ab}$ and $\dfrac{(c + a - b)(c - a + b)}{2ab}$.

6. Reduce $1 - \dfrac{a^2 - 2ab + b^2}{a^2 + b^2}$ and $1 + \dfrac{a^2 - b^2}{a^2 + b^2}$ to entire fractions.

Ans. $\dfrac{2ab}{a^2 + b^2}$ and $\dfrac{2a^2}{a^2 + b^2}$.

7. Reduce $ab + cd + \dfrac{abc - c^2d - 2cd^2}{c + 2d}$ and $xy - ab - \dfrac{2xy^2 - 2aby}{x + y}$ to entire fractions.

Ans. $\dfrac{2ab(c + d)}{c + 2d}$ and $\dfrac{(xy - ab)(x - y)}{x + y}$.

63. *To reduce fractions having different denominators to equivalent fractions having a common denominator.*

The principle employed in this transformation or reduction of fractions is, that if the terms of a fraction be both multiplied by any quantity, the value of the fraction will not be altered (59). Thus if it be required to reduce $\dfrac{a}{b}$, $\dfrac{c}{d}$, and $\dfrac{e}{f}$ to equivalent fractions having a common denominator, we have

$$\frac{a}{b} = \frac{a \times df}{b \times df} = \frac{adf}{bdf},\ \frac{c}{d} = \frac{c \times bf}{d \times bf} = \frac{bcf}{bdf},\ \frac{e}{f} = \frac{e \times bd}{f \times bd} = \frac{bde}{bdf},$$

where the terms of each fraction are multiplied by the product of the denominators of the other two fractions.

Again, let it be required to reduce $\frac{m}{a\,x}$, $\frac{n}{b\,x}$, and $\frac{p}{c\,x}$ to equivalent fractions, having a common denominator.

The least common multiple of the denominators $a\,x$, $b\,x$, and $c\,x$, is evidently $a\,b\,c\,x$, which is the least common denominator; hence we have only to multiply the terms of any one of the fractions by that factor which renders its denominator equivalent to the least common denominator. Thus the terms of the fraction $\frac{m}{a\,x}$ must be multiplied by $b\,c$, because the denominator $a\,x$, multiplied by $b\,c$, produces $a\,b\,c\,x$, the common denominator: therefore

$$\frac{m}{a\,x}=\frac{m\times b\,c}{a\,x\times b\,c}=\frac{m\,b\,c}{a\,b\,c\,x},\ \frac{n}{b\,x}=\frac{n\times a\,c}{b\,x\times a\,c}=\frac{n\,a\,c}{a\,b\,c\,x},$$
$$\frac{p}{c\,x}=\frac{p\times a\,b}{c\,x\times a\,b}=\frac{p\,a\,b}{a\,b\,c\,x}.$$

EXAMPLES.

1. Reduce $\frac{3\,x}{4\,a^2}$, $\frac{2\,y}{3\,a}$, and $\frac{z}{2\,a}$ to equivalent fractions having a common denominator.

The least common multiple of the denominators is $12\,a^2$; hence

$$\frac{3\,x\times 3}{4\,a^2\times 3}=\frac{9\,x}{12\,a^2},\ \frac{2\,y\times 4\,a}{3\,a\times 4\,a}=\frac{8\,a\,y}{12\,a^2},\ \frac{z\times 6\,a}{2\,a\times 6\,a}=\frac{6\,a\,z}{12\,a^2}.$$

2. Reduce $\frac{5\,a}{6}$, $\frac{3\,a}{4}$, and $2\,b+\frac{3\,a}{b}$ to equivalent fractions having a common denominator.

Here $2\,b+\frac{3\,a}{b}=\frac{2\,b}{1}+\frac{3\,a}{b}=\frac{2\,b^2+3\,a}{b}$, consequently the least common denominator is $12\,b$, and therefore

$$\frac{5\,a\times 2\,b}{6\times 2\,b}=\frac{10\,a\,b}{12\,b},\ \frac{3\,a\times 3\,b}{4\times 3\,b}=\frac{9\,a\,b}{12\,b},\ \frac{(2\,b^2+3\,a)\times 12}{b\times 12}$$
$$=\frac{24\,b^2+36\,a}{12\,b}$$ are the fractions required.

3. Reduce $\frac{x}{a\,b}$, $\frac{y}{a\,c}$, and $\frac{z}{b\,c}$ to equivalent fractions, having a common denominator.

Here $a\,b\,c$ is the least common denominator, and therefore

$$\frac{x}{a\,b}=\frac{c\,x}{a\,b\,c},\ \frac{y}{a\,c}=\frac{b\,y}{a\,b\,c},\ \text{and}\ \frac{z}{b\,c}=\frac{a\,z}{a\,b\,c}.$$

Reduce the following sets of fractions to equivalent sets, having common denominators.

4. $\frac{a}{2\,b\,x}$, $\frac{b}{3\,c\,x}$, and $\frac{c}{4\,a\,b\,x\,y}$.

Ans. $\frac{6\,a^2\,c\,y}{12\,a\,b\,c\,x\,y}$, $\frac{4\,a\,b^2\,y}{12\,a\,b\,c\,x\,y}$, and $\frac{3\,c^2}{12\,a\,b\,c\,x\,y}$.

5. $\frac{a}{b}, \frac{b}{a}, \frac{a+x}{b}$, and $\frac{a-x}{a}$. *Ans.* $\frac{a^2}{ab}, \frac{b^2}{ab}, \frac{a^2+ax}{ab}$, and $\frac{ab-bx}{ab}$.

6. $\frac{a}{b}, \frac{ab}{a^2-b^2}, \frac{a}{a+b}$, and $\frac{b}{a-b}$.

Ans. $\frac{a^3-ab^2}{b(a^2-b^2)}, \frac{ab^2}{b(a^2-b^2)}, \frac{a^2b-ab^2}{b(a^2-b^2)}$, and $\frac{ab^2+b^3}{b(a^2-b^2)}$.

7. $\frac{m}{4a^3(a+x)}, \frac{n}{4a^3(a-x)}$, and $\frac{p}{2a^2(a^2-x^2)}$.

Ans. $\frac{ma-mx}{4a^3(a^2-x^2)}, \frac{na+nx}{4a^3(a^2-x^2)}$, and $\frac{2ap}{4a^3(a^2-x^2)}$.

8. $\frac{a}{6}, \frac{2b}{15}, \frac{3c}{5}, \frac{4d}{3x}, \frac{5e}{21x}$, and $\frac{7f}{54x}$.

Ans. $\frac{315ax}{1890x}, \frac{252bx}{1890x}, \frac{1134cx}{1890x}, \frac{2520d}{1890x}, \frac{450e}{1890x}$, and $\frac{245f}{1890x}$.

9. $\frac{x}{1-x}, \frac{x^2}{(1-x)^2}$, and $\frac{x^3}{(1-x)^3}$.

Ans. $\frac{x(1-x)^2}{(1-x)^3}, \frac{x^2(1-x)}{(1-x)^3}$, and $\frac{x^3}{(1-x)^3}$.

ADDITION OF FRACTIONS.

64. If the fractions have a common denominator, it is very obvious that the sum of the numerators will be the numerator of the sum, and the common denominator will be its denominator.

Thus $\frac{5x}{a}+\frac{7x}{a}+\frac{9x}{a}=\frac{5x+7x+9x}{a}=\frac{21x}{a}=$ sum;

and $\frac{x}{a+x}+\frac{a}{a+x}+1=\frac{a+x}{a+x}+1=1+1=2.$

But if the fractions have different denominators, they must be reduced to equivalent ones, having a common denominator, and then their sum may be found as before. One fraction cannot be added to another unless the denominators of both are the same, for it is only then that the unit is divided into the same number of equal parts indicated by the common denominator, and therefore the quantities which are to be added are like or similar quantities.

$$\text{Thus } \frac{5x}{a}+\frac{7x}{b}+\frac{9x}{c}=\frac{5x\times bc}{a\times bc}+\frac{7x\times ac}{b\times ac}+\frac{9x\times ab}{c\times ab}$$
$$=\frac{5bcx}{abc}+\frac{7acx}{abc}+\frac{9abx}{abc}=\frac{5bcx+7acx+9abx}{abc}$$
$$=\frac{(5bc+7ac+9ab)x}{abc}.$$

Examples.

1. Find the sum of $\frac{a}{3}$ and $\frac{a}{4}$, and of $\frac{4x}{3a}$ and $\frac{2x}{5b}$.

Here $\frac{a}{3}+\frac{a}{4}=\frac{4a}{12}+\frac{3a}{12}=\frac{4a+3a}{12}=\frac{7a}{12}$;

and $$\frac{4x}{3a}+\frac{2x}{5b}=\frac{20bx}{15ab}+\frac{6ax}{15ab}=\frac{20bx+6ax}{15ab}.$$

2. Find the sum of $\frac{a-b}{ab}$, $\frac{c-a}{ac}$, and $\frac{b-c}{bc}$.

Here $$\frac{a-b}{ab}+\frac{c-a}{ac}+\frac{b-c}{bc}=\frac{a}{ab}-\frac{b}{ab}+\frac{c}{ac}-\frac{a}{ac}+\frac{b}{bc}-\frac{c}{bc}$$
$$=\frac{1}{b}-\frac{1}{a}+\frac{1}{a}-\frac{1}{c}+\frac{1}{c}-\frac{1}{b}=0.$$

3. Find the sum of $\frac{a+x}{a^2+ax+x^2}$ and $\frac{a-x}{a^2-ax+x^2}$.

As the denominators have no common factor, we must multiply the terms of the first fraction by a^2-ax+x^2, and those of the second fraction by a^2+ax+x^2:

Thus we have $$\frac{a+x}{a^2+ax+x^2}+\frac{a-x}{a^2-ax+x^2}$$
$$=\frac{(a+x)(a^2-ax+x^2)+(a-x)(a^2+ax+x^2)}{(a^2+ax+x^2)(a^2-ax+x^2)}$$
$$=\frac{2a^3}{a^4+a^2x^2+x^4}=\text{the sum required.}$$

4. Add $\frac{1}{a}$, $\frac{1}{2b}$, and $\frac{1}{3c}$. *Ans.* $\frac{2ab+3ac+6bc}{6abc}$.

5. Add $\frac{2}{a^2b^2}$, $\frac{3}{a^2b^2}$, and $\frac{4}{a^2b^2}$. *Ans.* $\frac{2a+3b+4}{a^2b^2}$.

6. Add $\frac{2a}{3x^2}$, $\frac{a+2x}{4x}$, and $\frac{a}{6x}$. *Ans.* $\frac{6x^2+5ax+8a}{12x^2}$.

7. Add $\frac{x+y}{2}$ and $\frac{x-y}{2}$. *Ans.* x.

8. Add $\frac{2x}{3}$, $\frac{3x}{5}$, and $\frac{5x}{7}$. *Ans.* $\frac{208x}{105}$ or $x+\frac{103}{105}x$.

9. Add $\frac{2x}{1-x^2}$ and $\frac{1}{x+1}$. *Ans.* $\frac{1}{1-x}$.

10. Add $\frac{a}{a+b}$ and $\frac{b}{a-b}$. *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

11. Add $\frac{1}{x}$, $\frac{2}{x+1}$, and $\frac{3}{x+2}$. *Ans.* $\frac{6x^2+10x+2}{x^3+3x^2+2x}$.

12. Add $\frac{x^2+xy+y^2}{x+y}$ and $\frac{y^2}{x-y}$. *Ans.* $\frac{x(x^2+y^2)}{x^2-y^2}$.

13. Add $\frac{2}{(x-1)^3}$, $\frac{3}{(x-1)^2}$, and $\frac{4}{x-1}$. *Ans.* $\frac{4x^2-5x+3}{(x-1)^3}$.

14. Add $\frac{b}{b+x}$, $\frac{bx}{(b+x)^2}$, and $\frac{x^2}{(b+x)^2}$. *Ans.* 1.

15. Add $\frac{x^2-xy+y^2}{x^2+xy+y^2}$ and $\frac{x^2+xy+y^2}{x^2-xy+y^2}$. *Ans.* $\frac{2(x^4+3x^2y^2+y^4)}{x^4+x^2y^2+y^4}$.

16. Add $\frac{a}{(1+a)(a+x)}$ and $\frac{x}{(1-x)(a+x)}$.

Ans. $\frac{1}{(1+a)(1-x)}$.

17. Add $\frac{1}{4(1+a)}$, $\frac{1}{4(1-a)}$, and $\frac{1}{2(1-a^2)}$. *Ans.* $\frac{1}{1-a^4}$.

18. Add $\frac{x^2}{x^2-x-2}$, $\frac{2x^2-1}{x(2-x)}$, and $\frac{11x-9}{6(x-2)}$. *Ans.* $\frac{5x^2-3}{6x(x+1)}$.

SUBTRACTION OF FRACTIONS.

65. If the fractions have a common denominator, it is obvious that the one numerator must be subtracted from the other, and this remainder placed over the common denominator will be the difference required.

Thus $$\frac{8x}{a}-\frac{3x}{a}=\frac{8x-3x}{a}=\frac{5x}{a}=\text{the difference,}$$

and $$\frac{x}{x-a}-\frac{a}{x-a}=\frac{x-a}{x-a}=1.$$

But if the fractions have different denominators, they must be reduced to equivalent fractions having a common denominator, and then proceed with the results as in the former case.

Thus, $$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd},\text{ and}$$

$$\frac{1}{a-b}-\frac{1}{a+b}=\frac{a+b}{a^2-b^2}-\frac{a-b}{a^2-b^2}=\frac{a+b-(a-b)}{a^2-b^2}=\frac{2b}{a^2-b^2}.$$

EXAMPLES.

1. From $\frac{4a+8}{5}$ take $\frac{2a+6}{9}$.

Here $$\frac{4a+8}{5}-\frac{2a+6}{9}=\frac{9(4a+8)-5(2a+6)}{45}$$
$$=\frac{36a+72-10a-30}{45}=\frac{26a+42}{45}.$$

2. From $\frac{a+2x}{a-2x}$ subtract $\frac{a-2x}{a+2x}$.

Here $$\frac{a+2x}{a-2x}-\frac{a-2x}{a+2x}=\frac{(a+2x)(a+2x)-(a-2x)(a-2x)}{(a-2x)(a+2x)}$$
$$=\frac{a^2+4ax+4x^2-a^2+4ax-4x^2}{a^2-4x^2}=\frac{8ax}{a^2-4x^2}.$$

3. Subtract $2a-\frac{a-3b}{c}$ from $4a+\frac{2a}{c}$.

Here $$4a+\frac{2a}{c}-\left\{2a-\frac{a-3b}{c}\right\}=4a+\frac{2a}{c}-2a+\frac{a-3b}{c}$$
$$=2a+\frac{2a}{c}+\frac{a-3b}{c}=2a+\frac{3(a-b)}{c}.$$

4. From $6a$ subtract $\frac{3a}{4}$. *Ans.* $\frac{21a}{4}$ or $5a + \frac{a}{4}$.

5. From $\frac{5x+3y}{4}$ take $\frac{x-2y}{5}$. *Ans.* $\frac{21x+23y}{20}$.

6. From $\frac{x+y}{2}$ take $\frac{x-y}{2}$. *Ans.* y.

7. Subtract $\frac{x}{a+x}$ from $\frac{a}{a-x}$. *Ans.* $\frac{a^2+x^2}{a^2-x^2}$.

8. Subtract $\frac{a}{a-x}$ from $\frac{2a^2-2ax+x^2}{a^2-ax}$. *Ans.* $\frac{a-x}{a}$.

9. Subtract $\frac{b^2}{yz}$ from $\frac{a^2}{xy}$. *Ans.* $\frac{a^2z-b^2x}{xyz}$.

10. Subtract $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$. *Ans.* $\frac{4xy}{x^2-y^2}$.

11. Subtract $\frac{x^2-y^2}{x^2+y^2}$ from 1. *Ans.* $\frac{2y^2}{x^2+y^2}$.

12. Subtract $\frac{ax}{b+c}$ from $\frac{ax}{b-c}$. *Ans.* $\frac{2acx}{b^2-c^2}$.

13. Subtract $a - \frac{a+x}{a-x}$ from $a + \frac{a-x}{a+x}$. *Ans.* $\frac{2a^2+2x^2}{a^2-x^2}$.

14. Subtract $\frac{a+x}{a(a-x)}$ from $a + \frac{a-x}{a(a+x)}$. *Ans.* $a - \frac{4x}{a^2-x^2}$.

15. Subtract $\frac{b}{(a-b)(x+b)}$ from $\frac{a}{(a-b)(x+a)}$.

Ans. $\frac{x}{(x+a)(x+b)}$.

16. From $3x + \frac{11x-10}{15}$ subtract $2x + \frac{3x-5}{7}$. *Ans.* $\frac{137x+5}{105}$.

MULTIPLICATION OF FRACTIONS.

66. Suppose it be required to multiply a fractional quantity by a whole number, as $\frac{a}{b}$ by c; then it is obvious that $\frac{a}{b}$ must be taken c times, and the product is $= \frac{a}{b} + \frac{a}{b} + \frac{a}{b} + \ldots$ to c terms $= \frac{a+a+a+\ldots\text{to } c \text{ terms}}{b} = \frac{ac}{b}$.

But if it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$, then since $\frac{a}{b} \times c = \frac{ac}{b}$ by the preceding case, the product $\frac{ac}{b}$ must be d times *greater* than the product of $\frac{a}{b}$ and $\frac{c}{d}$; because in the former case $\frac{a}{b}$ is multiplied by c, and in the latter $\frac{a}{b}$ is to be multiplied by the d^{th} part of c; consequently the

product $\frac{ac}{b}$ is d times too great, and hence (59) the denominator must be increased d times; therefore $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Hence, to multiply a fraction by an integer or an integer by a fraction, *multiply the numerator of the fraction by the integer, and divide by the denominator*; and to multiply one fraction by another, *multiply the numerators together for the numerator of their product, and the denominators for the denominator of their product.*

Hence also, *the product of any number of fractions is equal to the product of their numerators divided by the product of their denominators.*

In multiplying fractions it is frequently advantageous to *indicate* the operation, and to simplify the terms of the fractional product, before the final multiplication is effected.

EXAMPLES.

1. Multiply $\frac{3a}{b}$, $\frac{8ac}{b}$, and $\frac{4ab}{3c}$ together.

Here $$\frac{3a}{b} \times \frac{8ac}{b} \times \frac{4ab}{3c} = \frac{3a \times 8ac \times 4ab}{b \times b \times 3c} = \frac{a \times 8a \times 4a}{b}$$
$$= \frac{32a^3}{b}.$$

2. Find the product of $\frac{x^2-y^2}{5a}$ and $\frac{15a^2}{x+y}$.

$$\frac{x^2-y^2}{5a} \times \frac{15a^2}{x+y} = \frac{15a^2(x+y)(x-y)}{5a(x+y)} = 3a(x-y).$$

3. Find the product of $\frac{x^2+3x+2}{x^2+2x+1}$ and $\frac{x^2+5x+4}{x^2+7x+12}$.

$$\frac{x^2+3x+2}{x^2+2x+1} \times \frac{x^2+5x+4}{x^2+7x+12} = \frac{(x+1)(x+2)(x+1)(x+4)}{(x+1)(x+1)(x+3)(x+4)} = \frac{x+2}{x+3}.$$

4. Multiply $1 - \frac{x-y}{x+y}$ by $2 + \frac{2y}{x-y}$.

Here $$1 - \frac{x-y}{x+y} = \frac{x+y-(x-y)}{x+y} = \frac{2y}{x+y},$$

and $$2 + \frac{2y}{x-y} = \frac{2x-2y+2y}{x-y} = \frac{2x}{x-y}; \text{ therefore}$$

$$\left\{1-\frac{x-y}{x+y}\right\} \times \left\{2+\frac{2y}{x-y}\right\} = \frac{2y}{x+y} \times \frac{2x}{x-y} = \frac{2y \times 2x}{(x+y)(x-y)} = \frac{4xy}{x^2-y^2}.$$

5. Multiply $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{5ac}{2b}$ together. *Ans.* $15ax$.

6. Find the product of $\frac{2x-2}{3}$ and $\frac{5y}{4x-4}$. *Ans.* $\frac{5y}{6}$.

7. Find the product of $\frac{a^2-x^2}{2ay}$ and $\frac{2a}{a-x}$. *Ans.* $\frac{a+x}{y}$.

8. Find the product of $\frac{x(a-x)}{a^2+2ax+x^2}$ and $\frac{a(a+x)}{a^2-2ax+x^2}$. *Ans.* $\frac{ax}{a^2-x^2}$.

9. Find the product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{x(a+x)}$ and $a+\frac{ax}{a-x}$. *Ans.* $\frac{a^2(a-b)}{x}$.

10. Multiply $x-\frac{y^2}{x}$ by $\frac{x}{y}+\frac{y}{x}$. *Ans.* $\frac{x^4-y^4}{x^2y}$.

11. Multiply $\frac{a^2-x^2}{a^2+x^2}$ by $\frac{(a+x)^2}{(a-x)^2}$. *Ans.* $\frac{a^3+2a^2x+2ax^2+x^3}{a^3-2a^2x+2ax^2-x^3}$.

12. Find the product of $x+\frac{2xy}{x-y}$ and $x-\frac{2xy}{x+y}$. *Ans.* x^2.

13. Find the product of $\frac{a^4x}{a^2-x^2}-x(a^2+x^2)$ and $\frac{a+x}{x^4}$. *Ans.* $\frac{x}{a-x}$.

14. Find the continued product of $\frac{a^3-x^3}{a^3+x^3}$, $\frac{a^2-x^2}{a^2+x^2}$, $\frac{a-x}{a+x}$, and $\frac{a^2-ax+x^2}{a^2+ax+x^2}$. *Ans.* $\frac{a^3-3a^2x+3ax^2-x^3}{a^3+a^2x+ax^2+x^3}$.

DIVISION OF FRACTIONS.

67. Let it be required to divide $\frac{a}{b}$ by c, then by (59) we have $\frac{a}{b}\div c=\frac{a}{bc}$; and if $\frac{a}{b}$ is to be divided by $\frac{c}{d}$, then since $\frac{a}{b}\div c=\frac{a}{bc}$, it is evident that the quotient $\frac{a}{bc}$ is d times *less* than the quotient of $\frac{a}{b}\div\frac{c}{d}$; because in the former case $\frac{a}{b}$ is divided by c, and in the latter $\frac{a}{b}$ is to be divided by the d^{th} part of c; consequently the quotient $\frac{a}{bc}$ must be increased d times; and therefore (59) $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c}$.

Hence to divide one fraction by another, we have only to *invert the divisor and proceed as in the Multiplication of Fractions.*

The rules of Multiplication and Division of Fractions are sometimes proved in the following manner:

Let $\frac{a}{b}=x$ and $\frac{c}{d}=y$; then we have $a=bx$ and $c=dy$; hence $ac=bdxy$, and dividing each of these equals by bd, we get $\frac{ac}{bd}=xy$; but $xy=\frac{a}{b}\times\frac{c}{d}$; therefore $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}=\frac{\text{product of numerators}}{\text{product of denominators}}$.

Again, since $a=bx$ and $c=dy$; therefore $ad=bdx$ and $bc=bdy$; hence $\frac{ad}{bc}=\frac{bdx}{bdy}=\frac{x}{y}$; but $\frac{x}{y}=x\div y=\frac{a}{b}\div\frac{c}{d}$; therefore

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c}.$$

EXAMPLES.

1. Divide $\frac{a}{4}$ by $\frac{3a}{8}$, and $\frac{3a}{2b}$ by $\frac{5c}{4d}$.

Here $$\frac{a}{4} \div \frac{3a}{8} = \frac{a}{4} \times \frac{8}{3a} = \frac{8}{4 \times 3} = \frac{2}{3},$$

and $$\frac{3a}{2b} \div \frac{5c}{4d} = \frac{3a}{2b} \times \frac{4d}{5c} = \frac{3a \times 2d}{b \times 5c} = \frac{6ad}{5bc}.$$

2. Divide $\frac{3a^2}{a^2 - b^2}$ by $\frac{a}{a+b}$, and $\frac{(a+b)^2}{a-b}$ by $\frac{a+b}{(a-b)^2}$.

Here $$\frac{3a^2}{a^2 - b^2} \div \frac{a}{a+b} = \frac{3a^2}{a^2 - b^2} \times \frac{a+b}{a} = \frac{3a}{a-b},$$

and $$\frac{(a+b)^2}{a-b} \div \frac{a+b}{(a-b)^2} = \frac{(a+b)^2}{a-b} \times \frac{(a-b)^2}{a+b} = (a+b)(a-b)$$
$$= a^2 - b^2.$$

3. Divide $\frac{x^3 + y^3}{x^2 - y^2}$ by $\frac{x^2 - xy + y^2}{x - y}$, and $\frac{x^4 - b^4}{x^2 - 2bx + b^2}$ by $\frac{x^2 + bx}{x - b}$.

Here $$\frac{x^3+y^3}{x^2-y^2} \div \frac{x^2-xy+y^2}{x-y} = \frac{(x+y)(x^2-xy+y^2)}{(x+y)(x-y)} \times \frac{x-y}{x^2-xy+y^2} = 1,$$

and $$\frac{x^4 - b^4}{x^2 - 2bx + b^2} \div \frac{x^2 + bx}{x - b} = \frac{(x^2+b^2)(x+b)(x-b)}{(x-b)(x-b)} \times \frac{x-b}{x(x+b)}$$
$$= \frac{x^2 + b^2}{x}.$$

4. Divide $\frac{6x^2}{5}$ by $3x$, and $\frac{4x}{5}$ by $\frac{3a}{5b}$. *Ans.* $\frac{2x}{5}$ and $\frac{4bx}{3a}$.

5. Divide $\frac{3a^2}{a^3 + b^3}$ by $\frac{a}{a+b}$. *Ans.* $\frac{3a}{a^2 - ab + b^2}$.

6. Divide $\frac{2ax + x^2}{c^3 - x^3}$ by $\frac{x}{c - x}$. *Ans.* $\frac{2a + x}{c^2 + cx + x^2}$.

7. Divide $x + y + \frac{x^2}{y}$ by $y + x + \frac{y^2}{x}$. *Ans.* $\frac{x}{y}$.

8. Divide $\frac{x^2 - 9}{x^2 + 4x + 4}$ by $\frac{x+3}{x+2}$ and 1 by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Ans. $\frac{x-3}{x+2}$ and $\frac{abc}{ab + ac + bc}$.

9. Divide $\frac{x^2 - 9x + 20}{x^2 - 8x + 12}$ by $\frac{x^2 - 12x + 35}{x^2 - 7x + 6}$. *Ans.* $\frac{x^2 - 5x + 4}{x^2 - 9x + 14}$.

10. Divide $\frac{x^2 + xy + y^2}{x^2 - y^2} + \frac{1}{x+y}$ by $\frac{x^2 + y^2}{x^2 - y^2}$. *Ans.* $\frac{2x}{x^2 + y^2}$.

11. Divide $a^4 - \frac{1}{a^4}$ by $a - \frac{1}{a}$. *Ans.* $a^3 + a + \frac{1}{a} + \frac{1}{a^3}$.

12. Divide $\frac{a+x}{a-x} - \frac{a-x}{a+x}$ by $\frac{a+x}{a-x} + \frac{a-x}{a+x}$. *Ans.* $\frac{2ax}{a^2 + x^2}$.

13. Divide $\frac{x^3 - 3x^2y + 3xy^2 - y^3}{x^2 - y^2}$ by $\frac{2xy - 2y^2}{3} \div \frac{x^2 + xy}{x - y}$. *Ans.* $\frac{3x}{2y}$.

MISCELLANEOUS EXAMPLES IN FRACTIONS.

68. Though the treatment of algebraic fractions is only the adaptation of the arithmetical rules to general symbols, still it is necessary that the student should work a variety of examples in order to obtain that facility in their management which is so essential to his future progress in the higher branches of analysis. For exercise in all the operations on fractions, we shall give a few additional examples.

Simplify as much as possible the subjoined examples:

1. $\dfrac{a^2+ab+b^2}{2}+\dfrac{a^2-ab+b^2}{2}$. *Ans.* a^2+b^2.

2. $\dfrac{x}{a^2+ax+x^2}+\dfrac{2ax+a^2}{x^3-a^3}$. *Ans.* $\dfrac{1}{x-a}$.

3. $\dfrac{1}{a-1}-\dfrac{2a}{a^2+1}+\dfrac{1}{a+1}$. *Ans.* $\dfrac{4a}{a^4-1}$.

4. $\dfrac{y-1}{2}+\dfrac{y-2}{3}+\dfrac{y+7}{6}$. *Ans.* y.

5. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^3-x^2y}{y^3-x^2y}$. *Ans.* $\dfrac{y}{x+y}$.

6. $\dfrac{12}{5(a+3)}-\dfrac{1}{15(a-2)}-\dfrac{4}{3(a+1)}$. *Ans.* $\dfrac{a^2-4a+3}{a^3+2a^2-5a-6}$.

7. $\dfrac{2x}{x^2-y^2}+\dfrac{1}{x+y}-\dfrac{1}{x-y}$. *Ans.* $\dfrac{2}{x+y}$.

8. $\dfrac{x-y}{y}+\dfrac{2x}{x-y}-\dfrac{x^3+x^2y}{x^2y-y^3}$. *Ans.* $\dfrac{y}{x-y}$.

9. $\left\{x+\dfrac{y-x}{1+xy}\right\}\div\left\{1-x\,\dfrac{y-x}{1+xy}\right\}$. *Ans.* y.

10. $\dfrac{1}{(a-b)(x+a)}+\dfrac{1}{(b-a)(x+b)}$. *Ans.* $\dfrac{-1}{(x+a)(x+b)}$.

11. $\dfrac{3}{2y-3}-\dfrac{2y+15}{4y^2+9}-\dfrac{2}{2y+3}$. *Ans.* $\dfrac{18(2y+15)}{16y^4-81}$.

12. $\dfrac{1}{x(a-x)}-\dfrac{1}{y(a-y)}-\dfrac{1}{x(a+x)}+\dfrac{1}{y(a+y)}$.
Ans. $\dfrac{2(x^2-y^2)}{(a^2-x^2)(a^2-y^2)}$.

13. $\dfrac{x}{1-x}-\dfrac{x^2}{(1-x)^2}+\dfrac{x^3}{(1-x)^3}$. *Ans.* $x+\dfrac{x^4}{(1-x)^3}$.

14. $\dfrac{1}{4a^3(a+x)}+\dfrac{1}{4a^3(a-x)}+\dfrac{1}{2a^2(a^2+x^2)}$. *Ans.* $\dfrac{1}{a^4-x^4}$.

15. $\dfrac{1}{(x+1)(x+2)}-\dfrac{1}{(x+1)(x+2)(x+3)}$. *Ans.* $\dfrac{1}{(x+1)(x+3)}$.

16. $\dfrac{1}{(x+1)(x+2)}-\dfrac{3}{(x+1)(x+2)(x+3)}$.
Ans. $\dfrac{x}{(x+1)(x+2)(x+3)}$.

17. $\frac{a+c}{(a-b)(x-a)} - \frac{b+c}{(a-b)(x-b)}$. *Ans.* $\frac{x+c}{(x-a)(x-b)}$.

18. $\frac{1}{x^3} + \frac{1}{x^2} - \frac{1}{x} - \frac{1}{(x^2+1)^2} + \frac{x-1}{x^2+1}$. *Ans.* $\frac{x^2+x+1}{x^3(x^2+1)^2}$.

19. $\frac{1}{b+a}\left\{a^2.\frac{a^2-b^2}{a+b} - b^2(2b-a) + b.\frac{a^3-b^3}{a-b} - \frac{a^4-b^4}{a+b}\right\}$. *Ans.* $a\,b$.

20. $\frac{a}{b} - \frac{a^2-b^2}{b^2}x + \frac{a(a^2-b^2)x^2}{b^2(b+ax)}$. *Ans.* $\frac{a+bx}{b+ax}$.

21. $\left\{\frac{x+2y}{x+y} + \frac{x}{y}\right\} \div \left\{\frac{x+2y}{y} - \frac{x}{x+y}\right\}$. *Ans.* 1.

22. $\dfrac{\frac{a^2+b^2}{2a^2} - \frac{2b^2}{a^2+b^2}}{\frac{a^2+b^2}{2b^2} - \frac{2a^2}{a^2+b^2}}$ and $\dfrac{1\frac{4}{5} - \frac{2}{3}(x+2)}{\frac{3}{10}(x+1)}$. *Ans.* $\frac{b^2}{a^2}$ and $\frac{2(7-10x)}{9(x+1)}$.

23. $\dfrac{1}{\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}} - \dfrac{1}{\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-2}}$. *Ans.* $\frac{6x^3(x^2-2)}{9x^4-24x^2+4}$.

24. $\frac{x^2-9x+20}{x^2-6x} \div \frac{x^2-5x}{x^2-13x+42}$. *Ans.* $\frac{(x-4)(x-7)}{x^2}$.

25. $\frac{x^{3n}}{x^n-1} - \frac{x^{2n}}{x^n+1} + \frac{1}{x^n+1} - \frac{1}{x^n-1}$ *Ans.* $x^{2n}+2$

26. $\frac{1}{(a-b)(a-c)(x+a)} + \frac{1}{(b-a)(b-c)(x+b)}$
$+\frac{1}{(c-a)(c-b)(x+c)}$. *Ans.* $\frac{1}{(x+a)(x+b)(x+c)}$.

INVOLUTION.

69. Involution is the process of finding the powers of any quantity, and the operation is nothing else than multiplication, where the factors are all the *same* as the quantity itself.

Since the n^{th} power of x^m, or $(x^m)^n = x^m \times x^m \times x^m \ldots$ to n factors.

$$= x^{m+m+m+\ldots\text{ to } n \text{ terms.}}$$
$$= x^{m\times n} = x^{mn};$$

and the m^{th} power of x^n, or $(x^n)^m = x^n \times x^n \times x^n \ldots$ to m factors.

$$= x^{n+n+n+\ldots\text{ to } m \text{ terms.}}$$
$$= x^{n\times m} = x^{mn};$$

therefore the n^{th} power of x^m is the same as the m^{th} power of x^n, or

$$(x^m)^n = (x^n)^m = x^{mn}.$$

Hence $(x^2)^3 = (x^3)^2 = x^6$; $(x^3)^4 = (x^4)^3 = x^{12}$,
and also $\{(x+a)^2\}^4 = \{(x+a)^4\}^2 = (x+a)^8$.

Thus any *power of a power* of a quantity is found by multiplying the indices of the two powers, for the index of the power required.

Also $(2x^2y^3)^2 = 2x^2y^3 \times 2x^2y^3 = 4x^4y^6$; $\left(\frac{3x}{2y}\right)^2 = \frac{3x}{2y} \times \frac{3x}{2y}$
$= \frac{9x^2}{4y^2}$; $(-3a^2x)^2 = -3a^2x \times -3a^2x = 9a^4x^2$; and

$$(-2x^3)^3 = (-2)^3x^{3\times3} = -8x^9.$$

The successive powers of a binomial quantity, $a+b$, may be found as follow:

$$\begin{array}{l}
a+b \\
\underline{a+b} \\
a^2+ab \\
\underline{\quad +ab \quad +b^2} \\
a^2+2ab+b^2=(a+b)^2 \\
\underline{a+b} \\
a^3+2a^2b+\ \ ab^2 \\
\underline{\quad +\ \ a^2b+2ab^2+b^3} \\
a^3+3a^2b+3ab^2+b^3=(a+b)^3 \\
\underline{a+b} \\
a^4+3a^3b+3a^2b^2+\ \ ab^3 \\
\underline{\quad +\ \ a^3b+3a^2b^2+3ab^3+b^4} \\
a^4+4a^3b+6a^2b^2+4ab^3+b^4=(a+b)^4.
\end{array}$$

If this last product be multiplied by $a+b$, we shall have

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5;$$

and multiplying this result by $a+b$, we get

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$$

In this manner any power of the binomial quantity $a+b$ can be found; but by considering attentively the several terms of any of the preceding powers of $a+b$, we may deduce the law of the formation of the successive terms, and thence obtain the result of any power without the operation of multiplication. Thus the index of the leading quantity, a, in the first term, is always the index of the given power, and it decreases by unity from term to term to the last, where it is 0, and a^0 being equivalent to 1 (40) is understood; hence

$$a^n,\ a^{n-1},\ a^{n-2},\ a^{n-3}, \ldots\ldots a^3,\ a^2,\ a^1,\ a^0, \ldots\ldots\ (\alpha)$$

are the $n+1$ factors involving the leading quantity, a. In like manner, since a and b are symmetrically involved in the quantity $a+b$, it is obvious that the powers of b will be the same as those of a, but in reverse order; hence

$$b^0,\ b^1,\ b^2,\ b^3, \ldots\ldots\ b^{n-3},\ b^{n-2},\ b^{n-1},\ b^n, \ldots\ldots\ (\beta)$$

are the $n+1$ factors containing the second term, b, of the binomial $a+b$. Hence, recollecting that $a^0=1$ and $b^0=1$, the terms of $(a+b)^n$, without their coefficients, are the products of the corresponding terms of (α) and (β), viz.:

$$a^n,\ a^{n-1}b,\ a^{n-2}b^2,\ a^{n-3}b^3, \ldots\ a^3b^{n-3},\ a^2b^{n-2},\ ab^{n-1},\ b^n \ldots\ (\gamma).$$

With reference to the coefficients of the several terms, we observe that the coefficients of the first and last terms are each unity, the coefficients of the second term and the last but one are each the same as the index of the given power, and the coefficient of any term is found by multiplying the coefficient of the preceding term by the index of the power of the leading quantity in it, and dividing the product either by the number of terms to that place, or by the index of the power of the second quantity in it increased by unity; thus in the expansion of $(a+b)^6$ given above, we have

$$\frac{1\times 6}{1}=6,\ \frac{6\times 5}{2}=15,\ \frac{15\times 4}{3}=20,\ \frac{20\times 3}{4}=15,\ \frac{15\times 2}{5}=6,$$

$\dfrac{6\times 1}{6}=1$; hence, generally, the coefficients of $(a+b)^n$ are

$$1,\ n,\ \frac{n(n-1)}{2},\ \frac{n(n-1)}{2}\cdot\frac{n-2}{3},\ \frac{n(n-1)(n-2)}{2.3}\cdot\frac{n-3}{4},\ \text{etc.};$$

consequently the complete expression for $(a+b)^n$ is

$$(a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2$$
$$+ \frac{n(n-1)(n-2)}{2.3} a^{n-3} b^3 + \dots + b^n \dots (1).$$

And if b is negative, then we have

$$(a-b)^n = a^n - n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2$$
$$- \frac{n(n-1)(n-2)}{2.3} a^{n-3} b^3 + \dots \pm b^n \dots (2);$$

where the last term is $+ b^n$ when n is even, and $- b^n$ when n is odd.

This is the binomial theorem of Newton, which, as is shown in Art. 136, holds true whether n be a whole number or a fraction, and either positive or negative.

The general term of the expansion of $(a+b)^n$ may be found by observing the law of formation of the several terms; thus the coefficient of the *fourth* term of (1) is $\frac{n(n-1)(n-2)}{2.3}$; therefore the coefficient of the p^{th} term will be $\frac{n(n-1)(n-2)\dots\{n-(p-2)\}}{1.2.3\dots(p-1)}$, and the

$$p^{\text{th}} \text{ term} = \frac{n(n-1)(n-2)\dots\{n-(p-2)\}}{1.2.3\dots(p-1)} a^{n-(p-1)} b^{p-1} \dots (3).$$

The whole number of terms is obviously one more than the index of the given power, and when the signs of both terms of the binomial are $+$, the signs of all the terms are $+$; but if the sign of the second term of the binomial is $-$, the signs of the odd terms are $+$, and the signs of the even terms $-$, or the signs of the terms are $+$ and $-$ alternately. Also the coefficients equidistant from the extremes are equal, and the sum of the indices in any term is equal to the index of the power.

Examples.

1. Find the square of $a+b$, and also the square of $a-b$.

$$\begin{array}{rl} & a+b \\ & \underline{a+b} \\ & a^2+ab \\ & \underline{\quad\ +ab+b^2} \\ (a+b)^2= & \underline{a^2+2ab+b^2} \end{array} \qquad \begin{array}{rl} & a-b \\ & \underline{a-b} \\ & a^2-ab \\ & \underline{\quad\ -ab+b^2} \\ (a-b)^2= & \underline{a^2-2ab+b^2} \end{array}$$

These two forms are of frequent occurrence, and ought to be recollected by the student, as by means of them we can at once write down the squares of other binomial quantities; thus, if it be required to find the square of $2a+3x$, we have

$$(2a+3x)^2 = (2a)^2 + 2(2a)(3x) + (3x)^2 = 4a^2 + 12ax + 9x^2;$$

Also $\left(\frac{a}{b}+\frac{b}{a}\right)^2 = \left(\frac{a}{b}\right)^2 + 2\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2 = \frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}$;

and $(3x-7y)^2 = (3x)^2 - 2(3x)(7y) + (7y)^2 = 9x^2 - 42xy + 49y^2$.

From these results we may also infer that *in any trinomial which is a complete square, four times the product of the extreme terms is equal to the square of the middle term;* since

$$4 \times a^2 \times b^2 = 4a^2b^2 = (\pm 2ab)^2.$$

2. Find the square of $1 + x - \frac{1}{x}$ and the cube of $2x - 1$.

$$\begin{array}{l}
1 + x - \frac{1}{x} \\
1 + x - \frac{1}{x} \\
\hline
1 + x - \frac{1}{x} \\
\quad + x + x^2 - 1 \\
\qquad - \frac{1}{x} - 1 + \frac{1}{x^2} \\
\hline
\quad 2x + x^2 - \frac{2}{x} - 1 + \frac{1}{x^2}
\end{array}
\qquad\qquad
\begin{array}{l}
2x - 1 \\
2x - 1 \\
\hline
4x^2 - 2x \\
\quad\;\; - 2x + 1 \\
\hline
4x^2 - 4x + 1 \\
2x - 1 \\
\hline
8x^3 - 8x^2 + 2x \\
\quad\;\; - 4x^2 + 4x - 1 \\
\hline
8x^3 - 12x^2 + 6x - 1
\end{array}$$

3. Find the fifth term of $(2x + 5y)^6$.

Comparing this with the binomial $(a + b)^n$, we have $a = 2x$, $b = 5y$, and $n = 6$; and as the fifth term is required, we must take $p = 5$, and by formula (3) we get the fifth term

$$= \frac{6 \times 5 \times 4 \times 3}{1.2.3.4}(2x)^2(5y)^4 = 15 \times 2^2 \times 5^4 \times x^2y^4 = 37500\,x^2y^4.$$

4. Find the square of $3a^2$, and the cube or third power of $-4a^2b^3$.
Ans. $9a^4$ and $-64a^6b^9$.

5. Find the fifth powers of $-2ax^2y$ and a^2xy^3.
Ans. $-32a^5x^{10}y^5$ and $a^{10}x^5y^{15}$.

6. Find the fourth power of $-\frac{a^2x}{2b^2}$ and the m^{th} power of $ax^2y^3z^n$.
Ans. $\frac{a^8x^4}{16b^8}$ and $a^mx^{2m}y^{3m}z^{mn}$.

7. Find the cube of $-\frac{4a^2x^4}{5y^2z}$ and the square of $\frac{3(a-x)}{4(a+x)}$.
Ans. $-\frac{64a^6x^{12}}{125y^6z^3}$ and $\frac{9a^2 - 18ax + 9x^2}{16a^2 + 32ax + 16x^2}$.

8. Find the fourth power of $mx + ny$.
Ans. $m^4x^4 + 4m^3nx^3y + 6m^2n^2x^2y^2 + 4mn^3xy^3 + n^4y^4$.

9. Find the *sixth* term of $(ay + cz^2)^8$ and the *fourth* term of $(a-2x)^4$.
Ans. $56a^3c^5y^3z^{10}$, and $-32ax^3$.

10. Find the fourth power of $a - 2x$ and the cube of $1 - 2x + 3x^2$.
Ans. $a^4 - 8a^3x + 24a^2x^2 - 32ax^3 + 16x^4$,
and $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

11. Find the fourth power of $2ax - x^2$, and the cube of $6a^2 - 5ax$.
Ans. $16a^4x^4 - 32a^3x^5 + 24a^2x^6 - 8ax^7 + x^8$,
and $216a^6 - 540a^5x + 450a^4x^2 - 125a^3x^3$.

12. Find the squares of $a + b + c$, and $a^2 - a + \frac{1}{4}$.
Ans. $a^2 + b^2 + c^2 + 2(ab + ac + bc)$,
and $a^4 - 2a^3 + \frac{3}{2}a^2 - \frac{1}{2}a + \frac{1}{16}$.

13. Find the square of the polynomial quantity, $a + b + c + d$.
Ans. $a^2 + b^2 + c^2 + d^2 + 2\,(a b + a c + a d + b c + b d + c d)$.

EVOLUTION.

70. EVOLUTION, or the extraction of roots, is the process of finding the quantity which, when raised to a proposed power, will produce a given quantity.

Since (69) any power of a power of a quantity is found by multiplying together the indices of the two powers, therefore, conversely, the root of any power of a quantity will be found by dividing the index of the power by the index of the root which is to be extracted. Thus, since the n^{th} power of x^m, or $(x^m)^n = x^{m \times n} = x^{mn}$, therefore, conversely, the n^{th} root of x^{mn} is $x^{\frac{mn}{n}} = x^m$. In a similar manner the square root of x^6 is $x^{\frac{6}{2}} = x^3$, the cube root of a^6 is $a^{\frac{6}{3}} = a^2$, the fourth root of $(a - x)^8$ is $(a - x)^{\frac{8}{4}} = (a - x)^2$, etc.

Also the cube root of $27\, a^3 x^6 y^{3m}$ is $= \sqrt[3]{27} \times a^{\frac{3}{3}} \times x^{\frac{6}{3}} \times y^{\frac{3m}{3}}$ $= 3\, a\, x^2 y^m$, for $(3\, a x^2 y^m)^3 = 3\, a x^2 y^m \times 3\, a x^2 y^m \times 3\, a x^2 y^m$ $= 3^3 \times a^3 \times x^{2 \times 3} \times y^{m \times 3} = 27\, a^3 x^6 y^{3m}$, and the fourth root of

$$\frac{16\, a^4}{81\, x^4} \text{ is } \frac{\sqrt[4]{16} \times a^{\frac{4}{4}}}{\sqrt[4]{81} \times x^{\frac{4}{4}}} = \frac{2\, a}{3\, x}, \text{ for } \left(\frac{2\, a}{3\, x}\right)^4 = \frac{2\, a}{3\, x} \times \frac{2\, a}{3\, x} \times \frac{2\, a}{3\, x} \times \frac{2\, a}{3\, x}$$
$$= \frac{2^4 \times a^4}{3^4 \times x^4} = \frac{16\, a^4}{81\, x^4}.$$

Hence to extract any root of a simple quantity, *divide the index of the power by the index of the root, prefixing the root of the numerical coefficient.* If the simple quantity is composed of several factors, the root will be the product of the roots of the several factors; and if the simple quantity is a fraction, the root will be the root of the numerator divided by the root of the denominator.

Any even root of a positive quantity will have the double sign $\pm$, thus the square root of $+ a^2$ is either $+ a$ or $- a$; because $(+ a)^2 = + a \times (+ a) = + a^2$, and $- a \times (- a) = + a^2$. But any odd root of a quantity will have the same sign as the quantity itself; thus the cube root of
$- x^3$ is $- x$, for $- x \times (- x) \times (- x) = + x^2 \times (- x) = - x^3$.
Hence there can be no *even* root of a negative quantity, for no quantity multiplied by itself can ever produce a negative quantity.

EXAMPLES.

1. Find the square root of $9\, a^2 b^4 x^2$, the cube root of $- 8\, a^6 y^3$, and the fourth root of $\frac{16\, x^4 y^8}{81\, a^8 z^{12}}$.

$\sqrt{9\, a^2 b^4 x^2} = \sqrt{9} \times a^{\frac{2}{2}} \times b^{\frac{4}{2}} \times x^{\frac{2}{2}} = \pm\, 3\, a b^2 x,$

$\sqrt[3]{- 8\, a^6 y^3} = - \sqrt[3]{8} \times a^{\frac{6}{3}} \times y^{\frac{3}{3}} = - 2 \times a^2 \times y = - 2\, a^2 y,$

$$\sqrt[4]{\frac{16x^4y^8}{81a^8z^{12}}}=\frac{\sqrt[4]{16}\times x^{\frac{4}{4}}\times y^{\frac{8}{4}}}{\sqrt[4]{81}\times a^{\frac{8}{4}}\times z^{\frac{12}{4}}}=\pm\,\frac{2\times x\times y^2}{3\times a^2\times z^3}=\pm\,\frac{2xy^2}{3a^2z^3}.$$

2. Find the square roots of $49\,a^2x^4y^6$, $\dfrac{a^4y^2z^8}{16}$, and $\dfrac{25x^6y^4}{81a^4z^8}$.

Ans. $7\,ax^2y^3$, $\dfrac{a^2yz^4}{4}$, and $\dfrac{5x^3y^2}{9a^2z^4}$.

3. Find the cube roots of $27\,a^3x^6y^3$, $\dfrac{a^3b^6c^9}{64x^{12}}$, and $-\dfrac{8a^3z^{3m}}{27x^3y^6}$.

Ans. $3\,ax^2y$, $\dfrac{ab^2c^3}{4x^4}$, and $-\dfrac{2az^m}{3xy^2}$.

4. Find the fourth root of $\dfrac{a^4(ax-x^2)^8}{b^8(ax+x^2)^4}$, and the m^{th} root of $2^m a^{3m}x^{2m}y^{2mn}$.

Ans. $\dfrac{a(ax-x^2)^2}{b^2(ax+x^2)}$ and $2\,a^3x^2y^{2n}$.

5. Find the square root of $a^{2m}x^{4n}$ and the n^{th} root of $a^{mn}x^{pn+n}$.

Ans. a^mx^{2n} and a^mx^{p+1}.

---

71. If the quantity whose root is to be extracted is a compound quantity, we may discover a process for the extraction of any root whatever, by a little consideration of the form of the several powers of a binomial or trinomial quantity. Thus since the square of $a+b$, or $(a+b)^2=a^2+2ab+b^2=a^2+(2a+b)\,b$, and the square of $a+b+c$, or $(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2$ $=a^2+(2a+b)\,b+(2a+2b+c)\,c$, we can readily derive the following process for extracting the square root of a compound quantity:

$$\begin{array}{lll}
 & a^2+2ab+b^2+2ac+2bc+c^2\ (a+b+c & \\
 & a^2 & \\
\hline
2a+b & 2ab+b^2 & \\
\quad\ \ b & 2ab+b^2 & \\
\hline
2a+2b+c & & 2ac+2bc+c^2 \\
 & & 2ac+2bc+c^2 \\
\hline
\end{array}$$

Here the square root of the first term a^2 is a, which is placed in the first term of the root on the right, and its square a^2 is written below a^2 and subtracted from the given quantity. The first term $2\,ab$ of the remainder is divided by $2\,a$, which is written on the left, and the quotient b is placed both in the root and added to $2\,a$ on the left; then the sum $2\,a+b$ is multiplied by b, and the product $2\,ab+b^2$ is subtracted from the preceding remainder, leaving $2\,ac$ for the first term of the second remainder. On the left, write twice the root thus obtained, viz., $2\,a+2\,b$, or, which is the same thing, add b to the former divisor $2\,a+b$; then $2\,ac$ divided by $2\,a$ gives c, the next term of the root. Add c to $2\,a+2\,b$ on the left, and multiply the sum by c, and write the product below the second remainder. Subtract and repeat the process until nothing remains, or the root be obtained to the required accuracy.

Examples.

1. Find the square root of $4x^4 - 4x^3 + 13x^2 - 6x + 9$.

$$\begin{array}{rl}
 & 4x^4 - 4x^3 + 13x^2 - 6x + 9\,(2x^2 - x + 3 \\
 & 4x^4 \\
 \hline
4x^2 - x) & -4x^3 + 13x^2 \\
 & -4x^3 + x^2 \\
 \hline
4x^2 - 2x + 3\ ) & 12x^2 - 6x + 9 \\
 & 12x^2 - 6x + 9 \\
 \hline
\end{array}$$

2. Extract the square root of $\frac{a^2x^2}{y^2} + \frac{1}{x^2y^2} + \frac{y^2}{x^2} - 2\left(a + \frac{1}{x^2} - \frac{a}{y^2}\right)$.

Arranging the terms in order, we have

$$\begin{array}{rl}
 & \frac{a^2x^2}{y^2} - 2a + \frac{y^2}{x^2} + \frac{2a}{y^2} - \frac{2}{x^2} + \frac{1}{x^2y^2}\left(\frac{ax}{y} - \frac{y}{x} + \frac{1}{xy}\right. \\
 & \frac{a^2x^2}{y^2} \\
 \hline
\left.\frac{2ax}{y} - \frac{y}{x}\right) & - 2a + \frac{y^2}{x^2} \\
 & - 2a + \frac{y^2}{x^2} \\
 \hline
\left.\frac{2ax}{y} - \frac{2y}{x} + \frac{1}{xy}\right) & + \frac{2a}{y^2} - \frac{2}{x^2} + \frac{1}{x^2y^2} \\
 & + \frac{2a}{y^2} - \frac{2}{x^2} + \frac{1}{x^2y^2} \\
 \hline
\end{array}$$

Find the square root of each of the following expressions:

3. $16x^2 - 56xy + 49y^2$. *Ans.* $4x - 7y$.

4. $a^2b^2 + ab + \frac{1}{4}$. *Ans.* $ab + \frac{1}{2}$.

5. $\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2\left(\frac{a}{b} + \frac{b}{a}\right) + 3$. *Ans.* $\frac{a}{b} + \frac{b}{a} + 1$.

6. $9x^4 + 12x^3 + 28x^2 + 16x + 16$. *Ans.* $3x^2 + 2x + 4$.

7. $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} + \frac{xy}{3} - \frac{xz}{4} - \frac{yz}{6}$. *Ans.* $\frac{x}{2} + \frac{y}{3} - \frac{z}{4}$.

8. $\frac{x^2}{9} + \frac{4y^2}{25} + \frac{9z^2}{16} + \frac{4xy}{15} - \frac{xz}{2} - \frac{3yz}{5}$. *Ans.* $\frac{x}{3} + \frac{2y}{5} - \frac{3z}{4}$.

9. $\frac{x^2}{y^2}\left(\frac{x^2}{4y^2} + 1\right) + \frac{4y^2}{x^2}\left(\frac{y^2}{x^2} + 1\right) + 3$. *Ans.* $\frac{x^2}{2y^2} + \frac{2y^2}{x^2} + 1$.

10. $x^4 + 2px^3 + (p^2 - 2q)x^2 - 2pqx + q^2$. *Ans.* $x^2 + px - q$.

11. $x^{2n}y^{4n} - 2x^ny^{2n}a^n + a^{2n}$. *Ans.* $x^ny^{2n} - a^n$.

12. $a^2 + x^2$ and $\frac{a + x}{a - x}$.

Ans. $a + \frac{x^2}{2a} - \frac{x^4}{2.4a^3} + \frac{3x^6}{2.4.6a^5} - \frac{3.5x^8}{2.4.6.8a^7} + \text{etc.}$

and $1 + \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{2a^3} + \frac{3x^4}{2.4a^4} + \text{etc.}$

72. To extract the cube root of a compound quantity.

Since $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+(3a^2+3ab+b^2)b$,
and $(a+b+c)^3 = a^3+(3a^2+3ab+b^2)b$
$$+\{3a^2+6ab+3b^2+3(a+b)c+c^2\}c,$$
we may extract the cube root in the following manner:

$$\begin{array}{lll}
 & & a^3+3a^2b+3ab^2+b^3\ (a+b \\
 & 3a^2 & a^3 \\
3a+b & \left.\begin{array}{r} 3ab+b^2 \\ \hline 3a^2+3ab+b^2 \\ b^2 \end{array}\right\} & \begin{array}{|l} 3a^2b+3ab^2+b^3 \\ 3a^2b+3ab^2+b^3 \\ \hline \end{array} \\
\hline
 & 3a^2+6ab+3b^2 = \text{second trial divisor.} &
\end{array}$$

Here the cube root of a^3 is a, which is the first term of the root. Subtract a^3, which is the cube of a, from the quantity, and bring down the remainder; write three times the square of a on the left, and three times a still farther towards the left, and one line lower than $3a^2$. Divide the first term of the remainder, $3a^2b$, by $3a^2$, and the quotient will give b, the second term of the root; then adding b to $3a$, and multiplying the sum, $3a+b$, by b, gives $3ab+b^2$, which place under $3a^2$. Adding $3ab+b^2$ to $3a^2$, and multiplying the sum $3a^2+3ab+b^2$ by b, gives $3a^2b+3ab^2+b^3$ to be placed under the first remainder. Subtract and bring down another part of the quantity; proceed in a similar manner to obtain the next term of the root, and so on until the process terminate, or be carried to a sufficient degree of approximation.

Examples.

1. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$\begin{array}{lll}
 & & x^6+6x^5-40x^3+96x-64\ (x^2+2x-4 \\
 & 3x^4 & x^6 \\
3x^2+2x & \left.\begin{array}{r} 6x^3+\ 4x^2 \\ \hline 3x^4+\ 6x^3+\ 4x^2 \\ \cdot\, 4x^2 \end{array}\right\} & \begin{array}{|l} 6x^5-40x^3 \\ \\ 6x^5+12x^4+8x^3 \\ \hline \end{array} \\
\hline
 & 3x^4+12x^3+12x^2 & \begin{array}{|l} -12x^4-48x^3+96x-64 \end{array} \\
3x^2+6x-4 & \qquad\quad -12x^2-24x+16 & \\
\hline
 & 3x^4+12x^3+\ 0\ \ -24x+16 & \begin{array}{|l} -12x^4-48x^3+96x-64. \end{array}
\end{array}$$

In the above it will be seen that the square of $2x$, viz., $4x^2$, is written in the second column, and the three lines which are braced are added together to give three times the square of x^2+2x, agreeably to the first step for finding the second term of the root, and $3x^2+6x$ is three times the root x^2+2x, which is also in accordance with the former process for finding the second term.

2. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
Ans. x^2-2x+1.

3. Extract the cube root of $64a^6-288a^5+1080a^3-1458a-729$.
Ans. $4a^2-6a-9$.

73. In a similar manner may any root be extracted, but the process is generally laborious, and may often be dispensed with, since, in many instances, the required root is easily found by inspection. Thus the fifth root of $32a^5 - 80a^4 + 80a^3 - 40a^2 + 10a - 1$ is $2a - 1$; because the fifth root of the first term is $2a$, and the fifth root of the last term is -1; hence $2a - 1$ is probably the fifth root of the proposed expression; and if $2a - 1$ be raised to the fifth power, the proposed expression will be produced. The fourth root may also be found by first extracting the square root, and then the square root of the first square root; and the sixth root is found by first extracting the square root, and then the cube root.

THEORY OF INDICES, AND IRRATIONAL OR SURD QUANTITIES.

74. We have seen (39) that powers of the same quantity are multiplied together by adding their indices for the index of the power of that quantity in the product; thus $a^5 \times a^3 = a^{5+3} = a^8$. We may now prove the truth of this principle in a general manner:

Since $a^m = a \times a \times a \times a \times \ \ldots$ to m factors,

and $a^n = a \times a \times a \times a \times \ \ldots$ to n factors;

$\therefore a^m \times a^n = a \times a \times a \times a \times \ \ldots$ to $(m + n)$ factors $= a^{m+n}$ . . (1).

Consequently $a^m \times a^n = a^{m+n}$, when m and n are any positive integers. And it has been shown (69) that

$$(a^m)^n = (a^n)^m = a^{mn} \ \ldots\ldots. (2);$$

hence *the n^{th} power of the m^{th} power of a is equal to the m^{th} power of the n^{th} power of a; and either of them is that power of a whose index is the product of the two indices.*

Also, if $\sqrt[n]{a^m} = x^m$; then raising both sides to the n^{th} power, we get

$$a^m = (x^m)^n = x^{mn} = (x^n)^m;$$

hence we have $\quad a = x^n$, and $\therefore \sqrt[n]{a} = x$;

consequently $(\sqrt[n]{a})^m = x^m$; but by hypothesis, $\sqrt[n]{a^m} = x^m$;

therefore $\quad \sqrt[n]{a^m} = (\sqrt[n]{a})^m \ \ldots\ldots\ldots \ (3)$;

and hence *the n^{th} root of the m^{th} power of a is equal to the m^{th} power of the n^{th} root of a.*

75. When any root of a quantity cannot be exactly obtained it is expressed by using the sign of evolution, and is called an *irrational* or *surd quantity*. Thus $\sqrt{5}$ is an irrational or surd quantity, but $\sqrt{4}$ is not a surd quantity, for though it is in the form of a surd, yet when the root is extracted it is rational, being $= 2$.

When the index of a power of a quantity is a multiple of the index of the root to be extracted, the root is found by dividing the former index by the latter.

Thus the square root of a^6 is $a^{\frac{6}{2}}$ or a^3, and if m is divisible by n, then the n^{th} root of a^m is $a^{\frac{m}{n}}$. But if m is not divisible by n, then the index, instead of being integer, will be fractional; and if quantities with fractional indices are to be treated in the same way as quantities with positive integer indices, we must ascertain what is the meaning of the symbol $a^{\frac{m}{n}}$ in this case.

Since $(a^m)^n = a^{mn}$, in the case of positive integers, then on this supposition we should have

$$\left(a^{\frac{p}{q}}\right)^q = a^{\frac{pq}{q}} = a^p;$$

hence it appears that the symbol $a^{\frac{p}{q}}$ denotes that quantity which when raised to the q^{th} power, becomes equal to a^p. But that quantity whose q^{th} power is equal to a^p, is *the q^{th} root of a^p*, or $\sqrt[q]{a^p}$; hence

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p;$$

and, therefore, in the case of a fractional index, *the numerator denotes the power to which the quantity is to be raised, and the denominator the root to be taken or extracted.* Thus $a^{\frac{1}{2}}$ means the second root of the first power of a, or $\sqrt{a}$; $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$. Also,

$a^{\frac{2}{3}}$ denotes the cube root of the square of a, viz. $(a^2)^{\frac{1}{3}}$;

or $a^{\frac{2}{3}}$ denotes the square of the cube root of a, viz. $\left(a^{\frac{1}{3}}\right)^2$.

In the same manner we see that

$$a^{\frac{1}{2}} = a^{\frac{2}{4}} = a^{\frac{3}{6}} = \text{etc., and } a^{\frac{m}{n}} = a^{\frac{rm}{rn}}.$$

In the division of one power of a quantity by another power of the same quantity, the index of the latter is subtracted from that of the former; thus $a^m \div a^n$ or $\dfrac{a^m}{a^n} = a^{m-n}$, where m is greater than n. But if n be greater than m, then $m - n$ will be a negative quantity, as $-p$; and if quantities with *negative* exponents are to be treated in the same way as quantities with positive integers, we must attach a meaning to the symbol a^{-p}.

Since $a^m \times a^n = a^{m+n}$, where m and n are positive integers; therefore, we should have $a^m \times a^{-p} = a^{m-p}$; but $a^m \times \dfrac{1}{a^p} = \dfrac{a^m}{a^p} = a^{m-p}$;

$$\therefore a^{-p} = \frac{1}{a^p}, \text{ or } a^p = \frac{1}{a^{-p}}.$$

Hence a quantity with a *negative* index denotes the reciprocal of the same quantity with the same *positive* index. By this principle we can remove any power from the numerator of a quantity into the denominator, and *vice versâ*, by changing the sign of its index.

Thus $a^2 b^{-2} = \dfrac{a^2}{b^2} = \dfrac{1}{a^{-2} b^2}$; $\dfrac{1}{(a+x)^2} = a + x)^{-2}$; $a^{-\frac{2}{3}} = \dfrac{1}{a^{\frac{2}{3}}}$, etc.

Lastly, if we take the symbol a^0, and subject it to the same treatment as if the index were an actual number, what would be the meaning of such a symbol?

Since $a^m \times a^n = a^{m+n}$ with integral positive indices;

$\therefore a^m \times a^0 = a^{m+0} = a^m$; but $a^m \times 1 = a^m$; hence $a^0 = 1$.

It follows from these results that

$a^{\frac{1}{2}} \times a^{\frac{3}{5}} = a^{\frac{1}{2}+\frac{3}{5}} = a^{\frac{11}{10}} = a^{1+\frac{1}{10}} = a \times a^{\frac{1}{10}}$;

$a^{\frac{2}{3}} \div a^{\frac{3}{5}} = a^{\frac{2}{3}-\frac{3}{5}} = a^{\frac{10}{15}-\frac{9}{15}} = a^{\frac{1}{15}}$,

$\left(a^{\frac{2}{3}}\right)^{\frac{4}{5}} = a^{\frac{2}{3}\times\frac{4}{5}} = a^{\frac{8}{15}}$, and so on, as in whole numbers.

IRRATIONAL QUANTITIES OR SURDS.

76. An *irrational* or *surd* quantity is one whose root cannot be exactly obtained, and the various transformations and operations connected with this class of quantities will be easily understood from the preceding results.

Similar surds are such as have the same radical sign, or index, and the same quantity under it; thus $\sqrt{2x}$ and $4\sqrt{2x}$, or $(2x)^{\frac{1}{2}}$ and $4(2x)^{\frac{1}{2}}$ are similar surds, but $4a^{\frac{2}{3}}$ and $6a^{\frac{1}{2}}$ are dissimilar surds.

A surd is reduced to its *simplest* form, when the quantity under the radical sign is of an integral form, and contains no factor whose root can be extracted; thus $\sqrt{12a^3} = 2a\sqrt{3a}$.

77. *To reduce surds of different denominations to equivalent ones having either a common or a given index.*

This will be effected by reducing the fractional indices either to a common or a given denominator; thus we have

$$x^{\frac{1}{n}} \text{ and } x^{\frac{1}{m}} = x^{\frac{m}{mn}} \text{ and } x^{\frac{n}{mn}} = (x^m)^{\frac{1}{mn}} \text{ and } (x^n)^{\frac{1}{mn}}.$$

EXAMPLES.

1. Reduce 2, $3^{\frac{1}{2}}$, and $a^{\frac{1}{3}}$ to equivalent surds having a common index, and also to others having the index $\frac{1}{5}$.

Here the indices are 1, $\frac{1}{2}$, and $\frac{1}{3}$, which, reduced to a common denominator, are $\frac{6}{6}$, $\frac{3}{6}$, and $\frac{2}{6}$; hence we get

$$2, 3^{\frac{1}{2}} \text{ and } a^{\frac{1}{3}} = 2^{\frac{6}{6}}, 3^{\frac{3}{6}}, \text{and } a^{\frac{2}{6}} = (2^6)^{\frac{1}{6}}, (3^3)^{\frac{1}{6}}, \text{and } (a^2)^{\frac{1}{6}}$$
$$= 64^{\frac{1}{6}}, 27^{\frac{1}{6}}, \text{ and } (a^2)^{\frac{1}{6}}.$$

Again, $\quad 1, \frac{1}{2}, \text{ and } \frac{1}{3} = \frac{1}{1}, \frac{\frac{1}{2}}{1} \text{ and } \frac{\frac{1}{3}}{1} = \frac{5}{5}, \frac{\frac{5}{2}}{5} \text{ and } \frac{\frac{5}{3}}{5}$; hence

$$2, 3^{\frac{1}{2}} \text{ and } a^{\frac{1}{3}} = 2^{\frac{5}{5}}, 3^{\frac{5}{2}\times\frac{1}{5}}, \text{ and } a^{\frac{5}{3}\times\frac{1}{5}} = (2^5)^{\frac{1}{5}}, \left(3^{\frac{5}{2}}\right)^{\frac{1}{5}}, \text{and } \left(a^{\frac{5}{3}}\right)^{\frac{1}{5}}.$$

2. Reduce $a^{\frac{1}{2}}$, b^2, $c^{\frac{1}{3}}$, and $d^{\frac{2}{3}}$ to equivalent surds having a common index.
Ans. $(a^3)^{\frac{1}{6}}$, $(b^{12})^{\frac{1}{6}}$, $(c^2)^{\frac{1}{6}}$, and $(d^4)^{\frac{1}{6}}$.

3. Express $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, and $(cx)^{\frac{1}{4}}$ as surds, having the simplest common index.
Ans. $(a^6)^{\frac{1}{12}}$, $(b^4)^{\frac{1}{12}}$, and $(c^3x^3)^{\frac{1}{12}}$.

4. Express $(a+x)^2$ and $\sqrt{(a-x)}$ as surds, having the index $\frac{1}{3}$, and also as surds having the simplest common index.
Ans. $\{(a+x)^6\}^{\frac{1}{3}}$ and $\{(a-x)^{\frac{3}{2}}\}^{\frac{1}{3}}$; $\{(a+x)^4\}^{\frac{1}{2}}$ and $(a-x)^{\frac{1}{2}}$.

5. Express $(xy^2)^{\frac{1}{m}}$, $(y^2z^3)^{\frac{1}{n}}$, and $(z^3v^4)^{\frac{1}{p}}$ as surds having the simplest common index.
Ans. $(x^{np}y^{2np})^{\frac{1}{mnp}}$, $(y^{2mp}z^{3mp})^{\frac{1}{mnp}}$, and $(z^{3mn}v^{4mn})^{\frac{1}{mnp}}$.

78. *To reduce surds to their simplest forms.*

It is easy to show that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, and $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

For since the square of $\sqrt{ab} = (a\,b)^{\frac{1}{2}} \times (a\,b)^{\frac{1}{2}} = (a\,b)^{\frac{1}{2}+\frac{1}{2}} = a\,b$, and the square of $\sqrt{a} \times \sqrt{b} = a^{\frac{1}{2}}\,b^{\frac{1}{2}} \times a^{\frac{1}{2}}\,b^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}}\,b^{\frac{1}{2}+\frac{1}{2}} = a\,b$;

$$\therefore \sqrt{ab} = \sqrt{a} \times \sqrt{b}.$$

In a similar manner it may be shown that $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

Also $\sqrt{abcd} = \sqrt{a} \times \sqrt{b} \times \sqrt{c} \times \sqrt{d}$; $\sqrt[3]{abc} = \sqrt[3]{a} \times \sqrt[3]{b} \times \sqrt[3]{c}$.

$$a\,(b^n\,c^n\,d)^{\frac{1}{n}} = a \times (b^n)^{\frac{1}{n}} \times (c^n)^{\frac{1}{n}} \times d^{\frac{1}{n}} = a\,b\,c\,d^{\frac{1}{n}};$$

and $a\left(\dfrac{b^n}{c}\right)^{\frac{1}{n}} = a \times \left(\dfrac{b^n\,c^{n-1}}{c^n}\right)^{\frac{1}{n}} = a \times \dfrac{(b^n)^{\frac{1}{n}} \times (c^{n-1})^{\frac{1}{n}}}{(c^n)^{\frac{1}{n}}} = \dfrac{a\,b}{c} \cdot c^{\frac{n-1}{n}}$.

By means of these principles the reductions in the following examples will readily be comprehended.

Thus $\sqrt{(48\,a^4\,x^2\,y)} = \sqrt{(16\,a^4\,x^2 \times 3\,y)} = \sqrt{(16\,a^4\,x^2)} \times \sqrt{3\,y}$
$= 4\,a^2\,x\sqrt{3\,y}$;

$$\sqrt[3]{\frac{2\,a^4}{3\,x^2}} = \sqrt[3]{\frac{2\,a^4 \times 9\,x}{27\,x^3}} = \sqrt[3]{\left(\frac{a^3}{27\,x^3} \times 18\,a\,x\right)}$$
$$= \sqrt[3]{\frac{a^3}{27\,x^3}} \times \sqrt[3]{18\,a\,x} = \frac{a}{3\,x}\sqrt[3]{18\,a\,x}.$$

$$\sqrt{(3\,a^2 - 6\,a\,x + 3\,x^2)} = \sqrt{(a^2 - 2\,a\,x + x^2) \times 3}$$
$$= \sqrt{(a^2 - 2\,a\,x + x^2)} \times \sqrt{3} = (a - x)\sqrt{3}.$$

EXAMPLES.

1. Reduce $\sqrt{125}$, $3\sqrt{75}$, $4\sqrt[3]{3\frac{3}{8}}$ and $\sqrt[3]{72}$ to their simplest forms.
Ans. $5\sqrt{5}$, $15\sqrt{3}$, 6 and $2\sqrt[3]{9}$.

2. Reduce $\sqrt{\dfrac{50\,a^2}{147\,b^2\,x}}$, $3\sqrt{56\,a^3\,x^3}$, and $\dfrac{1}{3}\sqrt[3]{\dfrac{16}{81}}$ to their simplest forms. *Ans.* $\dfrac{5\,a}{21\,b\,x}\sqrt{6\,x}$, $6\,a\,x\sqrt{14\,a\,x}$, and $\dfrac{2}{27}\sqrt[3]{18}$.

3. Reduce $\sqrt{98\,a^2\,x}$, $\dfrac{4}{7}\sqrt[3]{\dfrac{3}{16}}$, and $\sqrt{(a + x)\;(a^2 - x^2)}$ to their most simple forms. *Ans.* $7\,a\sqrt{2\,x}$, $\dfrac{1}{7}\sqrt[3]{12}$, and $(a + x)\sqrt{(a - x)}$.

4. Reduce $\dfrac{a}{b}\left(\dfrac{x^{2n}\,b^{3n}}{a^n\,y^{2n}}\right)^{\frac{1}{n}}$ and $\left\{\dfrac{(x + y)^m\;(x - y)^{2m}}{(a + b)^{4m}}\right\}^{\frac{1}{2m}}$ to their simplest forms. *Ans.* $\dfrac{b^2\,x^2}{y^2}$ and $\dfrac{x - y}{(a + b)^2}(x + y)^{\frac{1}{2}}$.

5. Reduce $\sqrt[3]{(2\,a^4\,x + a^3\,x^2)}$ and $\sqrt{\left(\dfrac{a^2\,b + a^2\,x}{b^3 - b^2\,x}\right)}$ to their simplest forms. *Ans.* $a\sqrt[3]{(2\,a\,x + x^2)}$ and $\dfrac{a}{b\,(b - x)}\sqrt{(b^2 - x^2)}$.

6. Reduce $\sqrt[3]{\frac{24}{25}}, \sqrt[3]{\frac{5}{72}}, \sqrt[3]{\frac{192}{125}}$, and $\sqrt[3]{\{16\,a^6\,(ax^4 - x^5)\}}$ to their simplest forms.

Ans. $\frac{2}{5}\sqrt[3]{15}$, $\frac{1}{6}\sqrt[3]{15}$, $\frac{4}{5}\sqrt[3]{3}$, and $2\,a^2x\sqrt[3]{2x\,(a-x)}$.

79. *To represent a rational quantity in an irrational or surd form.*

$$\text{Since } a = a^1 = a^{\frac{2}{2}} = a^{\frac{3}{3}} = a^{\frac{m}{m}} = (a^m)^{\frac{1}{m}},$$

we see that a rational quantity may be made to assume the *form* of a surd whenever it may be necessary.

Also a *mixed* surd, or the product of a rational quantity and a surd, may be represented in the form of an entire surd.

Thus $3 = 3^{\frac{2}{2}} = (3^2)^{\frac{1}{2}} = \sqrt{9}$; $3 = 3^{\frac{3}{3}} = (3^3)^{\frac{1}{3}} = \sqrt[3]{27}$;

$3\sqrt[3]{a} = \sqrt[3]{27} \times \sqrt[3]{a} = \sqrt[3]{27\,a}$; $a\sqrt{b} = \sqrt{a^2} \times \sqrt{b} = \sqrt{a^2 b}$.

This operation is only the converse of the transformation in the last article, and the forms just given will indicate the process to be adopted in all cases.

EXAMPLES.

1. Represent $2\sqrt{a}$, $3a\sqrt[3]{b}$ and $\frac{1}{2}\sqrt[3]{2c}$ as entire surds.

Here
$$2\sqrt{a} = \sqrt{4} \times \sqrt{a} = \sqrt{4\,a},$$
$$3a\sqrt[3]{b} = \sqrt[3]{27\,a^3} \times \sqrt[3]{b} = \sqrt[3]{27\,a^3 b},$$
$$\frac{1}{2}\sqrt[3]{2c} = \sqrt[3]{\frac{1}{8}} \times \sqrt[3]{2c} = \sqrt[3]{\frac{c}{4}}.$$

2. Represent $2\sqrt[3]{4}$, $2\sqrt{\frac{a}{2}}$, and $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$ as entire surds.

Ans. $\sqrt[3]{32}$, $\sqrt{2\,a}$, and $\sqrt[3]{\frac{2a}{3}}$.

3. Represent a^2 in the form of the sixth root, and $a + x$ in the form of the square root. *Ans.* $\sqrt[6]{a^{12}}$, and $(a^2 + 2ax + x^2)^{\frac{1}{2}}$.

4. Express $(a + x)\left\{\frac{a-x}{a+x}\right\}^{\frac{1}{2}}$ and $\frac{x+1}{x-1}\left\{\frac{x-1}{x+1}\right\}^{\frac{1}{2}}$ in simple radical forms.

Ans. $(a^2 - x^2)^{\frac{1}{2}}$ and $\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}$.

5. Express $-2x^2$ in the form of the cube root, and also in the form of the fourth root. *Ans.* $(-8x^6)^{\frac{1}{3}}$ and $(16x^8)^{\frac{1}{4}}$.

6. Express $2+\sqrt{3}$ in the form of the square root. *Ans.* $(7+4\sqrt{3})^{\frac{1}{2}}$.

ADDITION AND SUBTRACTION OF SURDS.

80. When the surd part is the same in all the quantities, then add or subtract their multipliers or coefficients in the usual manner, and

prefix the result to the common surd. Thus $a\sqrt{x} + b\sqrt{x} + c\sqrt{x} = (a + b + c)\sqrt{x}$, and $a\sqrt{x} - b\sqrt{x} = (a - b)\sqrt{x}$.

But if the surd parts be different, reduce them (78) to their simplest form; and if the surds are now similar, their sum or difference will be found as before; but if they are still different, the quantities must be connected by the sign + or −, as the question may direct.

Examples.

1. Find the sum of $\sqrt{48}$, $\sqrt{27}$, and $\sqrt{108}$.

Here
$$\begin{aligned}\sqrt{48} &= \sqrt{(16 \times 3)} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3},\\ \sqrt{27} &= \sqrt{(9 \times 3)} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3},\\ \sqrt{108} &= \sqrt{(36 \times 3)} = \sqrt{36} \times \sqrt{3} = 6\sqrt{3},\\ &\therefore \text{sum} = 13\sqrt{3}.\end{aligned}$$

2. Find the difference between $\sqrt{\frac{8}{27}}$ and $\sqrt{\frac{1}{6}}$.

Here,
$$\sqrt{\frac{8}{27}} = \sqrt{\frac{24}{81}} = \sqrt{\frac{4 \times 6}{81}} = \sqrt{\frac{4}{81}} \times \sqrt{6} = \frac{2}{9}\sqrt{6},$$
$$\sqrt{\frac{1}{6}} = \sqrt{\frac{6}{36}} = \sqrt{\frac{1 \times 6}{36}} = \sqrt{\frac{1}{36}} \times \sqrt{6} = \frac{1}{6}\sqrt{6},$$
$$\therefore \text{difference} = \frac{1}{18}\sqrt{6}.$$

Find the value of each of the following expressions:

3. $\sqrt{18} + \sqrt{32} + \sqrt{50} + \sqrt{72}$, $\sqrt{\frac{3}{5}} + \sqrt{\frac{1}{15}}$, and $\sqrt[3]{56} + \sqrt[3]{189}$ *Ans.* $18\sqrt{2}$, $\frac{4}{15}\sqrt{15}$, and $5\sqrt[3]{7}$.

4. $\sqrt{320} - \sqrt{80}$, $\sqrt{75} - \sqrt{48}$, $\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{3}}$, and $\sqrt[3]{128} - \sqrt[3]{54}$. *Ans.* $4\sqrt{5}$, $\sqrt{3}$, $\frac{1}{6}\sqrt{3}$, and $\sqrt[3]{2}$.

5. $2\sqrt{8} + 3\sqrt{50} - 6\sqrt{18}$, and $4a\sqrt[3]{a^3 b^4} + b\sqrt[3]{27a^6 b} - \sqrt[3]{216a^6 b^4}$. *Ans.* $\sqrt{2}$ and $a^2 b\sqrt[3]{b}$.

6. $\sqrt{(ax^2 - 2a^2x + a^3)} + \sqrt{a^3}$, and $\sqrt{(a^3 + 2a^2b + ab^2)} - \sqrt{(a^3 - 2a^2b + ab^2)}$. *Ans.* $x\sqrt{a}$, and $2b\sqrt{a}$.

7. $\sqrt{18a^5b^3} + \sqrt{50a^3b^3}$, and $\frac{2}{3}\sqrt[3]{\frac{2}{9}} - \frac{1}{6}\sqrt[3]{\frac{1}{36}} + \frac{3}{5}\sqrt[3]{\frac{3}{32}}$.

Ans. $(3a^2b + 5ab)\sqrt{2ab}$ and $\frac{31}{90}\sqrt[3]{6}$.

8. $x\sqrt{12a^4x} + 2a^2\sqrt{27x^3} - 3a\sqrt{48a^2x^3} + 5a^2x\sqrt{3x}$.

Ans. $a^2x\sqrt{3x}$.

9. $\sqrt[3]{(54a^{n+6}b^3)} - a\sqrt[3]{(16a^{n-3}b^6)} + \sqrt[3]{(2a^{4n+9})} + \sqrt[3]{(2c^3a^n)}$.

Ans. $(3a^2b - 2b^2 + a^{n+3} + c)\sqrt[3]{2a^n}$.

MULTIPLICATION AND DIVISION OF SURDS.

81. These operations are performed as in rational quantities, by means of the following formulas:

$$a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}, \text{ and } a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}};$$

$$a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}, \text{ and } a^{\frac{1}{n}} \div b^{\frac{1}{n}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

If the surds are of different denominations, reduce them to equivalent ones having a common index, and find their product and quotient by the preceding processes.

Examples.

1. Find the product of $3\sqrt{8}$, $2\sqrt{6}$, and $4\sqrt{48}$.

Here $3\sqrt{8} \times 2\sqrt{6} \times 4\sqrt{48} = 3 \times 2 \times 4 \times \sqrt{(8 \times 6 \times 48)}$
$= 24\sqrt{48^2} = 24 \times 48 = 1152.$

2. Divide $2\frac{1}{3}\sqrt[3]{\frac{2}{3}}$ by $3\frac{1}{4}\sqrt[3]{\frac{3}{4}}$, and also $\sqrt{a}$ by $\sqrt[3]{a}$.

Here

$$2\frac{1}{3}\sqrt[3]{\frac{2}{3}} \div 3\frac{1}{4}\sqrt[3]{\frac{3}{4}} = \frac{7}{3} \times \frac{4}{13} \times \sqrt[3]{\left(\frac{2}{3} \times \frac{4}{3}\right)} = \frac{28}{39}\sqrt[3]{\frac{8}{9}}$$
$$= \frac{28}{39}\sqrt[3]{\frac{24}{27}} = \frac{28}{39}\sqrt[3]{\left(\frac{8}{27} \times 3\right)}$$
$$= \frac{28}{39} \times \frac{2}{3} \times \sqrt[3]{3} = \frac{56}{117}\sqrt[3]{3}.$$

And $a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{2}-\frac{1}{3}} = a^{\frac{3}{6}-\frac{2}{6}} = a^{\frac{1}{6}}$.

3. Multiply $3\sqrt{2} - 2\sqrt{3}$ by $2\sqrt{2} + \sqrt{3}$, and divide $\sqrt{5} + \sqrt{3}$ by $\sqrt{5} - \sqrt{3}$.

$$\begin{array}{r} 3\sqrt{2} - 2\sqrt{3} \\ 2\sqrt{2} + \sqrt{3} \\ \hline 12 - 4\sqrt{6} \\ + 3\sqrt{6} - 6 \\ \hline \therefore \text{ product } = \quad 6 - \sqrt{6}. \end{array}$$

Also $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \dfrac{8 + 2\sqrt{15}}{5-3}.$
$= 4 + \sqrt{15}.$

Here we have multiplied the terms of the fraction $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ by $\sqrt{5}+\sqrt{3}$, because the product of $\sqrt{5}-\sqrt{3}$ and $\sqrt{5}+\sqrt{3}$ is $\sqrt{5^2} - \sqrt{3^2}$ or $5 - 3$, a rational quantity, and the quotient is obtained in the neatest possible form.

4. Multiply $4\sqrt{12}$ by $3\sqrt{2}$, and also $\frac{1}{3}\sqrt[3]{4}$ by $\frac{3}{4}\sqrt[3]{12}$.

Ans. $24\sqrt{6}$ and $\frac{1}{2}\sqrt[3]{6}$.

5. Multiply $2\sqrt[3]{14}$ by $3\sqrt[3]{4}$, and divide $6\sqrt{96}$ by $3\sqrt{8}$.

Ans. $12\sqrt[3]{7}$ and 4 3.

6. Multiply $2a^{\frac{2}{3}}$ by $a^{\frac{4}{3}}$, and divide $\frac{4}{5}a^{\frac{1}{2}}$ by $\frac{2}{3}a^{\frac{1}{3}}$. *Ans.* $2a^2$ and $\frac{6}{5}a^{\frac{1}{6}}$.

7. Multiply $\sqrt{2ab^3}$ by $\sqrt{8a^3b}$, and divide $5a\sqrt{ax}$ by $\frac{5}{2}\sqrt{bx}$.

Ans. $4a^2b^2$, and $\frac{2a}{b}\sqrt{ab}$.

8. Multiply together ab, $a\sqrt{(x+2\sqrt{a})}$ and $b\sqrt{(x-2\sqrt{a})}$.
Ans. $a^2b^2\sqrt{(x^2-4a)}$.

9. Multiply together $3+\sqrt{5}$, $\sqrt{2}+\sqrt{3}$, $\sqrt{2}-\sqrt{3}$ and $3-\sqrt{5}$.
Ans. -4.

10. Multiply $(a+b)^{\frac{1}{n}}$ by $(a+b)^{\frac{1}{m}}$, and divide $4x\sqrt{a}$ by $\frac{4}{7}\sqrt[3]{ax}$.

Ans. $(a+b)^{\frac{m+n}{mn}}$ and $7a^{\frac{1}{6}}x^{\frac{2}{3}}$.

11. Multiply $x^{\frac{1}{3}}+2x^{\frac{1}{6}}-4$ by $x^{\frac{1}{3}}-2x^{\frac{1}{6}}+4$, and divide $x^{\frac{2}{3}}+2x^{\frac{1}{3}}+9$ by $x^{\frac{1}{3}}+2x^{\frac{1}{6}}+3$.

Ans. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}+16x^{\frac{1}{6}}-16$ and $x^{\frac{1}{3}}-2x^{\frac{1}{6}}+3$.

12. Multiply $a^{\frac{3}{2}}+a^{\frac{1}{2}}x^{\frac{1}{2}}+a^{\frac{1}{2}}x+x^{\frac{3}{2}}$ by $a^{\frac{1}{2}}-x^{\frac{1}{2}}$, and divide $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$. *Ans.* $a-x^2$, and $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

13. Divide $(x^4-3x^3+2x^2)^{\frac{1}{2}}$ by $x^{\frac{1}{2}}$, and $6x(x-y)y^{\frac{1}{2}}-x^{\frac{3}{2}}y\sqrt{6}$ y $2x\sqrt{3}-3x^{\frac{1}{2}}y^{\frac{1}{2}}\sqrt{2}$. *Ans.* $(x^2-3x+2)^{\frac{1}{2}}$ and $xy^{\frac{1}{2}}\sqrt{3}+yx^{\frac{1}{2}}\sqrt{2}$.

14. Multiply $\frac{1}{2}a^{\frac{1}{2}}-\frac{1}{3}a^{\frac{1}{3}}+\frac{1}{4}a^{\frac{1}{4}}$ by $a^{\frac{1}{4}}-\frac{3}{5}a^{\frac{1}{5}}$.

Ans. $\frac{1}{2}a^{\frac{3}{4}}-\frac{3}{10}a^{\frac{7}{10}}-\frac{1}{3}a^{\frac{7}{12}}+\frac{1}{5}a^{\frac{8}{15}}+\frac{1}{4}a^{\frac{1}{2}}-\frac{3}{20}a^{\frac{9}{20}}$.

15. Divide $16x-\frac{y^4}{16}$ by $2x^{\frac{1}{4}}-\frac{y}{2}$, and a^2-b by $a^{\frac{1}{3}}-b^{\frac{1}{6}}$.

Ans. $8x^{\frac{3}{4}}+2x^{\frac{1}{2}}y+\frac{1}{2}x^{\frac{1}{4}}y^2+\frac{1}{8}y^3$,

and $a^{\frac{5}{3}}+a^{\frac{4}{3}}b^{\frac{1}{6}}+ab^{\frac{1}{3}}+a^{\frac{2}{3}}b^{\frac{1}{2}}+a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{5}{6}}$.

INVOLUTION AND EVOLUTION OF SURDS.

82. Since the $\left(\frac{p}{q}\right)^{\text{th}}$ power of $a^{\frac{m}{n}}$ is $a^{\frac{m}{n}\times\frac{p}{q}} = a^{\frac{mp}{nq}}$, and the $\left(\frac{p}{q}\right)^{\text{th}}$ root of $a^{\frac{m}{n}} = a^{\frac{m}{n}\div\frac{p}{q}} = a^{\frac{mq}{np}}$; therefore the operations of involution and evolution are expressed as in rational quantities, by the multiplication and division of indices.

EXAMPLES.

1. Find the square of $\frac{2}{3}a^{\frac{1}{3}}$ and the cube of $17\sqrt{21}$.

Here $$\left(\frac{2}{3}a^{\frac{1}{3}}\right)^2 = \frac{2}{3}\times\frac{2}{3}\times a^{\frac{1}{3}\times 2} = \frac{4}{9}a^{\frac{2}{3}},$$

and $(17\sqrt{21})^3 = 17^3 \times 21^{\frac{1}{2}\times 3} = 17^3 \times 21^{\frac{3}{2}} = 17^3 \times 21 \times 21^{\frac{1}{2}}$,

$$\therefore (17\sqrt{21})^3 = 103173\sqrt{21}.$$

2. Find the square root of 10^3 and the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.

Here $(10^3)^{\frac{1}{2}} = 10^{3\times\frac{1}{2}} = 10^{\frac{3}{2}} = 10 \times 10^{\frac{1}{2}} = 10\sqrt{10}$,

and $\left(\frac{a}{3}\sqrt{\frac{a}{3}}\right)^{\frac{1}{3}} = \left(\sqrt{\frac{a^3}{3^3}}\right)^{\frac{1}{3}} = \left(\frac{a^3}{3^3}\right)^{\frac{1}{2}\times\frac{1}{3}} = \left(\sqrt[3]{\frac{a^3}{3^3}}\right)^{\frac{1}{2}} = \left(\frac{a}{3}\right)^{\frac{1}{2}}$

$$= \frac{1}{3}\sqrt{3a}.$$

3. Find the square of $\frac{1}{2}a^{\frac{2}{3}}$, and the cube root of $\frac{1}{4}\sqrt{2}$.

Ans. $\frac{1}{4}a^{\frac{4}{3}}$ and $\frac{1}{2}\sqrt{2}$.

4. Find the cube of $\frac{1}{3}\sqrt{3}$, and the cube roots of $\frac{1}{27}\sqrt{3}$ and $\frac{1}{8}\sqrt{2ab^2}$.

Ans. $\frac{1}{9}\sqrt{3}$, $\frac{1}{3}\sqrt[6]{3}$, and $\frac{1}{2}\sqrt[6]{2ab^2}$.

5. Find the square root of $16a^{\frac{2}{3}}x^{2n}$ and the cube root of $-27a^{\frac{3}{2}}x^{-6n}$.

Ans. $4a^{\frac{1}{3}}x^n$ and $-3a^{\frac{1}{2}}x^{-2n}$.

6. Find the fourth power of $2 + \sqrt{3}$ and the square root of $a^2 - 6a\sqrt{b} + 9b$. *Ans.* $97 + 56\sqrt{3}$ and $a - 3\sqrt{b}$.

7. Extract the square root of $1 + \frac{41}{16}a - \frac{3+3a}{2}\sqrt{a} + a^2$.

Ans. $a - \frac{3}{4}\sqrt{a} + 1$.

8. Extract the square root of $\frac{9a^3}{4} - 5a^{\frac{5}{2}}b^{\frac{1}{2}} + \frac{179a^2b}{45} - \frac{4a^{\frac{3}{2}}b^{\frac{3}{2}}}{3} + \frac{4ab^2}{25}$.

Ans. $\frac{3a^{\frac{3}{2}}}{2} + \frac{2a^{\frac{1}{2}}b}{5} - \frac{5ab^{\frac{1}{2}}}{3}$.

9. Extract the cube root of $a^3 - 3a^2b^{\frac{1}{2}} + 3ab - b^{\frac{3}{2}}$. *Ans.* $a - b^{\frac{1}{2}}$.

EQUATIONS.

83. An *equation* is the expression of the equality of two different algebraical quantities, one or both of which contain some power or powers of the unknown quantity. Thus, $3x + 4 = 13$, $ax^2 + bx = c$, and $x^3 + ax^2 = bx + c$, are equations.

The *members* or *sides* of an equation are the two quantities between which the sign of equality is placed; thus in the equation $x - a = b - 2x$, the first side or member is $x - a$, and the second side or member is $b - 2x$.

A *simple equation* is one which contains the first power only of an unknown quantity, as $2x + 5 = 3x - 2$.

A *quadratic equation* is one which contains the second power, or both the first and second powers of the unknown quantity, as $x^2 = 25$, or $x^2 + 10x = 12$.

An equation is said to be of the *first*, *second*, *third*, or n^{th} degree, according as the highest power of the unknown quantity contained in it is the *first*, *second*, *third*, or n^{th} power of that quantity; thus,

$$x^n + a x^{n-1} + b x^{n-2} + \dots\dots + h x = k$$

is an equation of the n^{th} degree, or of n dimensions.

A *root of an equation* is that quantity which, when substituted for the unknown quantity in it, makes the values of the two sides of the equation the *same*, and thus verifies the equation.

A simple equation has one root only; for if the equation $2x + 3a = 4b$ can have two roots, as r_1 and r_2, then by substituting these for the unknown quantity x, we have,

$$2r_1 + 3a = 4b,$$
$$2r_2 + 3a = 4b.$$

Subtracting the latter from the former, we get,

$$2r_1 - 2r_2 = 0, \text{ or } r_1 - r_2 = 0, \therefore r_1 = r_2;$$

and the simple equation has consequently only one root.

An *identical equation* is one which can be verified by all values of the quantities contained in it; thus $(x + y)^2 = x^2 + 2xy + y^2$ is an identical equation, for if any values be given to x and y, the equation will be verified.

The *solution of an equation* is the method of separating the unknown quantity from the other known quantities in the equation, so that the unknown quantity may form one side of the equation, and a known quantity, or a combination of the known quantities, the other side.

SIMPLE EQUATIONS.

84. The unknown quantity is usually so involved in the different terms of an equation as to require several operations before the solution can be effected, and the value of the unknown quantity determined. The solution of all equations depends on the following axioms:—

1. If equals, or the same be added to equals, the sums are equal.
2. If equals, or the same be taken from equals, the remainders are equal.
3. If equals be multiplied by the same, or by equals, the products are equal.
4. If equals be divided by the same, or by equals, the quotients are equal.
5. If equals be raised to the same power, the powers are equal.
6. If equals have the same root extracted, the roots are equal.

The following remarks, which are founded on the preceding axioms, apply to all classes of equations.

85. *A quantity may be transposed from one side of an equation to the other by changing its sign from + to —, or from — to +, without destroying the equality.*

Thus if $x + 3 = 17$, then subtracting 3 from both sides, we have, by axiom 2, $x = 17 - 3 = 14$, where the $+ 3$ has been transferred from the one side to the other with its sign changed to $-$. Hence also if $x - a = b + c$, then adding $a - c$ to both sides, we have (axiom 1),

$$x - a + a - c = b + c + a - c, \text{ or } x - c = b + a;$$

where the $- a$ and $+ c$ have changed sides, and changed their signs.

If the signs of *all* the terms of both sides of an equation be changed, the equality will still remain, for every term has been transposed, and the sides afterwards interchanged.

Thus, if $16 - x = 2x - 11$, then $-2x + 11 = -16 + x$, or changing sides, we have $-16 + x = -2x + 11$.

Hence also if a quantity be found in both sides of an equation with the same sign, it may be removed from each side.

Thus if $x + c = a + c$, then will $x = a$.

86. *Every term of each side of an equation may be multiplied or divided by the same quantity, without affecting the equality.*

Thus if $\frac{x}{5} + 7 = 9$, then multiplying every term by 5, we get

$$x + 35 = 45, \text{ and transposing } x = 45 - 35 = 10.$$

And if $5x + 10 = 15$, then dividing every term by 5, we get

$$x + 2 = 3, \text{ and transposing } x = 3 - 2 = 1.$$

Let $\frac{x}{a} = b + c$, then multiplying every term by a, we get

$$x = ab + ac, \text{ or } x = a(b + c).$$

And if $ax = b + c$, then dividing every term by a, we have

$$x = \frac{b + c}{a}.$$

87. *An equation may be cleared of fractions by multiplying first by one of the denominators, then by another, and so on, until all the denominators have been taken away; or multiplying every term by the least common multiple of all the denominators.*

Thus if the equation be $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} = 30$; then

multiplying by 2, gives $x + \frac{2x}{3} + \frac{x}{2} + \frac{x}{3} = 60$.

Multiplying by 3, we have $3x + 2x + \frac{3x}{2} + x = 180$; and again multiplying by 2, $6x + 4x + 3x + 2x = 360$; hence $15x = 360$, and, dividing by 15, gives $x = 24$. But if we find the least common multiple of the denominators 2, 3, 4, and 6, viz., 12, and multiply every term of the equation by it, we get at once

$$6x + 4x + 3x + 2x = 360;$$

hence $15x = 360$, and $x = \frac{360}{15} = 24$, as before.

Again, if the equation be $x - \frac{2x - 3}{7} = 4$, then multiplying by 7, we get $7x - (2x - 3) = 28$, or removing the bracket, and changing the signs of $2x$ and 3 into $-$ and $+$, we get

$$7x - 2x + 3 = 28, \text{ or } 5x = 25; \text{ hence } x = 5.$$

88. The student should be particularly careful in equations of this kind, where a fraction is to be subtracted, whose numerator consists of two or more terms; because the line which separates the numerator from the denominator is a species of vinculum, and when that is removed by the removal of the denominator, the whole of the numerator must be subtracted, and the sign of each term must be changed in accordance with the principle of subtraction.

Hence also if every term of an equation be either multiplied or

divided by the same quantity, that common quantity may be removed without affecting the equality.

Thus, if $a x = a b + a c$, then will $x = b + c$,

or, if $\frac{x}{a} = \frac{b}{a} + \frac{c}{a}$, then will $x = b + c$.

89. *If the unknown quantity be either a surd, or appear in one or more terms of a surd expression, transpose the terms so that the surd quantity may stand alone on one side of the equation, and the remaining terms on the other; then raise both sides of the equation to the power denoted by the index of the surd, and the equality will still subsist.*

Thus if $\sqrt{(x+21)} = 3 + \sqrt{x}$, then, by squaring both members, we get

$$(x+21)^{\frac{1}{2}\times 2} = (3+\sqrt{x})^2, \text{ or } x + 21 = 9 + 6\sqrt{x} + x;$$

whence by transposition (85) we get

$$6\sqrt{x} = 21 - 9 = 12, \text{ or } \sqrt{x} = 2,$$

consequently by squaring we have $(\sqrt{x})^2 = 2^2$, or $x = 4$.

And if $\sqrt[3]{(x+23)} + 9 = 12$ be the proposed equation, then by transposition,

$$\sqrt[3]{(x+23)} = 12 - 9 = 3.$$

And, raising both sides to the third power, we get

$$(x+23)^{\frac{1}{3}\times 3} = 3^3, \text{ or } x + 23 = 27; \text{ hence } x = 4.$$

But if the equation contain more than one surd expression, let that which is most involved stand alone on one side of the equation, by transposing all the terms to the other side; then raise each side to the power corresponding to the index of the root quantity, which stands alone. Proceed in a similar manner with the remaining root quantities or surds until they are all removed.

Thus, if $\sqrt{(x+a)} + \sqrt{x} = b$, then by transposition we get

$$\sqrt{(x+a)} = b - \sqrt{x},$$

and, squaring both sides, $x + a = b^2 - 2b\sqrt{x} + x$;

transposing, $2b\sqrt{x} = b^2 - a$, and dividing by $2b$, $\sqrt{x} = \frac{b^2 - a}{2b}$;

and squaring, we get finally $x = \left(\frac{b^2 - a}{2b}\right)^2$.

If we had squared both members as they originally stood, we should have had

$$x + a + 2\sqrt{x}\sqrt{(x+a)} + x = b^2,$$
$$\text{or } 2x + a + 2\sqrt{(x^2 + ax)} = b^2;$$

hence, transposing, $2\sqrt{(x^2 + ax)} = b^2 - a - 2x = (b^2 - a) - 2x$,

and squaring, $4x^2 + 4ax = (b^2 - a)^2 - 4x(b^2 - a) + 4x^2$

$$= (b^2 - a)^2 - 4b^2x + 4ax + 4x^2;$$

whence, cancelling $4x^2$ and $4ax$ from both sides, we get by transposition

$$4b^2x = (b^2 - a)^2, \text{ or } x = \frac{(b^2 - a)^2}{4b^2}, \text{ as before.}$$

But this method is not so commodious as the other, for the transposition of one of the surds simplifies the solution, inasmuch as each member then involves only one surd, while in the original form both surds are on the same side.

90. *If the side of the equation which contains the unknown quantity be a complete power, the equation may be reduced to one of a lower degree by extracting the root of the power, and the same root of the other member of the equation.*

Thus if $x^2 = 16\,a^2$, then, extracting the square root, $x = \pm\,4\,a$;
if $8\,x^3 = 27\,a^3$, then, extracting the cube root, $2\,x = \quad 3\,a$;
and if $x^2 + 6\,x + 9 = 25$, then extracting the square root, we get
$x + 3 = 5$, and $x = 5 - 3 = 2$.

91. *A proportion containing the unknown quantity in any of its terms may be changed into an equation, by multiplying the two extremes together, and also the two means, and putting the products equal to one another.*

Thus if $x : y :: 3 : 11$; then since the ratio of $x : y$ is the same as the ratio of $3 : 11$, we have $\frac{x}{y} = \frac{3}{11}$, and clearing this equation of fractions, gives $11 \times x = 3 \times y$, or $11\,x = 3\,y$, where $11\,x$ is the product of the extreme terms, and $3\,y$ is the product of the means.

In a similar manner, if $12 - x : \frac{2}{3}\,x :: 3 : 1$, then we have $12 - x = \frac{2}{3}\,x \times 3 = 2\,x$; hence, by transposition, $3\,x = 12$, and $x = 4$.

SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

92. In the solution of simple equations, the first step is to clear the equation of fractions aud surds, if it contain them; then transpose all the terms which contain the unknown quantity to one side of the equation, and the known quantities to the other, and divide both sides by the coefficient of the unknown quantity; then will the unknown quantity stand alone on one side and its value on the other.

EXAMPLES.

1. Given $9\,x - 7 = 4\,x + 23$ to find the value of x.
By transposition, $9\,x - 4\,x = 23 + 7$, or $5\,x = 30$;
hence, dividing by 5, we get $x = 6$.

2. Let $\frac{x+1}{2} + \frac{x+2}{3} = 15 + \frac{1-x}{4}$, to find the value of x.

The least common multiple of the denominators 2, 3, 4, is 12; hence multiplying each term by 12 we get

$$6\,x + 6 + 4\,x + 8 = 180 + 3 - 3\,x,$$

and transposing, $$(6 + 4 + 3)\,x = 180 + 3 - 6 - 8,$$

or $$13\,x = 169, \quad \therefore\ x = \frac{169}{13} = 13.$$

We might have proceeded in the following manner. Since 4 is a multiple of 2, we may first multiply every term by 4, which will remove two of the denominators; hence we get

$$2\,x + 2 + \frac{4\,x + 8}{3} = 60 + 1 - x,$$

and transposing, $3\,x + \frac{4\,x + 8}{3} = 60 + 1 - 2 = 59.$

Now multiply by 3, and we get $9x + 4x + 8 = 177$;
hence $13x = 177 - 8 = 169$, and $x = 13$, as before.

3. Let $\dfrac{6x + 13}{15} - \dfrac{3x + 5}{5(x - 5)} = \dfrac{2x}{5}$, to find the value of x.

Here it will be advantageous to remove first the simple factor 5, in the denominator of the second fraction. Multiply every term by 15, then

$$6x + 13 - \frac{9x + 15}{x - 5} = 6x, \text{ or } 13 = \frac{9x + 15}{x - 5}.$$

Multiplying now by $x - 5$, gives $13x - 65 = 9x + 15$;
hence $13x - 9x = 65 + 15$, or $4x = 80$; $\therefore x = 20$.

4. Given $\dfrac{x - 3}{2} + \dfrac{x}{3} = 20 - \dfrac{x - 19}{2}$, to find the value of x.

Multiply first by 2, then $x - 3 + \dfrac{2x}{3} = 40 - x + 19$,

and transposing, $2x + \dfrac{2x}{3} = 40 + 19 + 3 = 62$.

Multiply now by 3, then $6x + 2x = 186$, or $8x = 186$;

hence $x = \dfrac{186}{8} = \dfrac{93}{4} = 23\frac{1}{4}$.

5. Let $\dfrac{ax - b}{4} + \dfrac{a}{3} = \dfrac{bx}{2} - \dfrac{bx - a}{3}$, to find the value of x.

Multiply first by 3, then $\dfrac{3ax - 3b}{4} + a = \dfrac{3bx}{2} - bx + a$,

or $$\frac{3ax - 3b}{4} = \frac{3bx}{2} - bx.$$

Multiplying by 4, gives $3ax - 3b = 6bx - 4bx = 2bx$;

hence $(3a - 2b)x = 3b$, or $x = \dfrac{3b}{3a - 2b}$.

6. Given $\sqrt{x} + \sqrt{(x + 7)} = \dfrac{28}{\sqrt{(x + 7)}}$, to find the value of x.

Clearing of fractions, by multiplying every term by $\sqrt{(x + 7)}$, we get $\sqrt{(x^2 + 7x)} + x + 7 = 28$, or $\sqrt{(x^2 + 7x)} = 21 - x$;
and squaring both sides, we get $x^2 + 7x = 441 - 42x + x^2$;
hence $7x + 42x = 441$, or $49x = 441$; $\therefore x = 9$.

7. Let $\dfrac{x - 9}{\sqrt{x} + 3} + \dfrac{x - 4}{\sqrt{x} - 2} = \dfrac{4(x - 16)}{\sqrt{x} + 4}$, to find x.

Since,

$x - 9 = (\sqrt{x} + 3)(\sqrt{x} - 3)$, it is easy to see that $\dfrac{x - 9}{\sqrt{x} + 3} = \sqrt{x} - 3$.

In a similar manner, $\dfrac{x - 4}{\sqrt{x} - 2} = \sqrt{x} + 2$, and $\dfrac{x - 16}{\sqrt{x} + 4} = \sqrt{x} - 4$;

hence the proposed equation is reduced to

$$\sqrt{x} - 3 + \sqrt{x} + 2 = 4(\sqrt{x} - 4) = 4\sqrt{x} - 16;$$
$$\therefore 16 - 3 + 2 = 4\sqrt{x} - 2\sqrt{x}, \text{ or } 15 = 2\sqrt{x};$$
$$\therefore \sqrt{x} = \frac{15}{2}, \text{ and } x = \left(\frac{15}{2}\right)^2 = \frac{225}{4} = 56\tfrac{1}{4}.$$

8. Given $\frac{a+c}{a+x} + \frac{a-c}{a-x} = \frac{2b^2}{a^2-x^2}$, to find x.

Here the least common multiple of the denominators is $a^2 - x^2$, since $a^2 - x^2$ is divisible by both $a + x$ and $a - x$; hence multiplying every term by $a^2 - x^2$, we get

$$(a+c)(a-x) + (a-c)(a+x) = 2b^2,$$

and effecting the multiplications indicated, we have

$$a^2 + ac - ax - cx + a^2 - ac + ax - cx = 2b^2.$$

Hence $\quad 2a^2 - 2cx = 2b^2$, or $a^2 - cx = b^2$;

therefore $\quad a^2 - b^2 = cx$, and $x = \frac{a^2 - b^2}{c}$.

Examples for Practice.

1. $7x - 18 = 4x + 6$. *Ans.* $x = 8$.
2. $92 - 10x = 8 - 4x$. *Ans.* $x = 14$.
3. $7x + 20 - 3x = 60 + 4x - 88 + 8x$. *Ans.* $x = 6$.
4. $5(x+1) - 6(x-2) = 8(x-5) - 6$. *Ans.* $x = 7$.
5. $20 - 4(x-2) = 5(x+2)$. *Ans.* $x = 2$.
6. $\frac{x}{2} - 2 = \frac{x}{3} + 3$. *Ans.* $x. = 30$.
7. $\frac{x-1}{2} + \frac{x-2}{3} - \frac{x-3}{4} = 6$. *Ans.* $x = 11$.
8. $\frac{x}{3} - \frac{x}{4} - \frac{1}{2} = \frac{x}{5} - \frac{x}{6}$. *Ans.* $x = 10$.
9. $\frac{3x-1}{7} + \frac{6-x}{4} - \frac{2x-4}{12} = 2 - \frac{x+2}{28}$. *Ans.* $x = 5$.
10. $\frac{2}{x+1} + \frac{5}{2x+2} + \frac{6(x-1)}{x^2-1} = \frac{21}{8}$. *Ans.* $x = 3$.
11. $\frac{5x-7}{3} - \frac{3x-2}{7} = \frac{x-5}{4}$. *Ans.* $x = \frac{67}{83}$.
12. $\frac{x}{8} - \frac{2(x-1)}{5} = \frac{3x-4}{15} + \frac{x}{12}$. *Ans.* $x = 1\frac{13}{67}$.
13. $\frac{x-a}{3} - \frac{2x-3b}{5} - \frac{a-x}{2} = 10a + 11b$. *Ans.* $x = 25a + 24b$.
14. $\frac{6x+a}{4x+b} = \frac{3x-b}{2x-a}$. *Ans.* $x = \frac{a^2-b^2}{b-4a}$.
15. $\sqrt{(a+x)} - \sqrt{(a-x)} = \sqrt{ax}$. *Ans.* $x = \frac{4a^2}{a^2+4}$.
16. $\sqrt{(4a+x)} = 2\sqrt{(b+x)} - \sqrt{x}$. *Ans.* $x = \frac{(a-b)^2}{2a-b}$.
17. $\sqrt{(4a+x)} + \sqrt{(a+x)} = 2\sqrt{(x-2a)}$. *Ans.* $x = \frac{17a}{8}$.
18. $\frac{x-1}{\sqrt{x}+1} = 4 + \frac{\sqrt{x}-1}{2}$. *Ans.* $x = 81$.
19. $\frac{1+x^3}{(1+x)^3} + \frac{1-x^3}{(1-x)^3} = a$. *Ans.* $x = \left(\frac{a-2}{a+4}\right)^{\frac{1}{2}}$.

20. $a+x+\sqrt{(a^2+bx+x^2)}=b.$ *Ans.* $x=\dfrac{b^2-2ab}{3b-2a}.$

21. $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=b.$ *Ans.* $x=\dfrac{2ab}{1+b^2}.$

22. $\sqrt{(x+a)}+\sqrt{(x-a)}=\dfrac{b}{\sqrt{(x+a)}}.$ *Ans.* $x=\dfrac{a^2+(a-b)^2}{2(b-a)}.$

23. $a+x=\sqrt{\{a^2+x\sqrt{(4b^2+x^2)}\}}.$ *Ans.* $x=\dfrac{b^2-a^2}{a}.$

24. $\sqrt{(a^2-x^2)}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}.$ *Ans.* $x=\left(\dfrac{a^2-1}{a^2+3}\right)^{\frac{1}{2}}.$

25. $(a+x)^{\frac{1}{m}}=(x^2+nax+b^2)^{\frac{1}{2m}}.$ *Ans.* $x=\dfrac{a^2-b^2}{(n-2)a}.$

26. $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b.$ *Ans.* $x=\left\{a^2-\left(\dfrac{b^3-2a}{3b}\right)^3\right\}^{\frac{1}{2}}.$

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

93. If we have an equation of the first degree containing *two* unknown quantities, the value of one of the unknown quantities may always be expressed in terms of the other and known quantities; thus if $ax+by=c$, then $ax=c-by$, and $x=\dfrac{c-by}{a}$. Now, if the value of y is unknown or arbitrary, the value of x will likewise be unknown or arbitrary; and from this equation alone we cannot determine the value of x. But if we have another equation containing the same unknown quantities, as $mx+ny=p$; then, as before, we have $x=\dfrac{p-ny}{m}$; and if the values of x and y are the same in both equations, then we shall have the equation

$$\frac{c-by}{a}=\frac{p-ny}{m}, \text{ or } m\,(c-by)=a(p-ny),$$

to determine the value of y; and then, by substitution, we may obtain the value of x. Such equations are called *simultaneous equations*, because the same values of x and y fulfil both the conditions expressed by these equations. Hence, if the values of two unknown quantities are to be determined, we must have two independent and consistent equations; and, by some process we must *eliminate* one of the unknown quantities, and the resulting equation will involve the other unknown, whose value may be found by the methods already employed for the solution of a simple equation containing only one unknown quantity.

The methods usually employed for this purpose are those of *comparison*, *substitution*, *equalizing of coefficients*, and *an arbitrary multiplier.*

First Method.

94. Find the value of either of the unknown quantities from each equation, in terms of the other and known quantities, by the rules already explained: put these two values equal to each other, and there

will be formed an equation involving only one unknown quantity, whose value can be found in the usual manner. This is the *method of comparison.*

Let it be required to determine the values of x and y from the equations

$$2x + 3y = 28 \ldots. (1), \text{and } 5x - 2y = 13 \ldots. (2).$$

From eq. (1), $x = \dfrac{28 - 3y}{2}$, and from eq. (2), $x = \dfrac{13 + 2y}{5}$;

hence we have $\dfrac{13 + 2y}{5} = \dfrac{28 - 3y}{2} \ldots\ldots (3).$

Clearing of fractions, by multiplying by 10, we get

$$26 + 4y = 140 - 15y,$$

and transposing $4y + 15y = 140 - 26$, or $19y = 114$;

hence $y = \dfrac{114}{19} = 6$, and $x = \dfrac{13 + 2y}{5} = \dfrac{13 + 12}{5} = 5.$

We might have first found the value of y from each equation; thus from eq. (1), $y = \dfrac{28 - 2x}{3}$, and from eq. (2) $y = \dfrac{5x - 13}{2}$;

$$\therefore \frac{5x - 13}{2} = \frac{28 - 2x}{3}, \text{ or } 15x - 39 = 56 - 4x;$$

hence $15x + 4x = 56 + 39$, or $19x = 95$; $\therefore x = 5$,

and $y = \dfrac{28 - 2x}{3} = \dfrac{28 - 10}{3} = \dfrac{18}{3} = 6$, as before.

Second Method.

95. Find the value of either of the unknown quantities from one equation in terms of the other and known quantities; substitute this value for it in the other equation, and there will arise a new equation involving only one unknown quantity, whose value may be found in the usual manner. This is the *method of substitution.*

Let it be required to determine the values of x and y from the equations

$$5x - 3y = 9 \ldots. (1), \text{ and } 2x + 5y = 16 \ldots. (2).$$

As the coefficient of x is least in eq. (2), we may find the value of x from that equation, viz., $x = \dfrac{16 - 5y}{2}$; and, if this value be substituted for x in the first equation, we get

$$5 \times \frac{16 - 5y}{2} - 3y = 9; \text{ or } \frac{80 - 25y}{2} - 3y = 9.$$

Cleairng of fractions, gives $80 - 25y - 6y = 18$,

and transposing $80 - 18 = 25y + 6y$, or $31y = 62$;

hence $y = \dfrac{62}{31} = 2$, and $x = \dfrac{16 - 5y}{2} = \dfrac{16 - 10}{2} = 3.$

Third Method.

96. Multiply or divide the two equations by such quantities that the coefficients of one of the unknown quantities may be the same in both equations; then the difference or sum of these resulting equations, according as the equalized terms have the same or different signs, will give an equation involving only one unknown quantity, as before. This is the *method of equalizing the coefficients.*

Let it be required to find the values of x and y from the equations

$$15x + 7y = 51 \ldots (1), \text{ and } 5x + 14y = 52 \ldots (2).$$

Multiply eq. (1) by 2, and we have

$$30x + 14y = 102,$$

but by eq. (2), $5x + 14y = 52$;

hence, by subtraction, $25x = 50$, and $x = 2$.

From (1) we get $7y = 51 - 15x = 51 - 30 = 21$; $\therefore y = 3$.

Or, if the second equation be multiplied by 3, we have

$$15x + 42y = 156,$$

but by (1), $15x + 7y = 51$;

therefore, by subtraction, $35y = 105$, or $y = 3$;

hence $5x = 52 - 14y = 52 - 42 = 10$; $\therefore x = 2$.

Fourth Method.

97. Multiply one of the equations by m, and add the members of the result to those of the other equation; put the coefficient of y in the resulting equation equal to nothing, so as to eliminate y, and find the value of x; and if the coefficient of x be put equal to nothing, so as to eliminate x, the value of y will be obtained. This is the *method of an indeterminate multiplier.*

Let it be required to find the values of x and y from the equations

$$3x - y = 5 \ldots (1), \text{ and } 7x + 3y = 33 \ldots (2).$$

Multiply eq. (1) by m, then $3mx - my = 5m$, and adding the members of this result to those of (2), we get

$$(3m + 7)x + (3 - m)y = 5m + 33 \ldots (3).$$

Now this equation is true, whatever be the value of m, and therefore we may give to m such a value as will eliminate one of the unknown quantities. If $3 - m = 0$, or $m = 3$, then y will be eliminated, and

we get $$x = \frac{5m + 33}{3m + 7} = \frac{15 + 33}{9 + 7} = \frac{48}{16} = 3;$$

hence by (1) $y = 3x - 5 = 9 - 5 = 4$.

Or, if we put $3m + 7 = 0$, then $m = -\frac{7}{3}$, and x is eliminated;

therefore $$y = \frac{5m + 33}{3 - m} = \frac{33 - \frac{35}{3}}{3 + \frac{7}{3}} = \frac{64}{16} = 4;$$

and (1) $$x = \frac{y + 5}{3} = \frac{4 + 5}{3} = \frac{9}{3} = 3.$$

EXAMPLES.

1. Given $x + \frac{1}{2}y = 14$ and $\frac{1}{2}\cdot x - \frac{1}{6}y = 2$, to find the values of x and y.

Clearing both these equations of fractions, we get

$$2x + y = 28 \ldots . (1).$$
$$3x - y = 12 \ldots . (2).$$

Adding these equations, we get $5x = 40$, and $x = 8$;

hence $y = 28 - 2x = 28 - 16 = 12$.

2. Given $x - \dfrac{y-2}{7} = 5$, and $4y - \dfrac{x+10}{8} = 3$, to find the values of x and y.

Clearing these equations of fractions, and simplifying the results, we get

$$7x - y + 2 = 35, \text{ or } 7x - y = 33 \;.\;.\;.\;. \;(1),$$
$$12y - x - 10 = 9, \text{ or } 12y - x = 19 \;.\;.\;.\;. \;(2).$$

From the first equation, $y = 7x - 33$, and this value substituted for y in the second, gives

$$12(7x - 33) - x = 19, \text{ or } 84x - 396 - x = 19;$$
$$\therefore 83x = 396 + 19 = 415 \text{ and } x = 5.$$

Hence $\quad y = 7x - 33 = 35 - 33 = 2.$

3. Given $\dfrac{3}{x} + \dfrac{2}{y} = 12$ and $\dfrac{7}{x} - \dfrac{3}{y} = 5$, to find the values of x and y.

Multiply the first of these equations by 3, and the second by 2; then

$$\frac{9}{x} + \frac{6}{y} = 36, \qquad \frac{14}{x} - \frac{6}{y} = 10.$$

Taking the sum of these, we get $\dfrac{23}{x} = 46$, or $23 = 46x$; hence $x = \dfrac{1}{2}$.

Again, multiply the first equation by 7 and the second by 3, then

$$\frac{21}{x} + \frac{14}{y} = 84, \qquad \frac{21}{x} - \frac{9}{y} = 15.$$

Subtracting, we get $\dfrac{23}{y} = 69$, or $23 = 69y$; hence $y = \dfrac{1}{3}$.

4. Given $ax + by = c$, and $a_1x + b_1y = c_1$, to find the values of x and y.

Multiply the first equation by a_1 and b_1 separately, and the second equation by a and b, separately; then we have

$$aa_1x + a_1by = a_1c \qquad ab_1x + bb_1y = b_1c$$
$$aa_1x + ab_1y = ac_1 \qquad a_1bx + bb_1y = bc_1;$$

hence, by subtraction, we get

$$(a_1b - ab_1)y = a_1c - ac_1 \quad \text{or,} \quad y = \frac{a_1c - ac_1}{a_1b - ab_1} = \frac{ac_1 - a_1c}{ab_1 - ab}$$

$$(ab_1 - a_1b)x = b_1c - bc_1 \quad \text{or,} \quad x = \frac{b_1c - bc_1}{ab_1 - a_1b}.$$

This may be considered as the *general solution* of simple equations containing two unknown quantities.

5. Let $\dfrac{3x + 4y + 3}{10} - \dfrac{2x + 7 - y}{15} = 5 + \dfrac{y - 8}{5}$,

and $\dfrac{9y + 5x - 8}{12} - \dfrac{x + y}{4} = \dfrac{7x + 6}{11}$, to find the values of x and y.

Clearing these equations of fractions, by multiplying the terms of the first by 30, and those of the second by 132, we get

$$9x + 12y + 9 - 4x - 14 + 2y = 150 + 6y - 48, \text{ or } 5x + 8y = 107,$$
$$99y + 55x - 88 - 33x - 33y = 84x + 72, \text{ or } 31x - 33y = -80.$$

To four times the first of these, viz., $20x + 32y = 428$,
add the second, $31x - 33y = -80$,
and we get $51x - y = 348$, or $y = 51x - 348$.
Substitute this value for y in the equation, $5x + 8y = 107$, and we get
$$5x + 8(51x - 348) = 107, \text{ or } 5x + 408x - 2784 = 107;$$
$$\therefore 413x = 2784 + 107 = 2891, \text{ and } x = \frac{2891}{413} = 7.$$
Hence also $\quad y = 51x - 348 = 357 - 348 = 9.$

6. Let $a(x^2 + y^2) - b(x^2 - y^2) = 2a$, and $(a^2 - b^2)(x^2 - y^2) = 4ab$, to find the values of x and y.

From the first we have $(a - b)x^2 + (a + b)y^2 = 2a \ldots\ldots (1)$,
and from the second $(a^2 - b^2)x^2 - (a^2 - b^2)y^2 = 4ab \ldots\ldots (2)$.

Multiply the terms of the former by $a + b$, then
$$(a^2 - b^2)x^2 + (a^2 + 2ab + b^2)y^2 = 2a^2 + 2ab;$$
but, $$(a^2 - b^2)x^2 - (a^2 - b^2)y^2 = 4ab.$$
Hence, subtracting the latter from the former, we have
$$(2a^2 + 2ab)y^2 = 2a^2 - 2ab, \text{ or } 2a(a + b)y^2 = 2a(a - b);$$
$$\therefore y^2 = \frac{a - b}{a + b}, \text{ and } y = \left(\frac{a - b}{a + b}\right)^{\frac{1}{2}}.$$
Again, multiply the terms of (1) by $a - b$, then
$$(a^2 - 2ab + b^2)x^2 + (a^2 - b^2)y^2 = 2a^2 - 2ab;$$
but, $$(a^2 - b^2)x^2 - (a^2 - b^2)y^2 = 4ab.$$
Hence, by addition,
$$(2a^2 - 2ab)x^2 = 2a^2 + 2ab, \text{ or } 2a(a - b)x^2 = 2a(a + b);$$
$$\therefore x^2 = \frac{a + b}{a - b}, \text{ and } x = \left(\frac{a + b}{a - b}\right)^{\frac{1}{2}}.$$

Examples for Practice.

1. $2x + 9y = 20$, $4x + y = 6$. *Ans.* $x = 1$, $y = 2$.
2. $2x - 9y = 2$, $3x - 5y = 20$. *Ans.* $x = 10$, $y = 2$.
3. $5x + 4y = 52$, $3x + 7y = 45$. *Ans.* $x = 8$, $y = 3$.
4. $2x + 3y = 22$, $7x - y = 31$. *Ans.* $x = 5$, $y = 4$.
5. $\frac{x}{2} + \frac{y}{3} = 12$, $\frac{x}{3} + \frac{y}{2} = 13$. *Ans.* $x = 12$, $y = 18$.
6. $x + y = a$, $x - y = b$. *Ans.* $x = \frac{a + b}{2}$, $y = \frac{a - b}{2}$.
7. $\frac{x + y}{3} + \frac{x - y}{4} = 59$, $5x = 33y$. *Ans.* $x = 99$, $y = 15$.
8. $\frac{1}{3x} + \frac{1}{5y} = \frac{2}{9}$, $\frac{1}{5x} + \frac{1}{3y} = \frac{1}{4}$. *Ans.* $x = 2$ [illegible], $y = 1$ [illegible].
9. $x + y = a$, $bx = cy$. *Ans.* $x = \frac{ac}{b + c}$, $y = \frac{ab}{b + c}$.
10. $\frac{x}{3} + 2y = 5$, $\frac{2x - 1}{5} - y + 1 = 0$. *Ans.* $x = 3$, $y = 2$.
11. $x - \frac{y - 2}{7} = 5$, $4y - \frac{x + 10}{3} = 3$. *Ans.* $x = 5$, $y = 2$.

12. $\frac{5x-4}{6}+2y=24, \frac{20-2y}{5}+5x=40\frac{2}{5}$. *Ans.* $x=8, y=9$.

13. $\left.\begin{array}{l} 3y+4x-20=\frac{9x-8(y-1)}{4}, \\ (x+7)(y-2)+3=2xy-(x+1)(y-1). \end{array}\right\}$ *Ans.* $x=4, y=3$.

14. $x^2-y^2=37, 3x-4y=0$. *Ans.* $x=4, y=3$.

15. $x^2+xy=a, y^2+xy=b$. *Ans.* $x=\frac{a}{\sqrt{(a+b)}}, y=\frac{b}{\sqrt{(a+b)}}$.

16. $\left.\begin{array}{l} x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2}, \\ y+\frac{y-3}{x-18}=30-\frac{73-3y}{3}. \end{array}\right\}$ *Ans.* $x=21, y=20$.

17. $\frac{10+6y-4x}{6x-9y+3}=\frac{4}{3}, \frac{126+8x-17y}{100-12x+7y}=\frac{35}{13}$. *Ans.* $x=8, y=5$.

18. $ax+by=c^2, \frac{a(a+x)}{b(b+y)}=1$.

Ans. $x=\frac{b^2+c^2-a^2}{2a}, y=\frac{a^2+c^2-b^2}{2b}$.

19. $\sqrt{(x^2+2y-1)}-1=x, \sqrt{(y^2+3x)}-1=y$. *Ans.* $x=3, y=4$.

20. $\left.\begin{array}{l} \sqrt{(y-x)}+\sqrt{(a-x)}=\frac{5}{2}\sqrt{(a-x)}, \\ \sqrt{y}-\sqrt{(a-x)}=\sqrt{(y-x)}. \end{array}\right\}$ *Ans.* $x=\frac{4a}{5}, y=\frac{5a}{4}$.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

98. The methods to be employed in the solution of equations containing several unknown quantities are precisely similar to those which have been employed in the solution of equations involving two unknown quantities.

It may readily be shown that if the values of three unknown quantities are to be determined, it is necessary that there should be given three independent and consistent equations, and in general there must be given as many independent and consistent equations as there are unknown quantities to be determined. By eliminating one of the unknown quantities from the three equations, there will arise two equations involving the two other unknown quantities, and these may be treated by the methods already detailed. Proceed in a similar manner with equations containing several unknowns.

EXAMPLES.

1. Given $x+y+2z=19$, $2x+3y-5z=7$, and $5x-2y+7z=48$, to find the values of x, y, and z.

Multiply the first equation by 2, then will

$$2x+2y+4z=38,$$

but

$$2x+3y-5z=7.$$

Subtracting the latter from the former gives

$$-y+9z=31, \text{ or } y-9z=-31 \quad . \; . \; . \; . \; . \quad (1).$$

Again, multiply the first equation by 5, then will

$$5x+5y+10z=95,$$

but $\quad 5x-2y+\ 7z=48;$

hence, by subtraction, $\quad 7y+\ 3z=47\ .\ .\ .\ (2).$

Eq. (1), multiplied by 7, gives $7y-63z=-217,$

and subtracting, we get $\quad 66z=264$, and $z=4$.

From (1) we get $\quad y=9z-31\ =36-31=5,$

and thence $\quad x=19-y-2z=19\ -\ 5-\ 8=6.$

2. Let $x+2y=17$, $y+2z=11$, and $3x-4z=9$, to find the values of x, y, and z.

From the second equation, $2y+4z=22,$

but (1) $\quad x+2y=17;$

and subtracting, we get $\quad -x+4z=5$, or $x-4z=-5$.

Subtracting this last equation from the third equation, gives

$$2x=14, \text{ and hence } x=7;$$

consequently $\quad 4z=x+5=7+5=12$, hence $z=3$,

and $\quad y=11-2z=11-6=5.$

Examples for Practice.

1. $x+y=30$, $x+z=21$, $y+z=27$. *Ans.* $x=12$, $y=18$, $z=9$.
2. $x-y=6$, $x-z=4$, $y+z=34$. *Ans.* $x=22$, $y=16$, $z=18$.
3. $5x+3y=84$, $4x+6z=108$, $5y+7z=110$.
Ans. $x=12$, $y=8$, $z=10$.
4. $\frac{x}{6}+\frac{y}{5}=10$, $\frac{x}{4}-\frac{z}{6}=\frac{7}{2}$, $\frac{y}{5}+\frac{z}{3}=13$.
Ans. $x=30$, $y=25$, $z=24$.
5. $3x+4y+5z=38$, $4x+3y-4z=1$, $6x-2y+7z=34$.
Ans. $x=2$, $y=3$, $z=4$.
6. $x+y+z=15$, $x+y-z=3$, $x-y+z=5$.
Ans. $x=4$, $y=5$, $z=6$.
7. $3x+2y-z=20$, $2x+3y+6z=70$, $x-y+6z=41$.
Ans. $x=5$, $y=6$, $z=7$.
8. $\frac{x}{2}+\frac{y}{3}=12-\frac{z}{6}$, $\frac{y}{2}+\frac{z}{3}=8+\frac{x}{6}$, $\frac{x}{2}+\frac{z}{3}=10$.
Ans. $x=12$, $y=12$, $z=12$.
9. $\frac{x}{2}+\frac{y}{3}+\frac{z}{7}=22$, $\frac{x}{3}+\frac{y}{5}+\frac{z}{2}=31$, $\frac{x}{6}+\frac{y}{3}+\frac{z}{6}=19$.
Ans. $x=12$, $y=30$, $z=42$.
10. $\frac{2}{x}+\frac{3}{y}=\frac{1}{12}+\frac{4}{z}$, $\frac{3}{x}+\frac{5}{z}=\frac{19}{24}+\frac{4}{y}$, $\frac{7}{y}+\frac{5}{z}=\frac{3}{8}+\frac{5}{x}$.
Ans. $x=6$, $y=12$, $z=8$.
11. $\frac{xy}{x+y}=a$, $\frac{xz}{x+z}=b$, $\frac{yz}{y+z}=c$.

Ans. $x=\frac{2abc}{bc+ac-ab}$, $y=\frac{2abc}{ab+bc-ac}$, $z=\frac{2abc}{ab+ac-bc}$.

12. $\frac{2x+z-4}{12}+\frac{3y-6z+1}{13}=\frac{x-2}{4}$, $\frac{x}{9}-y+3z=2$,
$\frac{3x-2y+5}{5}-\frac{4x-5y+7z}{7}=\frac{2}{7}+\frac{3y-9z+6}{6}$.
Ans. $x=18, y=12, z=4$.

13. $2x-3y+2z=13$, $2v-x=15$, $2y+z=7$, $5y+3v=32$. *Ans.* $x=3, y=1, z=5, v=9$.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

99. When a question is proposed to be solved by the principles of algebra, its meaning ought to be perfectly understood, and its condition or conditions exhibited in the clearest manner. If there is only one unknown quantity to be determined, one condition will be sufficient, and if several quantities are to be determined, we must have as many independent conditions as there are unknown quantities to be found. In reducing problems to algebraical equations no general rule can be given, but the following hint may be useful. Suppose that the values of the quantities to be determined are actually found, and consider by what operations the truth of the solution may be verified. Then, instead of known numbers, make use of unknown symbols for the quantities to be determined, and let the same operations be performed with these letters as were performed with the numbers, and we shall obtain certain equations expressing the conditions of the question. The solution of these equations will give the values of the unknown quantities.

Thus if the question were to find the number which exceeds its fourth part by 27, then supposing the number were found, viz. 36, the process of verification would be as follows:

$$36-\frac{36}{4}=27.$$

Now if x denote the number required, we should, by following the same steps, arrive at the equation

$$x-\frac{x}{4}=27,$$

and the solution of this equation would give the number required; for, multiplying by 4, we get $4x-x=108$, or $3x=108$; hence $x=36$.

100. When several quantities are to be determined, there must be given an equal number of independent conditions, and for the purpose of expressing these conditions we may find it convenient to satisfy one or more of them by a judicious selection of an unknown quantity which has some known relation to the required quantity. Thus, if the sum of two quantities be given equal to $2s$, we may denote their difference by $2x$; then $s+x$ and $s-x$ would denote the greater and less numbers. In a similar manner, if the difference of two numbers be given equal to $2d$, then $x+d$, and $x-d$ might be employed to denote the two numbers, for the condition of their having a given difference would be fulfilled. Also, if the ratio of two numbers be given, as m to n, then mx and nx might be put for the numbers, and one condition would be fulfilled; and if the product of two numbers be given equal to p, then x and $\frac{p}{x}$ might represent the two numbers, for then $x\times\frac{p}{x}=p=$ the given product, and so on.

1. What number is that to which if 8 be added, one-third of the sum is equal to 5.

Let x represent the number required, then if 8 be added, the sum is $x + 8$, and one-third of this sum is $\frac{x+8}{3}$. Hence, by the condition of the question, we have

$$\frac{x+8}{3} = 5 \text{; therefore } x + 8 = 15, \text{ and } x = 7.$$

2. What two numbers are those whose sum is 48, and difference 18.

Let x denote the less number, then $x + 18$ will denote the greater, and one condition is fulfilled; and we have only to satisfy the other by adding the two expressions together; thus

$$x + x + 18 = 48, \text{ or } 2x = 48 - 18 = 30;$$

hence $x = 15$, the less number, and $x + 18 = 15 + 18 = 33$, the greater number.

Or thus, by two unknown Quantities.

Let $x =$ the greater number and $y =$ the less; then the two conditions of the question expressed in algebraical language are

$$x + y = 48 \ldots\ldots (1),$$
$$x - y = 18 \ldots\ldots (2).$$

By addition, $2x = 66$, therefore $x = 33$, the greater number;
by subtraction, $2y = 30$, therefore $y = 15$, the less number.

3. A bill of 25 guineas was paid with crowns and half-guineas; and twice the number of half-guineas exceeded three times that of the crowns by 17; how many were there of each?

Let x denote the number of crowns, and y the number of half-guineas, then the second condition of the question gives

$$2y - 3x = 17 \ldots\ldots (1).$$

And as we have employed two unknown quantities, we must have another condition to furnish a second equation. The other condition is obviously this:

$$x \text{ crowns} + y \text{ half-guineas} = 25 \text{ guineas}.$$

And here it is necessary to reduce these different coins to the same denomination, and thus divest the terms of the equality of their concrete character. The highest common denomination is sixpences; the value of x crowns is therefore $10x$ sixpences, the value of y half-guineas is $21y$ sixpences, and 25 guineas is $= 25 \times 42 = 1050$ sixpences; hence the preceding equality becomes in the abstract

$$10x + 21y = 1050 \ldots\ldots (2).$$

To resolve these equations, multiply (2) by 3, and (1) by 10, then we have

$$30x + 63y = 3150,$$
$$20y - 30x = 170;$$

hence, by adding, we get $83y = 3320$, and $y = 40$, the number of half-guineas. Also from (1) we get $3x = 2y - 17 = 80 - 17 = 63$; therefore $x = 21$, the number of crowns. These numbers will be found to fulfil both the conditions expressed in the question.

4. A garrison of 1000 men was victualled for 30 days; after 10 days

it was reinforced, and then the provisions were exhausted in 5 days: find the number of men in the reinforcement.

Let x denote the number of men in the reinforcement; then since the consumption varies directly as the product of the men and time,

we have $1000 \times 30 = 1000 \times 10 + (1000 + x) \times 5$

or $30000 = 10000 + 5000 + 5x;$

$\therefore\ 5x = 15000$, and $x = 3000$ men.

5. There is a number consisting of two digits, which, when divided by their sum, gives the quotient 4; but if the digits be reversed, and the number thus formed be increased by 12, and then divided by their sum, the quotient is 8; find the number.

Let x = the digit in the ten's place, and y = the digit in the unit's place; then $10x + y$ = the number, and $10y + x$ = the number when the digits are reversed; hence the two conditions of the question furnish these two equations:

$$\frac{10x + y}{x + y} = 4 \ldots (1); \quad \frac{10y + x + 12}{x + y} = 8 \ldots\ldots (2).$$

Clearing these equations of fractions, we get

$$10x + y = 4x + 4y, \text{ or } 6x = 3y, \text{ or } 2x = y,$$

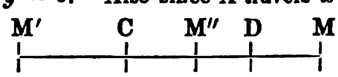
$$10y + x + 12 = 8x + 8y, \text{ or } 2y = 7x - 12;$$

but $2y = 4x$; hence $4x = 7x - 12$, or $3x = 12$;

hence $x = 4, y = 2x = 8$, and the number is $10x + y = 48$.

6. Two couriers, A and B, start at the same time from two towns, C and D, distant c miles from each other, and travel in the direction of the straight line passing through C and D. Now supposing A travels at the rate of a miles, and B at the rate of b miles per hour, when and where will they be together?

Let A and B travel in the same direction from C and D towards M, the place where the couriers will be together. Let C M $= x$, and D M $= y$; then we have $x - y = c$. Also since A travels x miles at the rate of



a miles per hour, it is obvious that $\frac{x}{a}$ = the number of hours that A travels. In a similar manner $\frac{y}{b}$ = the number of hours that B travels; but these times are equal by the condition of the question; hence,

$$\frac{x}{a} = \frac{y}{b}, \text{ or } ay = bx.$$

Multiply the terms of the former equation $x - y = c$ by a, then will $ax - ay = ac$, or $ax - bx = ac$, since $ay = bx$; consequently

$$(a - b)x = ac, \text{ and } x = \frac{ac}{a - b}.$$

Hence also $$y = \frac{bx}{a} = \frac{b}{a} \times \frac{ac}{a - b} = \frac{bc}{a - b}.$$

To apply this result to particular examples, let the distance C D = 10 miles = c, and let $a = 8$, and $b = 7$; then from the preceding values of x and y we get

$$x = \frac{ac}{a-b} = 80, \text{ and } y = \frac{bc}{a-b} = 70 \text{ miles},$$

which are the spaces travelled by A and B before they come together.

If $a = 16$ and $b = 15$, then $x = 160$, and $y = 150$ miles; hence it appears that the nearer their rates of travelling approach to equality the greater will be the spaces to be travelled before they come together. And if the rates of travelling are equal, then $x = \infty$ and $y = \infty$; whence it appears that they cannot come together, as is manifest from the equality of their rates of travelling.

Now if we suppose b greater than a, then $a - b$ would be negative, and the values of x and y would be negative. Thus if $a = 12$ and $b = 13$, then $x = -120$ and $y = -130$. To interpret the meaning of these *negative* results, we may observe that, if the couriers travel in the direction C D M, they can never come together, for B travels at a greater rate than A does, and therefore the distance between them is continually increasing. The meaning of the negative result will be easily understood, by considering the couriers to have been travelling at the specified rates towards C and D from some point M′ where they had been together. Travelling from M′ in the direction M′C D at the rates of 12 and 13 miles respectively, A would have travelled 120 miles and B 130 miles before their distance apart had been 10 miles; hence if M′ be 120 miles from C and 130 from D, the couriers, starting together from the point M′, and travelling at the proposed rates, would be simultaneously at C and D. The true interpretation of the negative results -120 and -130 is, that they denote distances in a direction directly opposite to those contemplated in the question.

Lastly if $a = 8$, and $b = -2$, then will $x = 8$ and $y = -2$; and as the supposition $b = -2$ signifies that B is travelling in a direction opposite to that of A, so the result $y = -2$ corresponds to such a supposition, and denotes that the place of meeting M″ is 2 miles to the left of D and 8 miles from C towards D.

Examples for Practice.

1. What number is that which exceeds its sixth part as much as 26 exceeds its fourth part? *Ans.* 24.

2. What number is that to which its third part being added, the sum shall be equal to its half added to 10? *Ans.* 12.

3. Divide 1000 into two parts, so that one of them shall be three-fifths of the other? *Ans.* 375 and 625.

4. If 2 be added to the numerator of a certain fraction, its value will be $\frac{3}{4}$; and if 2 be added to the denominator, its value will be $\frac{1}{2}$; what is the fraction? *Ans.* $\frac{7}{12}$.

5. A farmer rents 150 acres of lands for 237*l.* 10*s.*, part at 25*s.* and part at 45*s.* per acre; how many acres are there of each kind? *Ans.* 100 acres at 25*s.*, and 50 at 45*s.*

6. A person after spending 10*l.* more than a fifth of his income, had remaining 35*l.* more than half of it; what is his income? *Ans.* 150*l.*

7. In a naval engagement, one-third of the fleet was taken, one-sixth sunk, and two ships were burnt. In a storm after the action, one-seventh of the remainder was lost, and only 24 ships are left; of how many ships did the fleet consist? *Ans.* 60 ships.

8. A person bought two cups, one of gold and the other of silver; the price of gold is 3*l.* 17*s.* 10½*d.* per ounce, and that of silver 5*s.* per ounce, and the sum paid for both was 26*l.* 7*s.* 3*d.* But if the latter had been gold, and the former silver, their value would have been increased by 21*l.* 17*s.* 3*d.*; what was the weight of each? *Ans.* 6 and 12 ounces.

9. The pay of a second lieutenant for x days, at 5*s.* 6*d.* per day, together with the pay of a first lieutenant for y days, at 7*s.* per day, amounted to 34*l.* 4*s.* Now if the second lieutenant is promoted to the rank of first lieutenant, and the first lieutenant to that of second captain, whose pay is 11*s.* per day, their pay together for x and y days respectively would be increased by 15*l.* 12*s*; find the values of x and y.

Ans. $x = 48$ and $y = 60$ days.

10. Two labourers, A and B, received 5*l.* 17*s.* for their wages, A having been employed 15 and B 14 days; and A received, for working four days, 11*s.* more than B did for three days; what were their daily wages? *Ans.* 5*s.* and 3*s.*

11. A can do a piece of work in 10 days which A and B can do together in 7 days; how long would B take to do it alone?

Ans. 23⅓ days.

12. A mass of copper and tin weighs 80 lbs., and for every 7 lbs. of copper there are 3 lbs. of tin; how much copper must be added to the mass that for every 11 lbs. of copper there may be 4 lbs. of tin?

Ans. 10 lbs.

13. A garrison consists of 2100 men; there are 10 times as many foot soldiers, and three times as many artillery as there are cavalry; how many were there of each?

Ans. 150 cavalry, 450 artillery, and 1500 foot.

14. A person possesses 650*l.* stock in the 3 and 3½ per Cents.; find the amount of each kind of stock, when his annual income thence arising is 20*l.* *Ans.* 550*l.* in the 3 per Cents., and 100 in the 3½ per Cents.

15. A courier, passing through a certain town, travels at the rate of 13 miles in 2 hours; 12 hours afterwards another passes through the same town, travelling the same road, at the rate of 26 miles in 3 hours; how long and how far must the second courier travel before he overtakes the first? *Ans.* 36 hours, 312 miles.

16. A waterman finds that he can row with the tide from M to N, a distance of 18 miles, in 1½ hours, and that to return from N to M against the same tide, though he rows back along the shore, where the stream is only three-fifths as strong as in the middle, takes him 2¼ hours; find the rate at which the tide runs in the middle, where it is strongest.

Ans. 2½ miles per hour.

17. A body of 1905 troops consists of three battalions, and ½ the first battalion is to ⅓ of the second as 7 to 5, whilst ⅘ of the second battalion is to ⅘ of the third as 9 to 10; find the strength of each battalion.

Ans. 630, 675, and 600.

18. Three soldiers, P, Q, R, divided a certain quantity of cartridges among them in the following manner: P took 2 as often as Q took 3, and R took 5 as often as Q took 4; and when they were all divided, it was found that R had 700 more than P. How many did each take?

Ans. P 800, Q 1200, and R 1500.

19. A general having detached 400 men to take possession of a strong

post, and $\frac{4}{5}$ of the remainder of his troops to watch the movements of the enemy, finds that he has only $\frac{5}{17}$ of his army left; what was his whole force? *Ans.* 850 men.

20. A garrison had provisions for 30 months, but at the end of 4 months the number of troops was doubled, and 3 months afterwards it was reinforced with 400 men more, and the provisions were exhausted in 15 months; find the strength of the garrison before any augmentation took place. *Ans.* 800 men.

QUADRATIC EQUATIONS.

101. A *quadratic equation* is one which contains the square or second power of the unknown quantity, and is either pure or adfected.

A *pure* quadratic equation contains the square only of the unknown quantity, as $x^2 = 36$, or $a x^2 = b$.

An *adfected* quadratic equation contains both the first and second powers of the unknown quantity, as $x^2 - 6x = 5$, or $a x^2 + b x = c$.

A pure quadratic equation is resolved by collecting the unknown quantities on one side, and the known quantities on the other, as in a simple equation; then taking the square root of both sides, and prefixing the double sign ($\pm$) to the second side. Thus, if $a x^2 = b$, then $x^2 = \dfrac{b}{a}$, and hence

$$x = \pm\sqrt{\frac{b}{a}} = \pm\sqrt{\frac{a b}{a^2}} = \pm\,\frac{1}{a}\,.\,\sqrt{a b}.$$

Hence every pure quadratic equation has two equal roots with contrary signs.

The general form of an adfected or compound quadratic equation is

$$a x^2 + b x = c \;\ldots\ldots\; (1),$$

which may be resolved in different ways. One method is to divide every term of the equation by a, the coefficient of the second power of the unknown quantity, and it will then be reduced to the form

$$x^2 + p x = q \;\ldots\ldots\; (2).$$

And since the square of $x + a$ is $= x^2 + 2 a x + a^2$, it is obvious that the square of some quantity must be added to the first side of equation (2), so as to make that side a complete square. Let m^2 be the quantity to be added, then the first side will be $x^2 + p x + m^2$, and its square root will be found thus:

$$
\begin{array}{rl}
 & x^2 + p x + m^2 \,\left(x + \dfrac{p}{2}\right. \\
 & x^2 \\
\hline
2x + \dfrac{p}{2}\Big) & p x + m^2 \\
 & \underline{p x + \frac{1}{4} p^2}
\end{array}
$$

Hence we see that when $x^2 + p x + m^2$ is a complete square, we must have the condition

$$m^2 = \tfrac{1}{4} p^2 = (\tfrac{1}{2} p)^2, \text{ or } m = \tfrac{1}{2} p.$$

We have then the following method for the solution of a quadratic equation of the form $x^2 + a x = b$.

Add the square of half the coefficient of the first or simple power of

the unknown quantity to both sides, and the first side will be a complete square. Extract the square root of each side, prefix the double sign ($\pm$) *to the second, and the two values of the unknown quantity will be readily obtained by transposition.*

Note.—The square root of the first side of the equation is the square root of the first term, increased or diminished by half the coefficient of the second term, according as its sign is $+$ or $-$.

The solution of the equation $x^2 + ax = b$, is effected in the following manner:

Adding $(\frac{1}{2}a)^2$ or $\frac{a^2}{4}$ to both sides, we get

$$x^2 + ax + \frac{a^2}{4} = \frac{a^2}{4} + b,$$

extracting the root, $x + \frac{a}{2} = \pm \sqrt{\left(\frac{a^2}{4} + b\right)}$,

and by transposition, $x = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}$.

102. There is a second method of resolving the equation, $ax^2 + bx = c$, which may frequently be employed with advantage, as it avoids the introduction of fractions until the last step of the operation.

Multiply every term of the equation by four times the coefficient of the first term, viz., $4a$, then we have

$$4a^2x^2 + 4abx = 4ac,$$

and let m^2 be the square to be added to the first side to make it a complete square; then take the square root of $4a^2x^2 + 4abx + m^2$, thus:

$$\begin{array}{r|l}
 & 4a^2x^2 + 4abx + m^2\,(2ax + b \\
 & \underline{4a^2x^2} \\
4ax + b) & 4abx + m^2 \\
 & \underline{4abx + b^2}
\end{array}$$

Hence, if the first side is a square, we must have the condition $m^2 = b^2$, or $m = b$; and, therefore to resolve a quadratic of the form $ax^2 + bx = c$, we have the following process.

Multiply every term of the equation by four times the coefficient of the first term; add the square of the coefficient of the second term to both sides; then extract the square root of both sides, prefixing the double sign ($\pm$) *to the second, and there will arise two simple equations, from which the values of the unknown quantity may be found.*

Thus, if $ax^2 + bx = c$, then multiplying every term by $4a$, we get

$$4a^2x^2 + 4abx = 4ac;$$

adding b^2 to both sides, $4a^2x^2 + 4abx + b^2 = b^2 + 4ac$,

extracting the root, $2ax + b = \pm \sqrt{(b^2 + 4ac)}$

$$\therefore\ 2ax = -b \pm \sqrt{(b^2 + 4ac)}, \text{ or } x = \frac{-b \pm \sqrt{(b^2 + 4ac)}}{2a}.$$

103. If the first term of any quadratic equation be a square, or if it be multiplied or divided by such a quantity as will render it a square, the solution can always be effected as in the following example:

Given $9x^2 - 5x = 66$, to find the values of x.

Employing the process of extracting the square root of the first side,

in order to ascertain the quantity to be added to complete the square, we have

$$\begin{array}{rl} & 9x^2 - 5x \ \left(3x - \frac{5}{6}\right. \\ & 9x^2 \\ \hline 6x - \frac{5}{6}\Big) & -5x \\ & -5x + \frac{25}{36} \\ \hline \end{array}$$

Hence, adding $\frac{25}{36}$ to both sides, the first side will necessarily be a complete square, and we have at once,

$$3x - \frac{5}{6} = \pm\sqrt{\left(66 + \frac{25}{36}\right)} = \pm\sqrt{\frac{2401}{36}} = \pm\frac{49}{6};$$

$$\therefore\ 3x = \frac{5 + 49}{6} = 9, \text{ or } 3x = \frac{5 - 49}{6} = -\frac{22}{3};$$

Hence $x = \frac{9}{3} = 3$, or $x = -\frac{22}{9}$, the two values of x.

104. We have seen that in a quadratic equation the unknown quantity has always two different values, in consequence of the square root of a quantity having either the sign $+$ or $-$ prefixed to it. This may be shown in a more general manner.

Let r be a root of the equation $x^2 + a\,x + b = 0$, then by the definition of a root (83), the equation will be satisfied by the substitution in it of r for x; hence we have

$$r^2 + a\,r + b = 0,$$

but $$x^2 + a\,x + b = 0;$$

therefore, subtracting $x^2 - r^2 + a\,(x - r) = 0$, or $(x + r)\,(x - r) + a\,(x - r) = 0$;

$$\therefore\ (x - r)\,(x + r + a) = 0 \ \ldots\ldots\ (A).$$

Now this equation is satisfied by making either $x - r = 0$, or $x + r + a = 0$; from the former $x = r$, and from the latter $x = -a - r$, which is also a root of the equation $x^2 + a\,x + b = 0$. But besides these two values of x, viz., r and $-a-r$, no other value of it can be found which will satisfy the equation (A), and consequently no other value of it will satisfy the equation $x^2 + a\,x + b = 0$. Hence if a quadratic equation have one *real* root, it will have two *real* roots, and no more.

Hence also *the sum of the two roots of a quadratic equation of the form $x^2 + a\,x + b = 0$ is equal to $-a$, the coefficient of the second term with its sign changed, and the product of the two roots is equal to b, the last term.*

For if r and r' be the two roots of the equation $x^2 + a\,x + b = 0$, then we have from above $r' = -a - r$; hence $r + r' = -a$.

Again $r\,r' = r\,(-a - r) = -a\,r - r^2$, but since r is a root, we have $r^2 + a\,r + b = 0$; therefore $b = -a\,r - r^2$; hence we get $r\,r' = b$.

105. If the last term of a quadratic equation be *negative*, as $x^2 + a\,x - b = 0$, then will $x^2 + a\,x = b$, and completing the square, $x^2 + a\,x + \frac{1}{4}a^2 = \frac{1}{4}a^2 + b$; hence $x + \frac{1}{2}a = \pm\sqrt{\left(\frac{a^2}{4} + b\right)}$, and the roots

are both *real*. But if the last term be *positive*, as $x^2 + a\,x + b = 0$ then we have in the same way $x + \frac{1}{2}a = \pm\sqrt{\left(\frac{a^2}{4} - b\right)}$.

Now the second side is *real* so long as $\frac{a^2}{4} - b$ is *positive*, or $\frac{a^2}{4} > b$; but if $\frac{a^2}{4} < b$; then $\frac{a^2}{4} - b$ is *negative*, and no even root of it can be extracted. The values of x in this case are *imaginary* or *impossible*, but still they are called the roots of the equation, since if substituted for x in it, the equation would be verified.

EXAMPLES IN PURE QUADRATIC EQUATIONS.

1. Let $x^2 + 5 = \frac{10\,x^2}{3} - 16$, to find the values of x.

Multiplying every term by 3 gives

$3\,x^2 + 15 = 10\,x^2 - 48$, or $15 + 48 = 10\,x^2 - 3\,x^2$;

hence $63 = 7\,x^2$, and $x^2 = 9$; therefore $x = +3$, or $x = -3$.

In taking the square roots of the sides of the equation $x^2 = 9$, it is of no use to prefix the double sign to both roots, for $\pm x = \pm 3$ will furnish only the two values $x = +3$ and $x = -3$.

2. $3\,x^2 - 4 = 28 + x^2$. *Ans.* $x = \pm 4$.

3. $\frac{3\,x^2 + 5}{8} - \frac{x^2 + 29}{3} = 117 - 5\,x^2$. *Ans.* $x = \pm 5$.

4. $x^2 + a\,b = 5\,x^2$. *Ans.* $x = \pm \frac{1}{2}\sqrt{a\,b}$.

5. $(x + a)^2 = 2\,a\,x + b$. *Ans.* $x = \pm\sqrt{(b - a^2)}$.

6. $\frac{x + 7}{x^2 - 7\,x} - \frac{x - 7}{x^2 + 7\,x} = \frac{7}{x^2 - 73}$. *Ans.* $x = \pm 9$.

7. $x\sqrt{(a + x^2)} = b + x^2$. *Ans.* $x = \pm \frac{b}{\sqrt{(a - 2\,b)}}$.

8. $\sqrt{\frac{x - 2}{x + 2}} + \sqrt{\frac{x + 2}{x - 2}} = 4$. *Ans.* $x = \pm \frac{4}{3}\sqrt{3}$.

EXAMPLES IN ADFECTED QUADRATIC EQUATIONS.

1. Given $x^2 - 14\,x = 120$, to find the values of x.

Half the coefficient of the second term is 7, which, squared and added to both sides, gives $x^2 - 14\,x + 49 = 169$;

and extracting the root, $x - 7 = \pm 13$;

$\therefore x = 7 + 13 = 20$, or $x = 7 - 13 = -6$.

2. Given $3\,x^2 - 20\,x = 7$, to find the values of x.

Multiplying by four times the coefficient of the first term, viz. 12, we get $36\,x^2 - 240\,x = 84$;

adding the square of the coefficient of the second term, viz. 20^2 or 400 to both sides, $36\,x^2 - 240\,x + 400 = 484$;

extracting the root, $6\,x - 20 = \pm 22$;

Hence $6\,x = 20 + 22 = 42$, or $6\,x = 20 - 22 = -2$;

$\therefore x = 7$, or $x = -\frac{1}{3}$.

3. Given $x + \sqrt{(5x + 10)} = 8$, to find x.
By transposition, $\sqrt{(5x + 10)} = 8 - x$, and squaring both sides,

$$5x + 10 = 64 - 16x + x^2, \text{ or } x^2 - 21x = -54.$$

Completing the square, we get

$$x^2 - 21x + \left(\frac{21}{2}\right)^2 = \frac{441}{4} - 54 = \frac{225}{4};$$

and extracting the root, $x - \frac{21}{2} = \pm\frac{15}{2}$; consequently

$$x = \frac{21 + 15}{2} = 18, \text{ or } x = \frac{21 - 15}{2} = 3.$$

These two values of x are the roots of the quadratic equation $x^2 - 21x = -54$; but they will not both verify the proposed equation $x + \sqrt{(5x + 10)} = 8$, from which the former was derived by transposition and involution. Since the square root of a quantity may have either the sign $+$ or $-$ prefixed to it, the proposed equation might have been $x \pm \sqrt{(5x + 10)} = 8$; because by the operations which have been employed, the same resulting equation $x^2 - 21x = -54$ would be obtained, whether the sign of the irrational part be $+$ or $-$.
Hence in the equation $x + \sqrt{(5x + 10)} = 8$, the value of x is 3;
but in the equation $x - \sqrt{(5x + 10)} = 8$, the value of x is 18.

106. The methods which have been given for the solution of a quadratic equation will be equally applicable to an equation of either of the forms,

$$x^{2n} + ax^n = b,$$

$$(x^2 + ax + b)^{2n} + p(x^2 + ax + b)^n = q,$$

or

$$x^n + ax^{\frac{n}{2}} = b;$$

where the same unknown quantity is found in two terms, having the index in the one double of the index in the other.

4. Given $5x^{\frac{4}{3}} + 7x^{\frac{2}{3}} = 108$, to find the values of x.

Multiplying by 20, gives $100x^{\frac{4}{3}} + 140x^{\frac{2}{3}} = 2160$,

adding 7^2 or 49 to both sides, $100x^{\frac{4}{3}} + 140x^{\frac{2}{3}} + 49 = 2209$.

Extracting the root, $10x^{\frac{2}{3}} + 7 = \pm 47$;

hence, $10x^{\frac{2}{3}} = 40$ or -54; $\therefore x^{\frac{2}{3}} = 4$ or $-\frac{27}{5}$.

Consequently, by cubing both sides, $x^2 = 64$, or $\left(-\frac{27}{5}\right)^3$;

$$\therefore x = \pm 8, \text{ or } x = \pm\left(-\frac{27}{5}\right)^{\frac{3}{2}}.$$

5. Let $x^2 + \sqrt{(x^2 - x - 6)} = x + 48$, to find the values of x.
By transposition, $x^2 - x - 6 + \sqrt{(x^2 - x - 6)} = 42$,

$$\text{or, } (x^2 - x - 6) + (x^2 - x - 6)^{\frac{1}{2}} = 42.$$

Completing the square gives

$$(x^2 - x - 6) + (x^2 - x - 6)^{\frac{1}{2}} + \frac{1}{4} = 42\tfrac{1}{4} = \frac{169}{4};$$

and extracting the root, $(x^2 - x - 6)^{\frac{1}{2}} + \frac{1}{2} = \pm\frac{13}{2}$;

$$\therefore (x^2 - x - 6)^{\frac{1}{2}} = 6, \text{ or } (x^2 - x - 6)^{\frac{1}{2}} = -7.$$

Squaring both sides of these equations, we get

$$x^2 - x - 6 = 36, \text{ or } x^2 - x = 42,$$
$$x^2 - x - 6 = 49, \text{ or } x^2 - x = 55.$$

Resolving the first of these quadratics, we have

$$x^2 - x + \frac{1}{4} = 42\tfrac{1}{4} = \frac{169}{4}; \therefore x - \frac{1}{2} = \pm\frac{13}{2}, \text{ and } x = 7 \text{ or } -6.$$

Resolving the second equation, gives

$$x^2 - x + \frac{1}{4} = 55\tfrac{1}{4} = \frac{221}{4}; \therefore x - \frac{1}{2} = \pm\frac{1}{2}\sqrt{221}, \text{ or } x = \frac{1 \pm \sqrt{221}}{2}.$$

6. Given $x^4 - 12x^3 + 44x^2 - 48x = 9009$, to find the values of x.

$$\begin{array}{rl}
 & x^4 - 12x^3 + 44x^2 - 48x \;(x^2 - 6x + 4 \\
 & x^4 \\
2x^2 - 6x) & \quad - 12x^3 + 44x^2 \\
 & \quad - 12x^3 + 36x^2 \\
2x^2 - 12x + 4) & \qquad\qquad 8x^2 - 48x \\
 & \qquad\qquad 8x^2 - 48x + 16.
\end{array}$$

Hence, adding 16 to both sides, and extracting the root, we get

$$x^2 - 6x + 4 = \pm\sqrt{(9009 + 16)} = \pm\sqrt{9025} = \pm 95;$$

therefore, $\quad x^2 - 6x + 9 = \pm 95 + 5 = 100, \text{ or } -90;$

hence, $\quad x - 3 = \pm 10, \text{ or } x - 3 = \pm\sqrt{-90} = \pm 3\sqrt{-10},$

$$\therefore x = 13 \text{ or } -7; \text{ or } x = 3 \pm 3\sqrt{-10}.$$

Or thus. The equation may be written in the form,

$$x^4 - 12x^3 + 36x^2 + 8x^2 - 48x = 9009, \text{ or } (x^2 - 6x)^2 + 8(x^2 - 6x) = 9009;$$

and completing the square,

$$(x^2 - 6x)^2 + 8(x^2 - 6x) + 16 = 9025;$$
$$\therefore x^2 - 6x + 4 = \pm 95, \text{ as before.}$$

Examples for Practice.

1. $x^2 - 12x + 30 = 3.$ *Ans.* $x = 9$, or 3.
2. $2x^2 + 3x - 6 = 21.$ *Ans.* $x = 3$, or $-4\frac{1}{2}$.
3. $7x^2 - 39x + 40 = -4.$ *Ans.* $x = 4$, or $1\frac{4}{7}$.
4. $x^2 - 6x = 6x + 28.$ *Ans.* $x = 14$, or -2.
5. $3x^2 + 10x = 57.$ *Ans.* $x = 3$, or $-6\frac{1}{3}$.
6. $(x - 1)(x - 2) = 1.$ *Ans.* $x = \frac{1}{2}(3 \pm \sqrt{5})$.
7. $17x^2 + 19x - 1848 = 0.$ *Ans.* $x = 9\frac{15}{17}$, or -11.
8. $2x = 4 + \frac{6}{x}.$ *Ans.* $x = 3$, or -1.
9. $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8.$ *Ans.* $x = 1\frac{1}{2}$, or $-\frac{5}{6}$.
10. $\frac{1}{3}x^2 + \frac{5}{2}x = 27.$ *Ans.* $x = 6$, or $-13\frac{1}{2}$.
11. $\frac{2x - 10}{8 - x} - \frac{x + 3}{x - 2} = 2.$ *Ans.* $x = 7$, or $\frac{4}{5}$.
12. $\frac{1}{x - 1} - \frac{1}{x + 3} = \frac{1}{35}.$ *Ans.* $x = 11$, or -13.

13. $x + \dfrac{24}{x-1} = 3x - 4.$ *Ans.* $x = 5$, or -2.

14. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}.$ *Ans.* $x = 2$, or -3.

15. $\dfrac{x-4}{\sqrt{x+2}} = x - 8.$ *Ans.* $x = 9$.

16. $x + 4 + \left(\dfrac{x+4}{x-4}\right)^{\frac{1}{2}} = \dfrac{12}{x-4}.$ *Ans.* $x = \pm 5$, or $\pm 4\sqrt{2}$.

17. $\dfrac{2x+9}{9} + \dfrac{4x-3}{4x+3} = 3 + \dfrac{3x-16}{18}.$ *Ans.* $x = 6$, or $-4\frac{3}{4}$.

18. $x^2 + x + 2\sqrt{(x^2+x+4)} = 20.$ *Ans.* $x = 3, -4$, or $\dfrac{-1 \pm \sqrt{129}}{2}$.

19. $\sqrt{\left(x - \dfrac{1}{x}\right)} + \sqrt{\left(1 - \dfrac{1}{x}\right)} = x.$ *Ans.* $x = \dfrac{1 \pm \sqrt{5}}{2}$.

20. $x^{4n} - 2x^{3n} + x^{n} - 6 = 0.$

Ans. $x = \left(\dfrac{1 \pm \sqrt{13}}{2}\right)^{\frac{1}{n}}$, or $\left(\dfrac{1 \pm \sqrt{-7}}{2}\right)^{\frac{1}{n}}$.

21. $\dfrac{x^4 + 2x^3 + 8}{x^2 + x - 6} = x^2 + x + 8.$ *Ans.* $x = 4$, or $-4\frac{2}{3}$.

22. $x^2 - 1 = 2 + \dfrac{2}{x}.$ *Ans.* $x = 2$, or -1.

23. $x + 1 = \dfrac{10}{\sqrt{x}}.$ *Ans.* $x = 4$, or $-3 \pm 4\sqrt{-1}$.

24. $x^{\frac{7}{3}} = 56\,x^{-\frac{2}{3}} + x^{\frac{5}{6}}.$ *Ans.* $x = 4$, or $\sqrt[3]{49}$.

SIMULTANEOUS EQUATIONS OF THE SECOND AND HIGHER DEGREES.

107. We shall now give some examples of quadratic equations involving two or more unknown quantities, as well as a few examples of equations of higher degrees.

If in one of the equations either of the unknown quantities can be expressed in terms of the other and known quantities, this value may be substituted for it in the other equation, which will contain only one unknown quantity. Thus if the two proposed equations be

$$a x + b y = c \quad \ldots. \ (1),$$
$$a' x^2 + b' x y + c' y^2 + d x + e y = f \quad \ldots \ (2);$$

then find the value of either x or y from (1), and substitute this value for it in (2); the resulting equation will be a quadratic from which the value of the unknown quantity may be found, and thence the value of the other from (1).

108. If the two equations of the second degree are ***homogeneous***, they may be resolved by a quadratic equation.

Let the two homogeneous equations be

$$a x^2 + b x y + c y^2 = d \quad \ldots. \ (1),$$
$$a' x^2 + b' x y + c' y^2 = d' \quad \ldots. \ (2).$$

Put $x = v y$, then substituting this value for x in both equations,

$$a v^2 y^2 + b v y^2 + c y^2 = d, \text{ or } y^2 (a v^2 + b v + c) = d \quad \ldots. \ (3),$$
$$a' v^2 y^2 + b' v y^2 + c' y^2 = d', \text{ or } y^2 (a' v^2 + b' v + c') = d' \quad \ldots. \ (4).$$

Dividing (3) by (4) gives $\frac{a\,v^2 + b\,v + c}{a'\,v^2 + b'\,v + c'} = \frac{d}{d'}$,

or $\quad d'\,(a\,v^2 + b\,v + c) = d\,(a'\,v^2 + b'\,v + c') \,\ldots\ldots (5)$;

a quadratic equation to determine the values of v, and from (3) or (4) we get the values of y, and thence the values of x from the assumption $x = v\,y$.

109. If the equations are *symmetrical* with respect to the two unknown quantities, they may frequently be solved by substituting for the two unknown quantities the sum and difference of two others.

Let the two symmetrical equations be

$$x^3 + y^3 = 18\,x\,y \,\ldots\ldots (1),$$
$$x + y = 12 \quad \ldots\ldots (2).$$

Put $x = v + w$ and $y = v - w$, then from (2) we have

$$x + y = (v + w) + (v - w) = 2\,v = 12, \therefore v = 6.$$

And from eq. (1) we get $(v + w)^3 + (v - w)^3 = 18\,(v + w)(v - w)$;
hence, $2\,v^3 + 6\,v\,w^2 = 18\,(v^2 - w^2)$ or $v^3 + 3\,v\,w^2 = 9\,(v^2 - w^2)$.
Writing 6 for v in this equation, we have

$$216 + 18\,w^2 = 9\,(36 - w^2), \text{ or } 27\,w^2 = 324 - 216 = 108;$$

hence $\quad w^2 = 4$, and $w = \pm 2$; consequently

$x = v + w = 6 \pm 2 = 8$ or 4, and $y = v - w = 6 \mp 2 = 4$ or 8.

110. These *general* methods for the solution of certain classes of equations will be found useful in several instances, but for the most part, we must have recourse to those artifices which experience alone can suggest.

Thus let the two following equations be proposed for solution, viz.:

$$x^2 + y^2 = 53 \,\ldots (1); \qquad x\,y = 14 \,\ldots (2).$$

Add twice the second to the first, then $x^2 + 2\,x\,y + y^2 = 81$;
subtract twice the second from the first, $x^2 - 2\,x\,y + y^2 = 25$.

Taking the square roots of these results, we have

$$x + y = \pm 9 \,\ldots\ldots (3),$$
$$x - y = \pm 5 \,\ldots\ldots (4).$$

By addition, $\quad 2\,x = 14, 4, -4, -14, \therefore x = 7, 2, -2$, or -7;
by subtraction, $2\,y = 4, 14, -14, -4, \therefore y = 2, 7, -7$, or -2

Again, let the two equations be

$$x + y = 4 \,\ldots\ldots (1), \text{ and } x^4 + y^4 = 82 \,\ldots (2).$$

Then raising both sides of (1) to the fourth power (69), we have

$$\begin{aligned} & x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,x\,y^3 + y^4 = 256, \\ \text{but (2) } & x^4 \qquad\qquad\qquad\qquad\qquad\quad + y^4 = 82; \\ \hline \therefore\ & 2\,x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,x\,y^3 + 2\,y^4 = 338; \\ \text{or } & x^4 + 2\,x^3y + 3\,x^2y^2 + 2\,x\,y^3 + y^4 = 169. \end{aligned}$$

Taking the square root, gives $\quad x^2 + x\,y + y^2 = 13 \,\ldots\ldots (3)$
Squaring equation (1), we get $\quad x^2 + 2\,x\,y + y^2 = 16$;
hence, subtracting, we get $x\,y = 3$; therefore $3\,x\,y = 9$, and subtracting this from (3), we get

$$x^2 - 2\,x\,y + y^2 = 4, \text{ or } x - y = \pm 2;$$

hence we have the equations $x + y = 4$, and $x - y = \pm 2$, to determine x and y, as in the previous solution.

The values of x are 3 or 1, and those of y are 1 or 3.

EXAMPLES FOR PRACTICE.

1. $5\,x + 3\,y = 19$, $7\,x^2 - 2\,y^2 = 10$.

Ans. $x = 2, y = 3$, or $x = -\frac{406}{13}, y = \frac{759}{13}$.

2. $x + 4y = 14,\ y^2 + 4x = 2y + 11.$
Ans. $x = 2,\ y = 3$, or $x = -46,\ y = 15$.

3. $x^2 + 4y^2 = 256 - 4xy,\ 3y^2 - x^2 = 39.$
Ans. $x = \pm 6,\ y = \pm 5$, or $x = \pm 102,\ y = \mp 59$.

4. $x^2 - y^2 = 24,\ x^2 + xy = 84.$ *Ans.* $x = \pm 7,\ y = \pm 5$.

5. $x - y = 4, xy = 45.$ *Ans.* $x = 9, y = 5$, or $x = -5,\ y = -9$.

6. $x^2 + y^2 - x - y = 78,\ x + y + xy = 39.$
Ans. $x = 9,\ y = 3,\ x = 3,\ y = 9,$
or $x = \dfrac{-13 \pm \sqrt{-39}}{2},\ y = \dfrac{-13 \mp \sqrt{-39}}{2}.$

7. $xy + xy^2 = 12, x + xy^3 = 18.$ *Ans.* $x = 2,\ y = 2$, or $x = 16,\ y = \frac{1}{2}$.

8. $\dfrac{1}{x} + \dfrac{1}{y} = a,\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = b.$
Ans. $\dfrac{1}{x} = \dfrac{a \pm \sqrt{(2b - a^2)}}{2},\ \dfrac{1}{y} = \dfrac{a \mp \sqrt{(2b - a^2)}}{2}.$

9. $x^3 - y^3 = 117,\ x - y = 3.$ *Ans.* $x = 5,\ y = 2$, or $x = -2,\ y = -5$.

10. $\dfrac{x^2}{y^2} + \dfrac{4x}{y} = \dfrac{85}{9},\ x - y = 2.$
Ans. $x = 5,\ y = 3$, or $x = \dfrac{17}{10},\ y = -\dfrac{3}{10}$.

11. $x^4 + x^2y^2 + y^4 = 91,\ x^2 + xy + y^2 = 13.$
Ans. $x = \pm 3,\ y = \pm 1$, or $x = \pm 1,\ y = \pm 3$.

12. $x + y + x^2 + y^2 = 18,\ xy = 6.$
Ans. $x = 3,\ y = 2$, or $x = -3 \pm \sqrt{3},\ y = -3 \mp \sqrt{3}$.

13. $\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} + \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{27}{4},\ x - y = 2.$
Ans. $x = 4,\ y = 2,\ x = -2,\ y = -4$.

14. $x^2 + y^2 + z^2 = 84,\ x + y + z = 14,\ xz = y^2.$
Ans. $x = 2,\ y = 4,\ z = 8$.

15. $x + y + \sqrt{(x + y)} = 12,\ x^2 + y^2 = 41.$
Ans. $x = 5,\ y = 4$, or $x = 4,\ y = 5$, etc.

16. $x^2 + x\sqrt[3]{xy^2} = 208,\ y^2 + y\sqrt[3]{x^2y} = 1053.$
Ans. $x = \pm 8,\ y = \pm 27$.

17. $\left(\dfrac{3x}{x + y}\right)^{\frac{1}{2}} + \left(\dfrac{x + y}{3x}\right)^{\frac{1}{2}} = 2,\ xy - (x + y) = 54.$
Ans. $x = 6,\ y = 12$, or $x = -\dfrac{9}{2},\ y = -9$.

18. $x^2 + 4(x^2 + 3y + 5)^{\frac{1}{2}} = 55 - 3y,\ 6x - 7y = 16.$
Ans. $x = 5,\ -\dfrac{53}{7}$, or $\dfrac{-9 \pm 4\sqrt{317}}{7}$;
$y = 2,\ -\dfrac{430}{49}$, or $\dfrac{-166 \pm 24\sqrt{317}}{49}$.

19. $\dfrac{x^2}{y^2} + \dfrac{2x + y}{\sqrt{y}} = 20 - \dfrac{y^2 + x}{y},\ x + 8 = 4y.$
Ans. $x = 8,\ y = 4$, or $x = 152 \mp 64\sqrt{6},\ y = 40 \mp 16\sqrt{6}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

111. When a problem expressed in the symbols of algebra, by the method already explained (99), gives rise to an equation of the second degree, the solution of it may be effected by the principles which have been developed for the solution of such equations. Every quadratic has two roots, and these roots may be one or both positive or negative; one may be negative and the other positive, or both may be imaginary. If both roots be *positive*, they will be recognised as the solution to the question in its literal meaning; but if one or both of the roots be *negative*, the meaning or interpretation of such negative root or roots must be sought for in some modification of the enunciation of the question; and if both roots are *imaginary*, we may conclude that the conditions of the problem are inconsistent, and that no quantities can be found to fulfil them. These remarks will be better understood by a slight consideration of the solutions of the following examples.

EXAMPLES.

1. Find a number such that, if 15 be added to its square, the sum shall be 8 times the number.

Let x denote the number, then the condition gives

$$x^2 + 15 = 8x, \text{ or transposing } x^2 - 8x = -15;$$

hence $\quad x^2 - 8x + 16 = 1, \therefore x - 4 = \pm 1,$

and consequently $\quad x = 4 + 1 = 5, \text{ or } x = 4 - 1 = 3.$

In this example both roots are positive, and both are answers to the question in its literal meaning.

2. To divide a line of 20 inches in length into two parts such that the rectangle contained by the whole line and one of the parts shall be equal to the square of the other part.

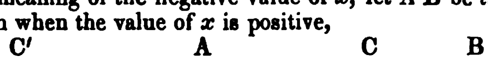
Let $x =$ the greater part; then $20 - x =$ the less part, and hence by the question we have

$$x^2 = 20(20 - x) = 400 - 20x, \text{ or } x^2 + 20x = 400;$$

therefore $\quad x^2 + 20x + 100 = 500, \text{ and } x + 10 = \pm 10\sqrt{5};$

$$\therefore x = -10 \pm 10\sqrt{5} = 10(-1 \pm \sqrt{5});$$

where one value of x is evidently positive and the other negative. To explain the meaning of the negative value of x, let A B be the line to be divided, then when the value of x is positive,



the point of division, C, falls between A and B, and A C is the positive value of x. But the line corresponding to the negative value of x must lie in a direction opposite to that of A C, and hence if B A be produced to C′, so that A C′ may be equal to $10(1 + \sqrt{5})$, then will A C′ represent the negative value of x.

3. A grazier bought as many sheep as cost him 60*l.*, and after reserving 15 out of the number, he sold the remainder for 54*l.*, and gained 2*s.* a-head by them; how many sheep did he buy?

Let x denote the number of sheep he bought; then

$$\frac{1200}{x} = \text{the buying price of each in shillings,}$$

and $$\frac{1080}{x - 15} = \text{the selling price of each in shillings;}$$

hence by the question $\dfrac{1080}{x-15} - \dfrac{1200}{x} = 2.$

Clearing of fractions, we get

$1080x - 1200x + 18000 = 2x^2 - 30x$, or $x^2 + 45x = 9000$;

and resolving this quadratic gives

$$x = -\frac{45}{2} \pm \sqrt{\left(\frac{2025}{4} + 9000\right)} = -\frac{45}{2} \pm \frac{195}{2} = 75 \text{ or } -120.$$

To interpret the meaning of the negative answer, let us write $-x$ for x in the equation $\dfrac{1080}{x-15} - \dfrac{1200}{x} = 2$, and it will become

$$\frac{1200}{x} - \frac{1080}{x+15} = 2,$$

which, translated into common language, gives the following problem:—

A grazier bought as many sheep as cost him 60*l.*, and after *increasing* the number by 15, he sold the whole for 54*l.*, and thereby *lost* 2*s.* a-head by them; how many sheep did he buy? *Ans.* 120.

Examples for Practice.

1. What number added to its square will make 42? *Ans.* 6 or -7.

2. The sum of two numbers is 40, and the sum of their squares 818; find the numbers. *Ans.* 23 and 17.

3. The difference of two numbers is 9, and their sum multiplied by the greater produces 266; find the numbers. *Ans.* 14 and 5.

4. A charitable person distributed 6*l.* among a certain number of poor people, and the same sum among a number consisting of 2 more than before. Each of the latter recipients got 2*s.* less than the former, how many were relieved the second time? *Ans.* 12.

5. What number is that which, divided by the product of its two digits, gives 2 for a quotient; and if 27 be added to it, the digits will be reversed. *Ans.* 36.

6. A person travelled 105 miles, and observed that if he had travelled slower by 2 miles an hour he would have been 6 hours longer in completing the same distance; how many miles did he travel per hour? *Ans.* 7 miles.

7. When 905 men were drawn up in two square columns, it was observed that one column contained 17 ranks more than the other; what was the strength of each column? *Ans.* 784 and 121.

8. A regiment was ordered to send 216 men on garrison duty, each company to furnish an equal number; but before the detachment marched, three of the companies were sent on another service, and then each of the remaining companies was obliged to supply 12 additional men, in order to make up the requisite number, 216. How many companies were in the regiment? *Ans.* 9 companies.

9. When 732 men were drawn up in column, the number in front and the number of ranks together made 73; find the number of ranks. *Ans.* 61 or 12.

10. Two detachments of foot are ordered to a station distant 39 miles, and they start at the same instant; but one detachment, by travelling $\frac{1}{4}$ of a mile an hour more than the other, arrived one hour before the other; find their rates of marching. *Ans.* 3 and $3\frac{1}{4}$ miles per hour.

11. Two partners, A and B, gained by trade 140*l.*; A's money was

3 months in trade, and his gain was 60*l.* less than his stock; and B's money, which was 50*l.* more than A's, was in trade 5 months; what was A's stock? *Ans.* 100*l.*

12. Find two numbers such that their product added to their sum may be 47, and their sum taken from the sum of their squares may leave 62. *Ans.* 5 and 7.

13. The continued product of four consecutive numbers is 3024; find the numbers. *Ans.* 6, 7, 8, 9.

14. A and B distribute 1200*l.* each among a number of persons; A gives to 40 persons more than B, and B gives 5*l.* a-piece to each person more than A; find the number of persons. *Ans.* 120 and 80.

15. From two towns, distant from each other 320 miles, two persons, A and B, set out at the same instant to meet each other. A travelled 8 miles a-day more than B, and the number of days in which they met was equal to half the number of miles B went in a day; how many miles did each travel per day? *Ans.* A 24 and B 16 miles.

16. Two mail trains start at the same time, the one from London to Edinburgh, and the other from Edinburgh to London, the speed of each train being uniform; now if the London train arrives in Edinburgh a hours, and the Edinburgh train in London b hours, after they passed each other, then the times of completing the whole distance by the London and Edinburgh trains will be respectively,

$$a^{\frac{1}{2}}\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right) \text{ and } b^{\frac{1}{2}}\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right).$$

BINOMIAL SURDS AND IMPOSSIBLE QUANTITIES.

112. *Binomial* or *compound* surd quantities are multiplied in the same manner as rational quantities, but the division of surds is best effected by multiplication. Thus, if $6\sqrt{2}-2\sqrt{3}+3\sqrt{6}-4$ is to be divided by $2\sqrt{6}$, then we have

$$\frac{6\sqrt{2}-2\sqrt{3}+3\sqrt{6}-4}{2\sqrt{6}}=\frac{6\sqrt{2}-2\sqrt{3}+3\sqrt{6}-4}{2\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}$$
$$=\frac{12\sqrt{3}-6\sqrt{2}+18-4\sqrt{6}}{12}=\sqrt{3}-\frac{1}{2}\sqrt{2}+\frac{3}{2}-\frac{1}{3}\sqrt{6}.$$

If the divisor be a compound quantity, the division is not easily performed unless the operation of division be converted into that of multiplication. To facilitate the operation of division, we may observe that if the dividend and divisor be multiplied by a quantity that will render the latter product a rational quantity, the value of the fraction expressing the quotient will not be altered. The surd divisors are commonly of the form $\sqrt{a}\pm b$, $\sqrt{a}\pm\sqrt{b}$, $\sqrt[3]{a}\pm\sqrt[3]{b}$, or $\sqrt[3]{a^2}\pm\sqrt[3]{ab}+\sqrt[3]{b^2}$; and from the known properties,

$$(x+y)\ (x-y) = x^2-y^2,$$
$$(x+y)\ (x^2-xy+y^2) = x^3+y^3,$$
$$(x-y)\ (x^2+xy+y^2) = x^3-y^3;$$

we may derive the following formulas, which are useful in the simplification of certain surd fractions with binomial or trinomial denominators, viz.,

$$(\sqrt{x}+\sqrt{y})\ (\sqrt{x}-\sqrt{y}) = x-y \quad \ldots \quad (1),$$
$$(\sqrt[3]{x}+\sqrt[3]{y})\ (\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}) = x+y \quad \ldots \quad (2),$$
$$(\sqrt[3]{x}-\sqrt[3]{y})\ (\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}) = x-y \quad \ldots \quad (3).$$

EXAMPLES.

1. Reduce the fractions $\frac{2}{\sqrt{5}}$ and $\frac{3}{\frac{1}{3}\sqrt[3]{7}}$ to equivalent ones with rational denominators.

Here $$\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5} = \frac{2}{5}\sqrt{5},$$

$$\frac{3}{\frac{1}{3}\sqrt[3]{7}} = \frac{3 \times 3 \times \sqrt[3]{7^2}}{\frac{1}{3}\sqrt[3]{7} \times 3 \times \sqrt[3]{7^2}} = \frac{9\sqrt[3]{7^2}}{\sqrt[3]{7^3}} = \frac{9}{7}\sqrt[3]{49}.$$

2. Reduce $\frac{2+\sqrt{3}}{3+\sqrt{3}}$ and $\frac{4\sqrt{7}+3\sqrt{2}}{7\sqrt{2}+3\sqrt{7}}$ to equivalent fractions, having rational denominators.

$$\frac{2+\sqrt{3}}{3+\sqrt{3}} = \frac{2+\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{3+\sqrt{3}}{6} = \frac{1}{2} + \frac{1}{6}\sqrt{3}.$$

$$\frac{4\sqrt{7}+3\sqrt{2}}{7\sqrt{2}+3\sqrt{7}} = \frac{4\sqrt{7}+3\sqrt{2}}{7\sqrt{2}+3\sqrt{7}} \times \frac{7\sqrt{2}-3\sqrt{7}}{7\sqrt{2}-3\sqrt{7}} = \frac{19\sqrt{14}-42}{25}.$$

3. Divide 2 by $\sqrt[3]{7} - \sqrt[3]{5}$.

Here $$\frac{2}{\sqrt[3]{7}-\sqrt[3]{5}} = \frac{2}{\sqrt[3]{7}-\sqrt[3]{5}} \times \frac{\sqrt[3]{7^2}+\sqrt[3]{(7\times 5)}+\sqrt[3]{5^2}}{\sqrt[3]{7^2}+\sqrt[3]{(7\times 5)}+\sqrt[3]{5^2}}$$
$$= \sqrt[3]{49} + \sqrt[3]{35} + \sqrt[3]{25}.$$

4. Divide 3 by $\sqrt{5} + \sqrt{2}$, and $2 + 4\sqrt{7}$ by $2\sqrt{7} - 1$.

Ans. $\sqrt{5} - \sqrt{2}$, and $\frac{8\sqrt{7}+58}{27}$.

5. Reduce the fractions $\frac{3}{\sqrt[3]{5}-\sqrt[3]{4}}$ and $\frac{2}{\sqrt[3]{16}-\sqrt[3]{12}+\sqrt[3]{9}}$ to equivalent fractions, having rational denominators.

Ans. $3(\sqrt[3]{25} + \sqrt[3]{20} + 2\sqrt[3]{2})$, and $\frac{2(\sqrt[3]{4}+\sqrt[3]{3})}{7}$.

6. Simplify the fractions $\frac{x}{\sqrt{(1+x^2)}-x}$ and $\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$.

Ans. $x\sqrt{(1+x^2)} + x^2$, and $\frac{a+\sqrt{(a^2-x^2)}}{x}$.

EXTRACTION OF THE SQUARE ROOT OF A BINOMIAL SURD.

113. *The product of two dissimilar quadratic surds is also a quadratic surd.*

For, if possible, let $\sqrt{x} \times \sqrt{y} = h$, a rational quantity; then $xy = h^2$;

$$\therefore y = \frac{h^2}{x} = \frac{h^2}{x^2} \times x, \text{ and } \sqrt{y} = \frac{h}{x}\sqrt{x};$$

and therefore $\sqrt{x}$ and $\sqrt{y}$ are *similar* quadratic surds, which is contrary to the hypothesis.

114. *A surd cannot be equal to the sum or difference of a rational quantity and a surd; nor can it be equal to the sum or difference of two dissimilar surds.*

For if $\sqrt{x} = a \pm \sqrt{y}$, then $x = a^2 \pm 2a\sqrt{y} + y$, and hence

$$\pm\sqrt{y} = \frac{x-y-a^2}{2a};$$

that is, a surd is equal to a rational quantity, which is impossible.

And if $\sqrt{x} = \sqrt{a} \pm \sqrt{y}$, then $x = a \pm 2\sqrt{ay} + y$,

$$\therefore \pm\sqrt{ay} = \frac{x - y - a}{2},$$

that is, the product of two dissimilar surds is equal to a rational quantity, which is absurd by Art. 113.

115. *In any equation $a + \sqrt{b} = x + \sqrt{y}$, consisting of rational quantities and surds, the rational quantities on each side are equal, and likewise the irrational quantities.*

For by transposition $\sqrt{b} = (x - a) + \sqrt{y}$, and if x be not equal to a, or if $x - a$ be not equal to zero, then we should have a surd equal to the sum of a rational quantity and a surd, which (114) is impossible; consequently, if $a + \sqrt{b} = x + \sqrt{y}$, then $x = a$, and $\sqrt{y} = \sqrt{b}$. By means of this principle we can extract the square root of a binomial surd of the form $a \pm \sqrt{b}$.

Let $\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}$; then $a + \sqrt{b} = x + y + 2\sqrt{xy}$;

$\therefore x + y = a \; . \; . \; . \; . \; (1), \quad 2\sqrt{xy} = \sqrt{b} \; . \; . \; . \; . \; (2).$

Squaring eq. (1) gives $x^2 + 2xy + y^2 = a^2$, and
squaring eq. (2) gives $\quad 4xy \quad = b$

$\therefore x^2 - 2xy + y^2 = a^2 - b$,

or $\quad x - y = \sqrt{(a^2 - b)} \; . \; . \; . \; . \; (3).$

But $\quad x + y = a$, by (1);

$$\therefore 2x = a + \sqrt{(a^2 - b)}, \text{ and } x = \frac{a + \sqrt{(a^2 - b)}}{2} = \frac{a + c}{2},$$

$$2y = a - \sqrt{(a^2 - b)}, \text{ and } y = \frac{a - \sqrt{(a^2 - b)}}{2} = \frac{a - c}{2},$$

if $\sqrt{(a^2 - b)} = c$; hence we have

$$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y} = \left(\frac{a + c}{2}\right)^{\frac{1}{2}} + \left(\frac{a - c}{2}\right)^{\frac{1}{2}}.$$

$$\text{Similarly, } \sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y} = \left(\frac{a + c}{2}\right)^{\frac{1}{2}} - \left(\frac{a - c}{2}\right)^{\frac{1}{2}}.$$

IMAGINARY OR IMPOSSIBLE QUANTITIES.

116. An *imaginary* quantity is one whose value cannot be assigned in consequence of the impossibility of performing the operation indicated. Thus $\sqrt{-c^2}$ is an imaginary or impossible quantity, for the operation of finding the square root of $-c^2$ cannot be performed. The square root of $-c^2$ is neither $+c$ nor $-c$, for the square of $+c$ is $+c^2$, and the square of $-c$ is $+c^2$; hence $\sqrt{-c^2}$ is an impossible quantity. If we separate the quantity c^2 from the factor -1, we get $\sqrt{-c^2} = \sqrt{c^2 \times (-1)} = \sqrt{c^2} \times \sqrt{-1} = c\sqrt{-1}$, and hence it is evident that all imaginary quantities will depend on the single imaginary expression $\sqrt{-1}$. The different powers of $\sqrt{-1}$ are as follow:—

$$(\sqrt{-1})^2 = \{(-1)^{\frac{1}{2}}\}^2 = (-1)^{\frac{1}{2}\times 2} = (-1)^1 = -1,$$
$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times (\sqrt{-1}) = -1 \times \sqrt{-1} = -\sqrt{-1},$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^3 \times (\sqrt{-1})^3 = -1 \times -1 = +1,$$
$$(\sqrt{-1})^5 = (\sqrt{-1})^4 (\sqrt{-1}) = +1 \times \sqrt{-1} = +\sqrt{-1},$$

and therefore every power of $\sqrt{-1}$ is included in one or other of these four forms, -1, $+1$, $-\sqrt{-1}$, $+\sqrt{-1}$.

EXAMPLES.

1. Find the square root of $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$; and multiply $6\sqrt{-3}$ by $\frac{1}{2}\sqrt{-12}$.

Let $\sqrt{(4\frac{1}{3}-\frac{4}{3}\sqrt{3})} = \sqrt{x} - \sqrt{y}$, then $4\frac{1}{3}-\frac{4}{3}\sqrt{3} = x+y-2\sqrt{xy}$;

$$\therefore x + y = 4\frac{1}{3} = \frac{13}{3}, \text{ and } 2\sqrt{xy} = \frac{4}{3}\sqrt{3};$$

$$\therefore x^2 + 2xy + y^2 = \frac{169}{9}, \text{ and } 4xy = \frac{16}{3};$$

hence $x^2 - 2xy + y^2 = \frac{121}{9}$, and $x - y = \frac{11}{3}$.

From the equations $x + y = \frac{13}{3}$ and $x - y = \frac{11}{3}$, we get readily

$$x = 4 \text{ and } y = \frac{1}{3};$$

$$\therefore \sqrt{(4\frac{1}{3} - \frac{4}{3}\sqrt{3})} = \sqrt{x} - \sqrt{y} = \sqrt{4} - \sqrt{\frac{1}{3}} = 2 - \frac{1}{3}\sqrt{3}.$$

Again, $6\sqrt{-3} \times \frac{1}{2}\sqrt{-12} = 6\sqrt{3} \times \sqrt{-1} \times \frac{1}{2}\sqrt{12} \times \sqrt{-1}$

$$= 3\sqrt{36} \times (-1) = -18.$$

2. Find the product of $x + a\sqrt{-1}$ and $x - a\sqrt{-1}$.
$(x+a\sqrt{-1})(x-a\sqrt{-1}) = x^2 - a^2(\sqrt{-1})^2 = x^2 - a^2 \times (-1) = x^2 + a^2$.

3. Extract the square roots of $23 \pm 8\sqrt{7}$ and $\frac{7+3\sqrt{5}}{2}$.

Ans. $4 \pm \sqrt{7}$ and $\frac{3+\sqrt{5}}{2}$.

4. Extract the square roots of $94 \pm 42\sqrt{5}$ and $\frac{47-21\sqrt{5}}{10}$.

Ans. $7 \pm 3\sqrt{5}$ and $\frac{7\sqrt{5}-15}{10}$.

5. Find the product of $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$, and divide $-12\sqrt{-15}$ by $6\sqrt{-5}$.
Ans. $(ac - bd) + (ad + bc)\sqrt{-1}$ and $-2\sqrt{3}$.

6. Divide $a + b\sqrt{-1}$ by $c + d\sqrt{-1}$.
Ans. $\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\sqrt{-1}$.

7. Find the square roots of $5 - 12\sqrt{-1}$ and of $2\sqrt{-1}$.
Ans. $3 - 2\sqrt{-1}$ and $1 + \sqrt{-1}$.

8. Find the cubes of $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$, and the fourth power of $3 - 2\sqrt{-1}$. *Ans.* 1, 1, and $-119 - 120\sqrt{-1}$.

9. Find the square root of $31 + 12\sqrt{-5}$ and the fourth root of -4. *Ans.* $6 + \sqrt{-5}$ and $1 + \sqrt{-1}$.

PROGRESSIONS.

117. When several quantities either increase or decrease according to a determinate law, the quantities are said to form a *progression* or *series*. Thus, if we take an expression, such as $2 + 3x + 4x^2$, and make successively $x = 0$, $x = 1$, $x = 2$, $x = 3$, ... $x = n - 1$, $x = n$, we get the following progression or series of numbers, viz.:

$$2, 9, 24, 47, \ldots 2 + 3(n-1) + 4(n-1)^2, 2 + 3n + 4n^2 \ldots (1).$$

If we take the first term from the second, the second from the third, and so on, the several remainders will evidently form a new series, which is called the *first order of differences*, with respect to the terms of (1). The new series, or first order of differences, will therefore be

$$7, 15, 23, \ldots. 7 + 8n \quad . \quad . \quad . \quad . \quad (2).$$

In the same manner, if we subtract each term from that which succeeds it, the several remainders will form another series or progression, called the *second order of differences*. In the present instance the second differences are constant, viz.:—

$$8, 8, 8, 8, \ldots. 8 \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Hence we see that if the general expression which furnishes the several terms of the progression or series be of the second degree, the second order of differences is constant. In the same manner it may be shown that if the general term of a series be of the third degree, the third order of differences is constant, and so on. From this principle we can determine the n^{th} or general term of any given series. Thus let

$$1, 3, 6, 10, 15, 21, 28, \ldots.$$

be a given series, to determine its n^{th} or general term.

Taking the differences, we get

1st order of differences, 2, 3, 4, 5, 6, 7, ...
2nd order of differences, 1, 1, 1, 1, 1, ...

Therefore the general term is of the second degree. Let it be $A + Bn + Cn^2$, where A, B, C are to be determined. Now we must have three equations to determine these three unknowns, and these will be obtained in the simplest possible form, by making $n = 1$, $n = 2$, and $n = 3$ in the general form, and putting the results equal to the *first*, *second*, and *third* terms of the series. Thus

if $n = 1$, then $A + B + C = 1$, the first term of the series,
$n = 2$, then $A + 2B + 4C = 3$, the second term of the series,
$n = 3$, then $A + 3B + 9C = 6$, the third term of the series.

Subtracting the first equation from the second gives $B + 3C = 2$,
and subtracting the second from the third, $B + 5C = 3$;
and again by subtraction we get $2C = 1$, or $C = \frac{1}{2}$; whence
$B = 2 - 3C = 2 - \frac{3}{2} = \frac{1}{2}$, and $A = 1 - B - C = 0$;
consequently the general or n^{th} term of the series proposed is

$$A + Bn + Cn^2 = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

ARITHMETICAL PROGRESSION.

118. An *arithmetical series*, or an *arithmetical progression*, is a series of quantities of which the difference between every two consecutive terms is the *same*. Thus

$$1, 3, 5, 7, \text{etc.}; \text{ and } 100a, 90a, 80a, 70a, \text{etc.}$$

are in arithmetical progression, the common difference of the former being 2, and that of the latter $- 10a$.

Let a, $a + d$, $a + 2d$, $a + 3d$, etc. be an arithmetical progression, where $a =$ the first term, $d =$ the common difference; and if $n =$ the number of terms, and $l =$ the n^{th} term; then since the coefficient of d in any term is one less than the number of that term, we get

$$l = a + (n - 1)\,d \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let $s = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$,
$\therefore s = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a$,
by taking the terms in the reverse order; hence, adding these,
$2s = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$.

Now since there are n equal terms in this series, we have

$$2s = (a + l) \times n, \text{ or } s = (a + l).\frac{n}{2} \quad . \quad . \quad . \quad . \quad (2).$$

Substituting the value of l from (1) in this equation, gives

$$s = \{2a + (n - 1)\,d\}\frac{n}{2} \quad . \quad . \quad . \quad . \quad (3).$$

If n terms are to be inserted between a and l, the first and n^{th} terms of the progression, then the whole number of terms will be $n + 2$, and hence, writing $n + 2$ for n in (1), we have

$$l = a + (n + 1)\,d, \text{ or } d = \frac{l - a}{n + 1} \quad . \quad . \quad . \quad (4),$$

which gives the common difference, and determines the series of terms or means to be inserted between a and l.

Examples.

1. Find the sum of 60 terms of the series 3, 5, 7, 9, 11, etc.

Here $a = 3$, $d = 2$, and $n = 60$; therefore we get

$$s = \{2a + (n - 1)\,d\}\,\frac{n}{2} = (6 + 59 \times 2) \times \frac{60}{2}$$
$$= (6 + 118) \times 30 \quad = 124 \times 30 = 3720.$$

2. The *fifth* term of an arithmetical progression is 19, and the *ninth* term is 35; what is the *fifteenth* term, and the sum of 20 terms of the series?

Here the *fifth* term $= a + (5 - 1)\,d = a + 4d = 19$,
and the *ninth* term $= a + (9 - 1)\,d = a + 8d = 35$.

Subtracting the former from the latter, gives

$4d = 16$, or $d = 4 =$ the common difference.

Hence $a = 19 - 4d = 19 - 16 = 3 =$ the first term of the series, and the *fifteenth* term $= a + (15 - 1)\,d = 3 + 14 \times 4 = 59$.

Also, if $n = 20$, we get

$$s = \{2a + (n - 1)\,d\}\,\frac{n}{2} = (6 + 19 \times 4) \times \frac{20}{2}$$
$$= (6 + 76) \times 10 = 820 = \text{the sum of 20 terms.}$$

3. What number of terms of the series 54, 51, 48, etc. must be taken so that their sum may be 513?

Here $a = 54$, $d = -3$, and $s = 513$; consequently by (3) we get

$$513 = \{108 - 3\,(n - 1)\}\,\frac{n}{2} = \frac{(111 - 3n) \times n}{2};$$

or $1026 = 111n - 3n^2$; $\therefore n^2 - 37n = -342$.

Solving this quadratic, we get $n = 18$ or 19, either of which fulfils the conditions of the question, since the 19th term is zero.

4. The first term of an arithmetical series is 3, and the last term is 27; find the series when the number of terms is 7. Or, which is the same thing, insert 5 arithmetical means between 3 and 27.

If we make use of formula (1), we must take $n = 7$; and if we make use of (4), then n must be 5. In either way we get

$$d = \frac{l-a}{6} = \frac{27-3}{6} = 4, \text{ the common difference;}$$

hence the five means are 7, 11, 15, 19, and 23.

Examples for Practice.

1. Find the tenth term of the series 1, 3, 5, 7, etc., and the sum of 10 terms. *Ans.* 19 and 100.

2. Find the sum of n terms of the natural numbers 1, 2, 3, 4, etc. *Ans.* $\frac{n(n+1)}{2}$.

3. Find the 365th term of the series 2, 4, 6, 8, etc. *Ans.* 730.

4. Find the 29th term of the series $\frac{5}{3}$, $\frac{1}{2}$, $-\frac{2}{3}$, etc. *Ans.* -31.

5. Find the 13th term of the series 4, -3, -10, etc. *Ans.* -80.

6. Find the sum of $3 + 7 + 11 +$ etc. to 12 terms. *Ans.* 300.

7. Find the sum of $6 + \frac{11}{2} + 5 +$ etc. to 25 terms. *Ans.* 0.

8. Find the sum of $2\frac{1}{2} + 2 + \frac{3}{2} +$ etc. to 20 terms. *Ans.* -45.

9. Sum $\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} +$ etc. to n terms. *Ans.* $\frac{n-1}{2}$.

10. The sum of an arithmetical series is 1455, the first term 5, and the common difference 3; find the number of terms. *Ans.* 30.

11. The *fifth* and *ninth* terms of an arithmetical series are 19 and 35; find the 20th term, and the sum of 12 terms of the series. *Ans.* 79 and 300.

12. Insert 3 arithmetical means between 2 and 14. *Ans.* 5, 8, 11.

13. Insert 9 arithmetical means between $-3\frac{2}{5}$ and $2\frac{3}{5}$.
Ans. $-2\frac{4}{5}$, $-2\frac{1}{5}$, $-1\frac{3}{5}$, -1, $-\frac{2}{5}$, $\frac{1}{5}$, $\frac{4}{5}$, $1\frac{2}{5}$, and 2.

14. Find three numbers in arithmetical progression whose sum is 21, and the sum of the first and second is equal to $\frac{5}{9}$ths that of the second and third. *Ans.* 3, 7, 11.

15. One hundred stones being placed on the ground in a straight line, at the distance of a yard from each other, how far will a person travel who shall bring them, one by one, to a basket which stands at the place of starting, 10 yards from the first stone.
Ans. 6 miles 6 furlongs 20 yards.

16. The first term of an arithmetical series is $n^2 - n + 1$, the common difference 2; find the sum of n terms, and thence show that

$$2^3 = 3 + 5, \quad 3^3 = 7 + 9 + 11, \quad 4^3 = 13 + 15 + 17 + 19, \text{ etc.}$$

Ans. n^3.

17. A party of foot begin their march at 6 in the morning, and travel $3\frac{1}{2}$ miles per hour; 3 hours after a troop of horse follows them from the same place, and travel $3\frac{1}{2}$ miles the first hour, 4 miles the second hour, $4\frac{1}{2}$ the third, and so on; in what time will they overtake the first?
Ans. In 7 hours.

18. A debt can be discharged in a year by paying one shilling the first week, three the second, five the third, and so on; find the last payment and the amount of the debt. *Ans.* 5*l.* 3*s.*, and 135*l.* 4*s.*

GEOMETRICAL PROGRESSION.

119. A *geometrical* progression is a series of quantities in which the ratio of every two consecutive terms is the *same*. Thus, 1, 3, 9, 27, etc., and a, ar, ar^2, etc., are in geometrical progressions, the common ratios being respectively 3 and r. When the common ratio is greater than unity, the series is called an *increasing* one, and a *decreasing* series when the common ratio is less than unity. The common ratio is found by dividing any term of the series by the term immediately preceding. Thus $ar^2 \div ar = r$, is the common ratio of the series of quantities a, ar, ar^2, ar^3, etc.

Let a be the first term of a series of quantities in geometrical progression, r the common ratio; l the last or n^{th} term, and s the sum of the series; then the series will be a, ar, ar^2, ar^3, ar^4, etc., where the index of r in any term is *less by one* than the number of the term: hence we have

$$l = ar^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let $s = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1}$,

$\therefore rs = \quad ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n$;

subtracting, we get $(r-1)\,s = ar^n - a = a\,(r^n - 1)$; consequently

$$s = a\,\frac{r^n - 1}{r - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If r is a proper fraction, then r will be less than 1, and the last equation may be written

$$s = a\,\frac{1 - r^n}{1 - r} \quad . \quad . \quad . \quad . \quad . \quad (2');$$

and if n be very great, r^n will be very small, and by increasing the value of n, the term r^n may be made as small as we please; hence the greater n is taken the more nearly will the sum of the series approach to $\frac{a}{1-r}$; therefore $s = \frac{a}{1-r}$ . . . . (3)
is the *limit* to which the sum approaches, but will never actually reach it.

If the number of terms, and the two extremes of a series of quantities in geometrical progression be given, then we have by equation (1)

$$\frac{l}{a} = r^{n-1}, \text{ and } r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} \quad . \quad . \quad . \quad . \quad (4);$$

and r being determined, the terms of the progression may be found.

EXAMPLES.

1. Find the sum of 11 terms of the series 1, 2, 4, 8, 16, etc.
Here $a = 1$, $r = 2$, and $n = 11$; therefore by (2)

$$s = a\,\frac{r^n - 1}{r - 1} = \frac{2^{11} - 1}{2 - 1} = 2^{11} - 1 = 2047.$$

2. Find the sum of 11 terms of the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc.; and also the limit of the sum of the series.

Here $a = \frac{1}{2}$, $r = \frac{1}{2}$ and $n = 11$; therefore by (2′)

$$s = \tfrac{1}{2}.\,\frac{1 - (\frac{1}{2})^{11}}{1 - \frac{1}{2}} = \frac{2^{11} - 1}{2^{11}} = \frac{2047}{2048}.$$

And when n is indefinitely increased, then we have by (3)

$$s = \frac{a}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 = \text{the } \textit{limit} \text{ of the sum.}$$

3. Insert 5 geometrical mean proportionals between 2 and 1458.

Here $a = 2$, $l = 1458$, and $n = 5$ means + 2 extremes = 7; therefore (1)

$$1458 = 2r^6 \quad \therefore r^6 = 729;\ r^3 = 27, \text{ and } r = 3;$$

hence 6, 18, 54, 162, 486, are the five means required.

4. Find the fractional value of the circulatng decimal $\cdot 34\dot{5}\dot{1}$.

$$\text{Here } \cdot 34\dot{5}\dot{1} = \frac{34}{100} + \frac{51}{10^4} + \frac{51}{10^6} + \frac{51}{10^8} + \text{etc.} \ldots$$

$$= \frac{34}{100} + \frac{51}{10^4}\left\{1 + \frac{1}{100} + \frac{1}{10000} + \text{etc.}\right\}$$

$$= \frac{34}{100} + \frac{51}{10^4} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{34}{100} + \frac{51 \times 10^2}{10^4 \times 99}$$

$$= \frac{34}{100} + \frac{17}{3300} = \frac{1139}{3300}, \text{ the value required.}$$

Or thus. Let $x = \cdot 34\dot{5}\dot{1}$; then multiplying by 100 in order to separate the non-recurring from the recurring decimal, we have

$$100x = 34 \cdot \dot{5}\dot{1}.$$

Again, multiplying by 100 to remove the decimal point to the beginning of the second period of decimals, we get

$$10000x = 3451 \cdot \dot{5}\dot{1};$$

$$\text{but } 100x = 34 \cdot \dot{5}\dot{1};$$

$$\therefore 9900x = 3417, \text{ or } x = \frac{3417}{9900} = \frac{1139}{3300}.$$

This is the foundation of the method usually adopted in arithmetic.

Examples for Practice.

1. Find the 7th term of the series 1, 4, 16, etc. *Ans.* 4096.

2. Find the 8th term of the series 9, − 6, 4, etc. *Ans.* $-\frac{128}{243}$.

3. Find the sum of 5 + 20 + 80 + etc. to 8 terms. *Ans.* 109225.

4. Find the sum of 2 − 4 + 8 − etc. to 9 terms. *Ans.* 342.

5. Find the sum of $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} +$ etc. to 7 terms. *Ans.* $\frac{2059}{1458}$.

6. Find the sum to which the series $2 - \frac{4}{3} + \frac{8}{9} -$ etc., approaches, when n is indefinitely great. *Ans.* $1\frac{1}{5}$.

7. Insert 3 geometrical means between 2 and $\frac{81}{8}$. *Ans.* 3, $\frac{9}{2}$, $\frac{27}{4}$.

8. Insert one geometrical mean between a and b. *Ans.* $\sqrt{ab}$.

9. The *third* term of a geometrical series is $\frac{9}{8}$, and the *sixth* is $\frac{243}{64}$; find the series. *Ans.* $\frac{1}{2}$, $\frac{3}{4}$, $\frac{9}{8}$, etc.

10. Find four numbers in geometrical progression, such that the sum of the means shall be = 36, and the sum of the extremes = 84. *Ans.* 3, 9, 27, 81.

11. Find the vulgar fraction equivalent to the decimal $\cdot 21333\ldots\ldots$

Ans. $\frac{16}{75}$.

12. The population of a town increases annually in geometrical progression, and in 3 years it was raised from 120000 to 138915 inhabitants; find by what part of itself it was annually increased.

Ans. By $\frac{1}{20}$th part.

13. From a vessel containing 10 gallons of brandy one gallon was drawn out and replaced by a gallon of water; a gallon of the mixture was then drawn out and replaced by a gallon of water. Now if this process were repeated 10 times, how much brandy would remain in the vessel, supposing the two fluids were thoroughly mixed each time.

Ans. 3·486784401 gallons.

14. The sum of three numbers in geometrical progression is 21, and the sum of their reciprocals is $\frac{7}{12}$; find the numbers. *Ans.* 3, 6, 12.

HARMONICAL PROGRESSION.

120. An *harmonical* progression is a series of which the first of any *three* consecutive terms has the same ratio to the third which the difference between the first and second has to the difference between the second and third. Thus a, b, c, d, e, etc., are in harmonical progression if $a : c :: a - b : b - c$; $b : d :: b - c : c - d$; $c : e :: c - d : d - e$, etc.

The reciprocals of quantities in harmonical progression are in arithmetical progression.

Let a, b, c, d, be four quantities in harmonical progression; then since $a : c :: a - b : b - c$, and $b : d :: b - c : c - d$;

$$\therefore\ ab - ac = ac - bc, \text{ and } bc - bd = bd - cd,$$

$$\text{or } \frac{ab - ac}{abc} = \frac{ac - bc}{abc} \text{ and } \frac{bc - bd}{bcd} = \frac{bd - cd}{bcd},$$

$$\therefore\ \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}, \text{ and } \frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b}.$$

Hence $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c}$, and therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in arithmetical progression. In consequence of this connexion between the terms of these two progressions, it is evident that all questions with reference to numbers in harmonical progression may be solved by the principles of arithmetical progression. Since an arithmetical series may have a term $= 0$, an harmonical series may have a term $= \frac{1}{0} = \infty$; and therefore no general expression for the *sum* of any number of terms of such a series can be given.

PILING OF SHOT AND SHELLS.

121. Shot and shells are usually piled in three different forms, two of which terminate at the top in a single shot, and the other terminates in a ridge, or single row of shot.

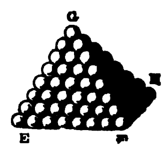
When the pile terminates in a single shot at top, and the horizontal courses are each of the form of an equilateral triangle, the pile is called

a triangular pile, and the number of shot in either side of the base course is equal to the number of horizontal rows or courses.

When the pile terminates in a single shot at top, and the horizontal courses are each of the form of a square, the pile is called *a square pile*, and the number of shot in each side of the base course is equal to the number of horizontal rows or courses.

When the pile terminates in a single row of shot, and the horizontal courses are each of the form of a rectangle, the pile is called a *rectangular* pile, and the number of shot in the *breadth* of the base course is equal to the number of horizontal rows or courses; and the number of shot in the top row is always one more than the difference between the length and breadth of the base course.

122. *To find the number of shot in a square pile.*



A square pile is formed of successive square horizontal courses, such that the number of shot in the sides of these courses decreases continuously by unity from the bottom to the single shot at top; consequently the number of shot (S) in a complete pile of n courses will be the sum of the squares of the natural numbers from 1 to n, or

$$S = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots\ldots + n^2.$$

Let $s = 1 + 2 + 3 + 4 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$, by Art. 118; then since

$$\begin{aligned} 2^3 &= (1+1)^3 = 1^3 + 3.1^2 + 3.1 + 1 \\ 3^3 &= (2+1)^3 = 2^3 + 3.2^2 + 3.2 + 1 \\ 4^3 &= (3+1)^3 = 3^3 + 3.3^2 + 3.3 + 1 \\ 5^3 &= (4+1)^3 = 4^3 + 3.4^2 + 3.4 + 1 \\ &\vdots \\ (n+1)^3 &= n^3 + 3.n^2 + 3.n + 1 \end{aligned}$$

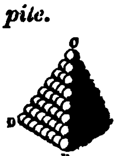
Adding all these equations together, and cancelling $2^3, 3^3, 4^3, \ldots\ldots n^3$, from both sides, we get at once

$$(n+1)^3 = 1^3 + 3(1^2+2^2+3^2+\ldots+n^2) + 3(1+2+3+\ldots+n) + n,$$

$$\text{or } (n+1)^3 = 1 + 3S + 3s + n;$$

$$\begin{aligned} \therefore S &= \frac{(n+1)^3 - 1}{3} - s - \frac{n}{3} \\ &= \frac{n^3 + 3n^2 + 3n}{3} - \frac{n(n+1)}{2} - \frac{n}{3} \\ &= \frac{n^3 + 3n^2 + 2n}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(2n+1)}{2.3} \ \ldots\ldots\ldots\ldots\ (1). \end{aligned}$$

123. *To find the number of shot in a triangular pile.*



A triangular pile is formed of successive triangular courses, such that the number of shot in the sides of these courses decreases continuously by unity from the bottom to the single shot at top; consequently the number of shot (S) in a complete triangular pile of n courses will be the sum of the series

$$1, 1+2, 1+2+3, 1+2+3+4, \text{etc.} \ldots\ldots 1+2+3+\ldots\ldots+n,$$

or $$1, 3, 6, 10, 15, \ldots\ldots\ldots \frac{n(n+1)}{2}.$$

Hence $$S = 1+3+6+10+15+21+\ \ldots + \frac{n(n+1)}{2}$$

$$= \frac{1.2}{2} + \frac{2.3}{2} + \frac{3.4}{2} + \frac{4.5}{2} + \frac{5.6}{2} + \ldots\ldots + \frac{n(n+1)}{2}$$

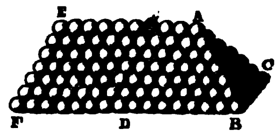
$$= \frac{1(1+1)}{2} + \frac{2(2+1)}{2} + \frac{3(3+1)}{2} + \frac{4(4+1)}{2} + \ldots + \frac{n(n+1)}{2}$$

$$= \frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \frac{4^2+4}{2} + \ldots\ldots + \frac{n^2+n}{2}$$

$$= \frac{1^2+2^2+3^2+4^2+\ldots\ldots+n^2}{2} + \frac{1+2+3+4+\ldots\ldots+n}{2}$$

$$= \frac{n(n+1)(2n+1)}{2.2.3} + \frac{n(n+1)}{2.2} = \frac{n(n+1)(n+2)}{2.3} \ldots\ldots (2).$$

124. *To find the number of shot in a rectangular pile.*



A rectangular pile is formed of successive rectangular courses, such that the number of shot in each of the sides of these courses decreases continuously by unity from the bottom to the single row of balls at the top; consequently if $m+1$ be the number of shot in the top row, the number of shot (S) in a complete rectangular pile of n courses will be the sum of the series $m+1$, $2(m+2)$, $3(m+3)$, $\ldots\ldots n(m+n)$.

Hence $$S = m+1+2(m+2)+3(m+3)+\ldots\ldots+n(m+n)$$

$$= m(1+2+3+4+\ldots\ldots+n) + 1^2+2^2+3^2+4^2+\ldots+n^2$$

$$= \frac{mn(n+1)}{2} + \frac{n(n+1)(2n+1)}{2.3}$$

$$= \frac{n(n+1)(3m+2n+1)}{2.3} \ldots\ldots (3).$$

The number of shot in the length of the base course of a rectangular pile is $m+n$, the number in the breadth of it is always n, and the number of shot in the top row is $m+1$, the difference between the length and breadth of the base course increased by 1.

The number of shot in an *incomplete pile* is evidently the difference between the number of shot in the pile considered as complete and the number in the pile which has been removed.

It will be observed that the three formulas (1), (2), (3), for the number of shot in the three different piles have a common factor, viz., $\frac{n(n+1)}{2}$, which expresses the number of shot in a triangular face of n courses, in either pile, and since

$$\frac{n(n+1)(n+2)}{2.3} = \frac{n(n+1)}{2}\cdot\frac{n+1+1}{3};\quad \frac{n(n+1)(2n+1)}{2.3}$$

$$= \frac{n(n+1)}{2}\cdot\frac{n+1+n}{3};\quad \frac{n(n+1)(2n+1+3m)}{2.3}$$

$$= \frac{n(n+1)}{2}\cdot\frac{(n+m)+(m+1)+(n+m}{3};$$

we have the following rule, which applies to all the piles:

Multiply the number of shot in a triangular face of the pile by one-third of the sum of the number of shot in the three parallel edges of the pile, and the product will be the number of shot in the pile.

In the triangular pile, one edge of the base, the shot at the top, and the shot at the vertex of the triangular base opposite to the edge, are considered the three parallel edges; and in the square pile, the shot at the top forms one of the parallel edges.

Examples for Practice.

1. Find the number of shot in a triangular pile of 15 courses.

Here $n = 15$, and $\dfrac{n(n+1)(n+2)}{2.3} = \dfrac{15 \times 16 \times 17}{2.3} = 680$.

2. Find the number of shot in an incomplete triangular pile of 15 courses, having 21 shot in the upper course.

Since $\dfrac{n(n+1)}{2}$ denotes the number of shot in a triangular course having n shot in the side, let

$$\frac{n(n+1)}{2} = 21, \text{ or } n^2 + n = 42;$$

hence, by solving the quadratic, we get $n = 6$, and therefore 5 courses have been removed from the pile; hence, by the formula,

$$\frac{20 \times 21 \times 22}{2.3} - \frac{5 \times 6 \times 7}{2.3} = 1540 - 35 = 1505;$$

which is the number of shot in the incomplete pile.

3. Find the number of shot in a square pile of 15 courses. *Ans.* 1240.

4. Find the number of shot in an incomplete square pile of 15 courses, having 49 shot in the upper course. *Ans.* 3220.

5. How many shot are in a rectangular pile, having 40 shot in the longer side, and 15 in the shorter side of the base course. *Ans.* 4240.

6. How many shot are in an incomplete rectangular pile of 8 courses, having 36 shot in the longer side, and 17 in the shorter side of the upper course. *Ans.* 6520.

7. Find the number of shot in an incomplete rectangular pile of 7 courses, having 7 shot in the shorter side of the upper course and 38 in the longer side of the bottom course. *Ans.* 2478.

8. The number of shot in the base and top courses of a square pile are 1521 and 169 respectively; how many are in the incomplete pile? *Ans.* 19890.

9. An incomplete rectangular pile has 13 courses left; the number of shot in the longer side of the base course is to the number in its shorter side as 7 to 5, and in the top course as 3 to 1. Find the number of shot in the original complete pile. *Ans.* 1960.

10. How many shot are required to complete a rectangular pile when the difference between the number of shot in the two sides of its base is 7 and the number of shot in the longer side of its upper course is 15? *Ans.* 336.

11. If p denote the number of courses in an incomplete rectangular pile, and $3h$ and h the number of shot in the longer and shorter sides of the upper course, show that the number of shot in the incomplete pile is

$$\frac{p\{2(p+3h)^2 - 3(p+4h) + 1\}}{6}.$$

NATURE AND SOLUTION OF EQUATIONS OF ALL DEGREES.

I. *Formation of Equations and Properties of their Roots.*

125. Every equation containing only one unknown quantity is either of the form

$$x^n + p x^{n-1} + q x^{n-2} + \ . \ . \ . \ . \ + t x + u = 0 \ . \ . \ . \ . \ (1),$$

where n is a whole positive number, or if it be not of this form, it can easily be reduced to it by fundamental operations. Let X denote the first member of eq. (1); and if X or $x^n + p x^{n-1} +$ etc., be divided by $x - a$, the quotient will be

$$x^{n-1} + (a + p) x^{n-2} + (a^2 + p a + q) x^{n-3} + \text{etc.} \ . \ . \ . \ . \ (2),$$

and the remainder

$$a^n + p a^{n-1} + q a^{n-2} + \ . \ . \ . \ . \ + t a + u \ . \ . \ . \ . \ (3),$$

will be precisely of the form of the original polynomial X, the only change being that a is substituted for x. The result thus far is entirely independent of the value of a, but if a be a root of the equation (1) then by Art. 83, the remainder (3) is equal to zero, and the original equation (1) is divisible by $x - a$ without remainder, the quotient (2) being a polynomial of the degree $n - 1$. We have therefore

$$\text{X} = (x - a)\{x^{n-1} + (a + p) x^{n-2} + \text{etc.}\} = 0 \ . \ . \ . \ . \ (4),$$

and this equation is evidently satisfied if either

$$x - a = 0 \text{ or } x^{n-1} + (a + p) x^{n-2} + \text{etc.} = 0.$$

Again, let the equation $x^{n-1} + (a + p) x^{n-2} + \text{etc.} = 0$ be divided by $x - b$; then if b is a root of this equation, it may be shown as before that the remainder will be zero, and the quotient will be of the degree $n - 2$; consequently

$$x^{n-1} + (a + p) x^{n-2} + \text{etc.} = (x - b)\,(x^{n-2} + \text{etc.}).$$

Substituting this in equation (4) we get

$$\text{X} = (x - a)\,(x - b)\,(x^{n-2} + \text{etc.}) = 0.$$

The proposed equation will consequently be satisfied if

$$x - a = 0\,;\ x - b = 0, \text{ or } x^{n-2} + \text{etc.} = 0.$$

Proceeding in this manner till the powers of x are exhausted, it will be seen that the original equation is divisible successively by $x - a$, $x - b$, ... $x - l$, where $a, b, c,$ . . . . l are the roots of the equation; and as the last of these divisors, and the last of the equations will involve only the single powers of x, the equation will admit of no further division. Hence

$$\text{X} = (x - a)\,(x - b)\,(x - c)\,(x - d) \ . \ . \ . \ . \ (x - l),$$

and the first member of equation (1) is resolved into n factors of the first degree. The original equation X $= 0$ will be satisfied, if any of the factors $x - a$, $x - b$, etc., be equal to zero, and in no other case. Hence each of the n quantities $a, b, c,$ . . . . l, when substituted for x in equation (1) will verify that equation, and it therefore follows that an equation has necessarily as many roots as there are units in the index of the highest power of the unknown quantity, and that it can have no greater number.

Conversely, if the roots of an equation be given, the equation will be

formed by the multiplication of certain factors obtained by connecting the roots severally, with their signs changed, to the unknown quantity. Thus if $x - a$, $x - b$, $x - c$, $x - d$, etc., be the factors, then taking only one of these as $x - a$, and putting it equal to 0, we get the simple equation

$$x - a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Take the product of the two factors $x - a$, $x - b$; and we get in like manner the quadratic equation

$$x^2 - (a + b)\,x + a\,b = 0 \quad . \quad . \quad . \quad . \quad (2).$$

The product of three factors $x - a$, $x - b$, $x - c$, will give the *cubic* equation

$$x^3 - (a + b + c)\,x^2 + (a\,b + a\,c + b\,c)\,x - a\,b\,c = 0 \quad . \quad . \quad (3).$$

And the product of four factors $x - a$, $x - b$, $x - c$, $x - d$, will give the *biquadratic* equation

$$x^4 - (a + b + c + d)\,x^3 + (a\,b + a\,c + a\,d + b\,c + b\,d + c\,d)\,x^2$$
$$- (a\,b\,c + a\,b\,d + a\,c\,d + b\,c\,d)\,x + a\,b\,c\,d = 0 \quad . \quad . \quad (4),$$

and so on to any extent.

The quantities a, b, c, d are called *roots* of the equation, a simple equation having only *one* root, a quadratic equation *two* roots, a cubic equation *three* roots, a biquadratic equation *four* roots, and generally an equation of the n^{th} degree has n roots.

If one of the quantities a, b, c, d, etc. be of the form $h + \sqrt{k}$; then another of them must be of the form $h - \sqrt{k}$; because the product of the two factors $x - (h + \sqrt{k})$ and $x - (h - \sqrt{k})$, viz., $x^2 - 2\,h\,x + h^2 - k$ does not involve either a surd or an imaginary quantity; hence surd and imaginary roots must occur in pairs. If k is positive, the two roots $h + \sqrt{k}$ and $h - \sqrt{k}$ will be real, and if k is negative, they will be imaginary.

From this method of formation of an equation we may deduce the following useful relations between the roots of the equation and the coefficients of the several terms.

(1). The coefficient of the second term is equal to the sum of the roots with their signs changed.

(2). The coefficient of the third term is equal to the sum of the products of the roots, taken two together, with their signs changed.

(3). The coefficient of the fourth term is equal to the sum of the products of the roots, taken three together, with their signs changed, and so on.

(4). The last term is equal to the product of all the roots with their signs changed.

Hence we see that

(α). If the second term of an equation is wanting, the sum of the positive roots is equal to the sum of the negative roots.

(β). If the signs of the terms of an equation be all positive, the roots will be negative; and if the signs be alternately positive and negative, the roots will be positive.

(γ). Every root of an equation is a divisor of the last term.

EXAMPLES.

1. Form the cubic equation whose roots are 1, 2, and -5.

Ans. $x^3 + 2\,x^2 - 13\,x + 10 = 0$.

3. Find the equation whose roots shall be *less* by 4 than those of the equation $x^3 - 15x^2 + 81x - 243 = 0$.

Ans. $y^3 - 3y^2 + 9y - 95 = 0$.

4. Find the equation whose roots shall be *greater* by 1 than those of the equation $x^4 - 4x^3 - 8x + 32 = 0$.

Ans. $y^4 - 8y^3 + 18y^2 - 24y + 45 = 0$.

5. Find the equation whose roots shall be *less* by $\cdot 3$ than those of the equation $x^4 + 9x^3 + 24x^2 + 18x - 11 = 0$.

Ans. $y^4 + 10\cdot 2y^3 + 32\cdot 64y^2 + 34\cdot 938y - 3\cdot 1889 = 0$.

III. *Solution of Equations.*

127. We have seen (125, γ) that every root of an equation is a factor of the last term, and therefore, in the case of integral roots, these may often be easily found by trial. Thus, in the equation

$$x^3 + 3x^2 + 9x - 38 = 0$$

the integral factors of the last term are $\pm 1, \pm 2, \pm 19, \pm 38$. It is easily seen that neither $+1$ nor -1 will verify the equation; but trying $+2$, it is found to succeed;

$$\text{for } 2^3 + 3 \times 2^2 + 9 \times 2 - 38 = 0.$$

But this is best effected by dividing the first side of the equation by $x - 2$, in the following manner, where the synthetic mode is adopted:

$$\begin{array}{rrrrl} 1 & +\ 3 & +\ \ 9 & -\ 38 & (2 \\ & 2 & +\ 10 & +\ 38 & \\ \hline 1 & +\ 5 & +\ 19 & +\ \ 0 & \end{array}$$

The remainder being 0, shows that 2 is a root of the equation, and also that the numbers 1, 5, 19 are the coefficients of the depressed equation containing the remaining roots.

For since 2 is a root of the equation, it is evident that $x - 2$ is one of the factors of the expression $x^3 + 3x^2 + 9x - 38$, and performing the division of this last expression by $x - 2$, either by the common process or the synthetic process above, we get

$$(x - 2)(x^2 + 5x + 19) = x^3 + 3x^2 + 9x - 38;$$

$$\therefore\ (x-2)(x^2 + 5x + 19) = 0.$$

To satisfy this equation we must have either $x - 2 = 0$, or $x^2 + 5x + 19 = 0$; the former gives $x = 2$, the root already found, and the quadratic $x^2 + 5x + 19 = 0$, will give the two other roots, which in this case are found to be imaginary. Hence if one root of an equation be known, the equation may be depressed to one of the next lower degree.

Examples.

1. If $x^3 - 12x^2 + 4x + 207 = 0$, what are the values of x?

Ans. $9, \dfrac{3 \pm \sqrt{101}}{2}$.

2. If $x^3 + 3x^2 - 6x - 8 = 0$, what are the values of x?

Ans. $2, -1$, or -4.

3. Given $x^3 + 9x = 1430$, to find the values of x.

Ans. $11, \dfrac{-11 \pm \sqrt{-399}}{2}$.

4. The number of shot in a complete triangular pile is 9139; find how many shot are in the bottom course. *Ans.* 703.

5. Find the three cube roots of 1, or solve the equation $x^3 - 1 = 0$.

Ans. $1, \tfrac{1}{2}(-1 + \sqrt{-3}), \tfrac{1}{2}(-1 - \sqrt{-3})$.

128. When the roots of an equation are incommensurable, and their initial figures have been found by trial, their values may be obtained to any degree of accuracy by the method of approximation devised by the late Mr. Horner, of Bath. This method is much the best that has ever appeared, and its principle consists simply in diminishing a root of the equation by its first figure, and repeating the process on each transformed equation, till the root be obtained to any extent.

Examples.

1. Given $x^3 + 9x - 16 = 0$, to find a near value of x.

Here the value of x is greater than 1, but less than 2, and the operation is as follows:

1 + 0	+ 9	−16 (1·4435452577
1	1	10
1	10	−6000
1	2	5344
2	1200	−656000
1	136	601984
30	1336	−54016000
4	152	45701307
34	148800	−8314693
4	1696	7624456
38	150496	−690237
4	1712	610050
420	15220800	−80187
4	12969	76257
424	15233769	−3930
4	12978	3050
428	$1524674\vert 7$	−880
4	$216\vert 4$	762
4320	$1524891\vert 1$	−118
3	$216\vert 4$	107
4323	$152510\vert 7$	−11
3	$1\vert 7$	11
4326	$152512\vert 4$	
3	$1\vert 7$	
$\vert 43\vert 29$	$1,5,2,5,1\vert 4$	

The coefficients of the first transformed equation whose roots are each less by 1 than those of the given equation are the numbers below

the first *dark* lines in each column, without the ciphers, and the equation itself is

$$y^3 + 3y^2 + 12y - 6 = 0 \;.\;.\;.\;.\;.\; (1).$$

The value of y in this equation is greater than $\cdot 4$, but less than $\cdot 5$, and in finding this value we are assisted by the equation $12y - 6 = 0$, as it gives $y = \frac{6}{12} = \cdot 5$, which is rather too great. Now, to avoid decimals, we may suppose the roots of (1) to be increased 10 times, by writing $\frac{z}{10}$ for y, and then clearing the equation of fractions; thus we get the equation

$$z^3 + 30z^2 + 1200z - 6000 = 0 \;.\;.\; (1'),$$

whose coefficients correspond exactly with the numbers below the first dark lines already referred to. The roots of this equation are now diminished by 4 as before, and this number is accounted as tenths in the root, because the roots were increased ten-fold. The coefficients of the second transformed equation are the numbers below the second dark lines in each column, and the equation itself is

$$v^3 + 4\cdot 2v^2 + 14\cdot 88v - \cdot 656 = 0 \;.\;.\;.\;.\; (2).$$

To find the value of v in this equation, take $14\cdot 88v - \cdot 656 = 0$; then we get $v = \frac{\cdot 656}{14\cdot 88} = \frac{656}{14880} = \cdot 04$, and thus the next figure in the root is 4. Proceeding with this figure as before, we effect a third transformation, and so on till the root be obtained to any extent. The numbers under the dark lines in the column next to that on the right speedily become available for determining the successive figures in the root, since the addends have little effect on the figures towards the left of these numbers, and to avoid unnecessary trouble in writing figures, the principle of contraction has been employed. Instead of annexing ciphers, cut off one figure from the last column but one, and two from the preceding column, which is equivalent to cutting off three figures from each column, and operate with the numbers on the left in the usual manner, taking care to allow for the units to be carried from the figures cut off.

129. The three roots of any cubic equation may be obtained very simply in the following manner:

Let $\alpha + \sqrt{\beta}$, $\alpha - \sqrt{\beta}$ and r denote the three roots of a cubic equation, then if the equation whose roots are r, $\alpha + \sqrt{\beta}$ and $\alpha - \sqrt{\beta}$ be formed by (125) it will be

$$x^3 - (2\alpha + r)x^2 + (\alpha^2 + 2r\alpha - \beta)x - r(\alpha^2 - \beta) = 0 \;.\;.\; (1).$$

Now reduce the roots of this equation by r; then we have the following process:—

$$\begin{array}{rrl}
1 \quad -(2\alpha + r) & +(\alpha^2 + 2r\alpha - \beta) & -r(\alpha^2 - \beta)\;(r \\
r & -2r\alpha & +r(\alpha^2 - \beta) \\
\hline
-2\alpha & \alpha^2 - \beta & \\
r & -2r\alpha + r^2 & \\
\hline
-2\alpha + r & \alpha^2 - 2r\alpha + r^2 - \beta & \\
r & 3\alpha^2 + 6r\alpha - \quad 3\beta & = \text{three times the coefficient of } x \\
\hline
2)\,\overline{-2\alpha + 2r} & (2\alpha + r)^2 - 4\beta & \\
-\quad \alpha + \quad r & -(2\alpha + r)^2 & = \text{the square of the coefficient} \\
\hline
-\quad r & 4)\,\overline{-4\beta} & \qquad \text{of } x^2 \\
\hline
-\quad \alpha & -\quad \beta &
\end{array}$$

Hence if the proposed cubic equation be $x^3 + a x^2 + b x + c = 0$, the type of solution will be as in the margin, where the values of α and β are obtained from the coefficients of the transformed equation whose roots are less by r than those of the proposed cubic, and r, α and β being thus found, the three roots of the equation are readily determined. When the value of β is positive, the roots are all real; and when it is negative, two of the roots are impossible. In the following example the additions are performed mentally, and the results alone are written down.

$1 + a$	$+\ b\ +$	$c\ (r$
r	$r a'$	$r b'$
a'	b'	0
r	$r a''$	
a''	b''	
r	$3 b$	
$2)\ a'''$	$\dots$	
$\dots$	$- a^2$	
$- r$	$4)\ -4\beta$	
$- \alpha$	$- \beta$	

Example.

Given $x^3 + 10 x^2 + 5 x - 2600 = 0$ to find all the roots.

1 + 10	+5	−2600(11·00679933972
21	236	2596
32	588	4000000000
43006	588258036	3529548216
43012	588516108	470451784
430187	58854622109	411982355
430194	58857633467	58469429
4302019	5885802064871	52972219
4302028	5885840783123	5497210
43020379	588584465495711	5297260
43020388	588584852679203	199950
430203973	588584865585322	176575
430203976	588584878491441	23375
4302039793	588584879782053	17657
4302039796	588584881072665	5718
43020397999	588584881459849	5297
43020398008	588584881847033	421
430203980177	588584881877147	412
430203980184	588584881907261	9
2)43·02039801916	588·584881908981	
21·51019900958	15	
− 11·00679933972	603·584881908981	
α = − 10·50339966986	100	
	4)503·584881908981	
	125·896220477245 = $-\beta$	

The three roots of the given cubic equation are thus found to be

11·00679933972,
$- 10·50339966986 + \sqrt{- 125·896220477245}$,
$- 10·50339966986 - \sqrt{- 125·896220477245}$.

Determine the roots of the following equations:

2. $x^2 + 10 x = 100$. *Ans.* $x = 6·18034$, $x = - 16·18034$.
3. $x^3 - 2 x = 5$. *Ans.* $x = 2·09455148$.
4. $x^3 + 2 x = 30$. *Ans.* $x = 2·89304$.
5. $x^3 + x^2 + x = 90$. *Ans.* $x = 4·10283$.

6. $x^3 + 9x^2 + 4x = 80.$ *Ans.* $x = 2\cdot47214.$
7. $x + \sqrt[3]{(x-5)} = 10.$ *Ans.* $x = 8\cdot4840198.$
8. $x^4 - 2x^3 - 3x^2 - 4x = -5.$
Ans. $x = 3\cdot182478, x = \cdot728727.$

SOLUTION OF EQUATIONS BY DOUBLE POSITION.

130. The roots of equations of all degrees may be determined to any degree of accuracy by the method of Double Position. It is equally applicable to all forms of equations.

Let $x^3 + ax^2 + bx = c$ be an equation of the third degree, and let s and s' be any two near values of x, found by trials; then if s and s' be substituted for x in the given equation, we shall have

$$x^3 + ax^2 + bx = c \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$s^3 + as^2 + bs = c' \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$s'^3 + as'^2 + bs' = c'' \quad . \quad . \quad . \quad . \quad (3).$$

Subtract (3) from (2) and (2) from (1); then we get

$$(s^3 - s'^3) + a(s^2 - s'^2) + b(s - s') = c' - c'',$$
$$(x^3 - s^3) + a(x^2 - s^2) + b(x - s) = c - c',$$

or, $$(s - s')\{s^2 + ss' + s'^2 + a(s + s') + b\} = c' - c'' \ldots (4),$$
$$(x - s)\{x^2 + xs + s^2 + a(x + s) + b\} = c - c' \ldots (5).$$

Now, since the values of x, s, and s' are nearly equal to each other, the bracketted expressions in (4) and (5) may be considered nearly equal to each other, and dividing (5) by (4) on this supposition, we get

$$\frac{x - s}{s - s'} = \frac{c - c'}{c' - c''}, \text{ or } x - s = \frac{s - s'}{c' - c''}(c - c') \ldots (6).$$

This expression affords the following rule:

Find by trials two numbers nearly equal to the root required, and substitute them for the unknown quantity in the given equation, noting the results that arise from each substitution. Then the difference of these results $(c' - c'')$ *is to the difference of the assumed numbers* $(s - s')$, *as the difference between the true result given by the equation and either of the former* $(c - c')$ *to the correction of the number belonging to the result used* $(x - s)$. *Apply this correction to the assumed number, according as that number is too little or too great, and an approximate value of the root will be obtained.*

With this value and the nearer of the two former values, or with any other values that appear to be more accurate, repeat the operation, and a second approximation to the true value will be obtained; and so on.

EXAMPLES.

1. Given $x^3 + 3x^2 = 500$, to find an approximate value of x.

By trials it is found that the value of x is greater than 7, but less than $7\cdot1$. Take these as the assumed numbers; then

(7)	(x)	(7·1)
343	x^3	357·911
147	$3x^2$	151·23
490	results	509·141
509·141	7·1	509·141
490	7	500

Then $19\cdot141 : \cdot1 :: 9\cdot141 : \cdot047;$

whence $x = 7 \cdot 1 - \cdot 047 = 7 \cdot 053$, which is the first approximate value of x.

Taking $7 \cdot 05$ and $7 \cdot 06$ for the assumed numbers, then

(7·05)	. . . . (x)	(7·06)
350·402625	. . . x^3	351·895816
149·1075	. . . $3x^2$ . . .	149·5308
499·510125	results	501·426616
501·426616	7·06	500
499·510125	7·05	499·510125
Then 1·916491 :	·01 ::	·489875 : ·00255;

whence $x = 7 \cdot 05 + \cdot 00255 = 7 \cdot 05255$, which is a near value of x, and correct as far as the last place of decimals inclusive.

Let the root thus found be denoted by r, then $x = r$, and if this value be substituted for x in the given equation, we have

$$x^3 + 3x^2 = 500 \quad . \; . \; . \; . \quad (1), \qquad r^3 + 3r^2 = 500 \quad . \; . \; . \; . \quad (2).$$

Subtracting (2) from (1) gives

$$(x^3 - r^3) + 3(x^2 - r^2) = 0,$$

or $$(x - r)(x^2 + xr + r^2) + 3(x - r)(x + r) = 0;$$

$$\therefore (x - r)\{x^2 + (r + 3)x + r(r + 3)\} = 0 \quad . \; . \; . \; . \quad (3).$$

Now this equation is fulfilled by making

$$x^2 + (r + 3)x + r(r + 3) = 0;$$

hence, if in the equation

$$x^2 + (r + 3)x = -r(r + 3) \quad . \; . \; . \; . \quad (4)$$

we substitute the value of r, viz., $7 \cdot 05255$, the values of the remaining roots may be found.

Find the roots of the following equations:

2. $x^3 + 10x^2 + 5x = 2600$. *Ans.* $x = 11 \cdot 00679$.
3. $x^3 + x^2 + x = 100$. *Ans.* $x = 4 \cdot 26443$.
4. $2x^3 + 3x^2 - 4x = 10$. *Ans.* $x = 1 \cdot 62482$.
5. $x^4 - x^3 + 2x^2 + x = 4$. *Ans.* $x = 1 \cdot 14699$.
6. $x^4 + x^3 + 2x^2 - x = 4$. *Ans.* $x = 1 \cdot 09059$.
7. $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$. *Ans.* $x = 8 \cdot 41445$.
8. $\sqrt[3]{7x^3 + 4x^2} + \sqrt{10x(2x - 1)} = 28$. *Ans.* $x = 4 \cdot 51066$.

CUBIC EQUATIONS.-CARDAN'S METHOD OF SOLUTION.

131. Let the general form of a cubic equation $x^3 + px^2 + qx + r = 0$ be transformed into the form $x^3 + ax + b = 0$, where the second power of the unknown quantity is absent; then if $x = y + z$, we get, by substitution,

$$(y + z)^3 + a(y + z) + b = 0,$$

or $$y^3 + 3yz(y + z) + z^3 + a(y + z) + b = 0.$$

Let $3yz = -a$; then the last equation becomes $y^3 + z^3 = -b$; whence $y^6 + 2y^3z^3 + z^6 = b^2$;

but $$4y^3z^3 \quad = -\frac{4a^3}{27}, \text{ since } yz = -\frac{a}{3};$$

$$\therefore y^6 - 2y^3z^3 + z^6 = b^2 + \frac{4a^3}{27} = 4\left(\frac{b^2}{4} + \frac{a^3}{27}\right).$$

The square root being taken, gives

$$y^3 - z^3 = 2\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}.$$

Combining this equation with $y^3 + z^3 = -b$, by addition and subtraction, we get

$$y^3 = -\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}};\ z^3 = -\frac{b}{2} - \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}};$$

$$\therefore x = y + z = \left\{-\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}} - \left\{\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}},$$

or since $z = -\frac{\frac{1}{3}a}{y}$, we have

$$x = \left\{-\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}} - \frac{\frac{1}{3}a}{\left\{-\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}}.$$

Ex. Given $x^3 - 6x = 12$, to find the value of x.
Here $a = -6$ and $b = -12$; hence by the formula

$$x = \sqrt[3]{\{6 + \sqrt{(36 - 8)}\}} + \frac{2}{\sqrt[3]{\{6 + \sqrt{(36 - 8)}\}}}$$

$$= \sqrt[3]{(6 + 2\sqrt{7})} + \frac{2}{\sqrt[3]{(6 + 2\sqrt{7})}}$$

$$= 2\cdot24345 + \frac{2}{2\cdot24345} = 2\cdot24345 + \cdot89148 = 3\cdot13493.$$

This method of solution applies only to those cubic equations which have one real root and two impossible ones.

For let $h + \sqrt{k}$, $h - \sqrt{k}$, and $-2h$ be the three roots of a cubic equation; then (125) that equation will be $x^3 - (3h^2 + k)x + 2h(h^2 - k) = 0$. Now if we substitute $-(3h^2 + k)$ for a and $2h(h^2 - k)$ for b in the expression $\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}$ which occurs in both terms of the value of x, we get

$$\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}} = \sqrt{\left(-3h^4k + \frac{2}{3}h^2k^2 - \frac{1}{27}k^3\right)}$$

$$= \sqrt{\left\{\left(h^4 - \frac{2}{9}h^2k + \frac{1}{81}k^2\right) \times -3k\right\}}$$

$$= \sqrt{\left\{\left(h^2 - \frac{1}{9}k^2\right)^2 \times -3k\right\}} = \left(h^2 - \frac{1}{9}k^2\right)\sqrt{(-3k)}.$$

Now this expression is always *impossible* when k is *positive*, and when k is positive, the roots $h + \sqrt{k}$ and $h - \sqrt{k}$ are *real;* and on the contrary, when k is *negative*, the radical value above is real, and the roots $h + \sqrt{k}$ and $h - \sqrt{k}$ are *impossible*. Cardan's method of solution is hence useless in practice when the roots are all real, as they cannot be computed by means of it, on account of the occurrence of an imaginary or impossible quantity, and it is only when the cubic equation has one real root that it is applicable.

BIQUADRATIC EQUATIONS.—SIMPSON'S METHOD OF SOLUTION.

132. Let the general form of a biquadratic equation $x^4 + p x^3 + q x^2 + r x + s = 0$ be transformed into the form $x^4 + a x^2 + b x + c = 0$, where the third power of the unknown quantity is absent; then if we assume

$$x^4 + a x^2 + b x + c = (x^2 + y)^2 - (h x + k)^2$$
$$= x^4 + (2 y - h^2) x^2 - 2 h k x + y^2 - k^2,$$

and equate the coefficients of the same powers of x on both sides, then

$$2 y - h^2 = a, \text{ or } \quad h^2 = 2 y - a \quad . \quad . \quad . \quad . \quad (1)$$
$$- 2 h k = b, \text{ or } 2 h k = - b \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$y^2 - k^2 = c, \text{ or } \quad k^2 = y^2 - c \quad . \quad . \quad . \quad . \quad (3).$$

Now it is obvious that the square of (2) is equal to four times the product of (1) and (3); hence

$$4 (2 y - a) (y^2 - c) = b^2, \text{ or } y^3 - \frac{1}{2} a y^2 - c y = \frac{1}{8} (b^2 - 4 a c).$$

Remove the second term of this cubic, and find the value of y by the previous method; hence $h = \pm \sqrt{(2 y - a)}$ and $k = \mp \sqrt{(y^2 - c)}$ will be known, and they must have opposite signs, for $2 h k$ is a negative quantity by (2). Then the first side $x^4 + a x^2 + b x + c$ of the first equation being equal to 0, we have

$$(x^2 + y)^2 = (h x + k)^2, \text{ or } x^2 + y = \pm (h x + k);$$

hence, since y, h, and k are all known, the solution of the quadratics

$$x^2 - h x + y - k = 0, \text{ and } x^2 + h x + y + k = 0$$

will afford the four values of x.

These methods may be applied to any of the examples which have already been given in (129) and (130).

INDETERMINATE ANALYSIS.

133. We have seen that, when the conditions of a question furnish as many independent and consistent equations as there are unknown quantities to be determined, the values of these unknowns may be found; but if the number of unknown quantities exceeds the number of equations, the question will admit of various solutions, and it is hence said to be ***indeterminate***. If integer values only are required, the number of solutions will be greatly restricted; and if negative values be excluded, the number of solutions in some cases may be very limited, while in others the number of positive integer solutions is ***unlimited***.

Indeterminate analysis of the first degree.

Indeterminate equations of the first degree are of the form

$$a x + b y = c, \text{ or } a x + b y + c z = d,$$

where a, b, c, d are given whole numbers, and x, y, z are limited to positive integer values. When the formula $a x \pm b y = c$ admits of integer values of x and y, the coefficients a and b cannot have a common divisor which is not also a divisor of c. For if $a = m d$, and $b = m e$, then

$$a x \pm b y = m (d x \pm e y) = c, \text{ hence } d x \pm e y = \frac{c}{m};$$

but d, e, x, y are all integers; therefore $\frac{c}{m}$ must be an integer, and m is a divisor of c.

If a and b are prime to each other, the solution of an equation of the form $a x - b y = \pm c$ is always possible, and the number of solutions

is unlimited, but if the equation be of the form $ax + by = c$, the number of solutions is always limited, and in some cases no solution can be obtained.

Let $ax - by = c$ be the equation in which a is less than b; then we have

$$x = \frac{by + c}{a} = my \pm n \pm \frac{dy \pm e}{a};$$

where my and n are the *nearest* quotients, whether in excess or defect, arising from dividing by and c by a; hence d and e are each less than $\frac{1}{2}a$. Let the last fraction, which must obviously be a whole number, be put $= v$, then we have $dy \pm e = av$, and as before

$$y = \frac{av \mp e}{d} = m'v \pm n' \pm \frac{fv \pm g}{d},$$

where f and g are each less than $\frac{1}{2}d$. Put the last fractional expression, which must also be a whole number, $= v_1$, and repeat the process until we arrive at a fractional expression in which the coefficient of the subsidiary unknown quantity v_n is unity. Let this fraction $= p$, then we have an expression of the form, $v_n = hp \pm k$; and by reversing the steps, we get in succession the values of the subsidiary unknown quantities which have been employed, and finally those of x and y.

Examples.

1. Given $7x + 12y = 50$, to find the values of x and y in whole numbers.

Here the smaller coefficient is 7, and therefore we have

$$x = \frac{50 - 12y}{7} = 7 - 2y + \frac{2y + 1}{7};$$

let $\frac{2y+1}{7} = v$, then $2y = 7v - 1$, and consequently

$$y = \frac{7v - 1}{2} = 3v + \frac{v - 1}{2};$$

let $\frac{v-1}{2} = p$, then $v = 2p + 1$, and hence

$$y = 3v + p = 7p + 3;$$
$$x = 7 - 2y + v = 2 - 12p.$$

Consequently we see that if x is to be a whole positive number, the value of p is limited to zero; hence when $p = 0$, we have

$$x = 2 \text{ and } y = 3,$$

which are the only integer values of the two unknown quantities.

2. A person purchased between 50 and 60 horses and oxen; he paid 31 dollars for each horse, and 20 dollars for each ox, and he found that the oxen cost him seven dollars more than the horses; how many of each did he buy?

Let x denote the number of horses, and y the number of oxen; then

$$20y - 31x = 7;$$

hence we have $y = \frac{31x + 7}{20} = 2x - \frac{9x - 7}{20};$

let $\frac{9x - 7}{20} = v$; then will $9x = 20v + 7$; hence

$$x = \frac{20v + 7}{9} = 2v + 1 + \frac{2v - 2}{9} = 2v + 1 + 2 . \frac{v - 1}{9};$$

let $\frac{v-1}{9} = p$, then $v = 9p + 1$, and consequently

$$x = 2v + 1 + 2p = 20p + 3,$$
$$\therefore y = 2x - v = 31p + 5.$$

In these expressions for x and y, we see that p may have all possible values from 0 to any extent; hence making $p = 0, p = 1, p = 2$, etc., we get $x = 3, 23, 43, 63, 83, 103$, etc.;
$y = 5, 36, 67, 98, 129, 160$, etc.;
therefore by the limitation in the question, we have $x = 23$ and $y = 36$, the respective numbers of horses and oxen that were purchased. The total number of solutions is unlimited.

3. How many ounces of gold, of 17 and 22 carats fine, must be mixed with 5 ounces of 18 carats fine, so that the composition may be 20 carats fine?

Let $x =$ the number of ounces of 17 carats fine, and $y =$ the number of 22 carats fine, then by the condition of the question we get

$$17x + 22y + 5 \times 18 = 20(x + y + 5);$$
$$\text{or } 2y = 3x + 10;$$

hence $y = x + 5 + \frac{x}{2}$, and if $\frac{x}{2} = p$, then will

$$x = 2p, \text{ and } y = x + 5 + p = 3p + 5.$$

If $p = 1, 2, 3, 4$, etc., then we get

$$x = 2, x = 4, x = 6, x = 8, \text{ etc.}$$
$$y = 8, y = 11, y = 14, y = 17, \text{ etc.}$$

134. If there be only one equation to determine three unknown quantities, as $ax + by + cz = d$; then by transposition, $ax + by = d - cz$, and by giving to z all its different integer values, we shall obtain in the usual manner all the corresponding values of x and y. The values of x and y cannot be less than unity; therefore the *highest* value of z cannot exceed the value derived from the equation

$$z = \frac{d - a - b}{c}.$$

Examples for Practice.

1. Given $14x = 5y + 7$, to find the least values of x and y.
Ans. $x = 3, y = 7$.

2. Given $27x + 16y = 1600$, to find the least values of x and y.
Ans. $x = 48, y = 19$.

3. Given $11x + 5y = 254$, to find all the possible values of x and y.
Ans. $x = 19, 14, 9, 4$; $y = 9, 20, 31, 42$.

4. In how many ways can $20l.$ be paid without using any other coin than half-guineas and half-crowns. *Ans.* In 7 different ways.

5. Given $x - 2y + z = 5$ and $2x + y - z = 7$, to find the values of x, y, z.

$$\textit{Ans.} \begin{cases} x = 5, \ 6, \ 7, \ 8 \ \ldots. \\ y = 3, \ 6, \ 9, 12, \ \ldots. \\ z = 6, 11, 16, 21, \ \ldots. \end{cases}$$

6. A person bought 100 animals for $100l.$, namely, oxen at $5l.$ each, sheep at $1l.$, and fowls at 1 shilling each; how many of each kind did he purchase? *Ans.* 19 oxen, 1 sheep, and 80 fowls.

7. I owe a person 3 shillings, and have nothing about me but guineas, and he has nothing but crowns; how must I discharge the debt?

Ans. I must give 3 guineas, and receive 12 crowns.

8. A jeweller requires to mix gold of 14, 11, and 9 carats fine, so as to make a composition of 20 ounces of 12 carats fine; find the quantities of each kind of gold to form the required mixture.

Ans. 8, 10, 2 ounces, or 10, 5, 5 ounces.

9. Given $5x + 7y + 11z = 224$, to find all the possible values of x, y, and z in whole numbers. *Ans.* The number of ways is 59.

INDETERMINATE COEFFICIENTS.

135. If two series of quantities be equal to each other, as

$$a + bx + cx^2 + dx^3 + \ldots = A + Bx + Cx^2 + Dx^3 + \ldots,$$

whatever be the value of x, and where the coefficients are independent of x; then will

$$A = a,\ B = b,\ C = c,\ D = d, \text{ etc.}$$

For by transposition we have

$$a - A = (B - b)\,x + (C - c)\,x^2 + (D - d)\,x^3 + \ldots;$$

and if a is not equal to A, then $a - A$ will be some constant quantity; but the second side of this equation varies as x varies, and may be made *less* than the fixed quantity $a - A$ by taking a sufficiently small value for x, which is absurd, and therefore we must have $a = A$. The proposed equation, then, separates itself into two equations, viz.,

$$A = a, \text{ and } bx + cx^2 + dx^3 + \ldots = Bx + Cx^2 + Dx^3 + \ldots$$

Dividing each side of the latter by x gives

$$b + cx + dx^2 + ex^3 + \ldots = B + Cx + Dx^2 + Ex^3 + \ldots,$$

which by similar reasoning separates into the two equations,

$$B = b, \text{ and } c + dx + ex^2 + \ldots = C + Dx + Ex^2 + \ldots,$$

and the latter gives, as before, $C = c$, and so on; hence we have

$$A = a,\ B = b,\ C = c,\ D = d, \text{ etc.}$$

Examples.

1. Required the development of $\sqrt{(a - x)}$ in a series.

$$\begin{array}{rl}
\text{Let } \sqrt{(a - x)} = & A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \\
\therefore\ \sqrt{(a - x)} = & A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \\
\hline
a - x = & A^2 + ABx + ACx^2 + ADx^3 + \ldots \\
& \quad + ABx + B^2x^2 + BCx^3 + \ldots \\
& \quad\quad + ACx^2 + BCx^3 + \ldots \\
& \quad\quad\quad + ADx^3 + \ldots
\end{array}$$

Hence $A^2 = a$, $2AB = -1$, $2AC + B^2 = 0$, $2AD + 2BC = 0$, etc.;

$$\therefore\ A = a^{\frac{1}{2}},\ B = -\frac{1}{2a^{\frac{1}{2}}},\ C = -\frac{B^2}{2A} = -\frac{1}{8a^{\frac{3}{2}}},$$

$$D = -\frac{BC}{A} = -\frac{1}{16a^{\frac{5}{2}}}, \text{ etc.};$$

and $$\sqrt{(a - x)} = a^{\frac{1}{2}} - \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} - \frac{x^3}{16a^{\frac{5}{2}}} - \text{etc.}$$

$$= a^{\frac{1}{2}}\left\{1 - \frac{x}{2a} - \frac{x^2}{2.4\,a^2} - \frac{1.3\,x^3}{2.4.6\,a^3} - \frac{1.3.5\,x^4}{2.4.6.8\,a^4} - \text{etc.}\right\}$$

2. Separate $\frac{2a^2}{a^2 - x^2}$ into two fractions whose denominators shall be $a + x$ and $a - x$.

Let $\frac{2a^2}{a^2 - x^2} = \frac{\text{A}}{a + x} + \frac{\text{B}}{a - x} = \frac{\text{A}a + \text{B}a - (\text{A} - \text{B})x}{a^2 - x^2}$;

$$\therefore\ 2a^2 = (\text{A} + \text{B})a - (\text{A} - \text{B})x,$$

and equating the coefficients of the like powers of x, we have

$$2a^2 = (\text{A} + \text{B})a, \text{ and } \text{A} - \text{B} = 0;$$

$$\therefore\ \text{A} = \text{B}, \text{ and } \text{A} + \text{B} = 2a, \text{ or } 2\text{B} = 2a; \text{ hence } \text{B} = \text{A} = a,$$

$$\therefore\ \frac{2a^2}{a^2 - x^2} = \frac{\text{A}}{a + x} + \frac{\text{B}}{a - x} = \frac{a}{a + x} + \frac{a}{a - x}.$$

3. Required the development of $\frac{a}{a - x}$ in a series.

Ans. $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \text{etc.}$

4. Required the development of $\sqrt{(a^2 + x^2)}$ in a series.

Ans. $a + \frac{x^2}{2a} - \frac{x^4}{2.4\,a^3} + \frac{1.3\,x^6}{2.4.6\,a^5} - \text{etc.}$

5. Separate $\frac{x^2 - x + 2}{(x - 1)(x - 2)(x - 3)}$ into three fractions.

Ans. $\frac{1}{x - 1} - \frac{4}{x - 2} + \frac{4}{x - 3}.$

6. Find the value of $\frac{1}{(a - x)^2}$ in a series.

Ans. $\frac{1}{a^2} + \frac{2x}{a^3} + \frac{3x^2}{a^4} + \frac{4x^3}{a^5} + \text{etc.}$

BINOMIAL THEOREM.

136. By actual multiplication we have $(1+x)^2 = 1 + 2x + x^2$; $(1+x)^3 = 1 + 3x + 3x^2 + x^3$; $(1 + x)^4 = 1 + 4x + \text{etc.}$, and hence generally we see that the form of the expansion of $(1 + x)^n$, when n is a positive integer, is

$$1 + nx + \text{A}x^2 + \text{B}x^3 + \text{C}x^4 + \ .\ .\ .\ .$$

where the coefficient of the second term is n, and the coefficients A, B, C, etc., are entirely independent of x.

But if the exponent be a positive fraction, as $\frac{m}{n}$, then we have

$$(1 + x)^{\frac{m}{n}} = \sqrt[n]{(1 + x)^m} = \sqrt[n]{(1 + mx + \text{A}'x^2 + \text{B}'x^3 + \text{C}'x^4 + \ldots)}.$$

And since $(1 + y)^n = 1 + ny + \text{A}y^2 + \text{B}y^3 + \text{C}y^4 + \ldots,$

if $\qquad y = ax + bx^2 + cx^3 + \ldots,$

$\therefore (1 + ax + bx^2 + cx^3 + \ldots)^n = 1 + nax + \text{A}_1x^2 + \text{B}_1x^3 + \ldots$

Hence, conversely, the n^{th} root of

$$1 + nax + \text{A}_1x^2 + \text{B}_1x^3 + \ldots = 1 + ax + bx^2 + cx^3 + \ldots$$

Let $na = m$, then will $a = \frac{m}{n}$, and therefore

the n^{th} root of $1 + mx + Px^2 + Qx^3 + \ldots$ is of the form

$$1 + \frac{m}{n}x + px^2 + qx^3 + \ldots$$

Again, if the exponent be negative, as $-n$, where n may be either integer or fractional, then we have

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \frac{1}{1 + nx + Ax^2 + Bx^3 + Cx^4 + \ldots}$$
$$= 1 - nx + P'x^2 + Q'x^3 + \ldots$$

by actual division; therefore generally the form of the expansion of

$$(1+x)^n = 1 + nx + Ax^2 + Bx^3 + Cx^4 + \ldots \quad (1).$$

Now if we substitute $x + y$ for x in equation (1) we have

$$(1 + x + y)^n = 1 + n(x+y) + A(x+y)^2 + B(x+y)^3 + \ldots$$
$$\left.\begin{array}{l} = 1 + nx + \quad Ax^2 + \quad Bx^3 + \quad Cx^4 + \ldots \\ \quad + ny + 2Axy + 3Bx^2y + 4Cx^3y + \ldots \\ \quad\quad + \quad Ay^2 + 3Bxy^2 + 6Cx^2y^2 + \ldots \\ \quad\quad\quad + \quad By^3 + 4Cxy^3 + \ldots \\ \quad\quad\quad\quad + \quad Cy^4 + \ldots \end{array}\right\} \ldots (2).$$

$$\text{But } (1 + x + y)^n = \{(1+y) + x\}^n = (1+y)^n\left\{1 + \frac{x}{1+y}\right\}^n$$
$$= (1+y)^n\left\{1 + n\frac{x}{1+y} + A\frac{x^2}{(1+y)^2} + B\frac{x^3}{(1+y)^3} + \ldots\right\}$$
$$= (1+y)^n + nx(1+y)^{n-1} + Ax^2(1+y)^{n-2} + Bx^3(1+y)^{n-3}$$
$$+ Cx^4(1+y)^{n-4} + \ldots$$
$$\left.\begin{array}{l} = 1 + ny + \qquad Ay^2 + \qquad By^3 + \qquad Cy^4 + \ldots \\ \quad + nx + n(n-1)xy + \quad nA_1xy^2 + \quad nB_1xy^3 + \ldots \\ \quad\quad + \qquad Ax^2 + A(n-2)x^2y + \quad AA_2x^2y^2 + \ldots \\ \quad\quad\quad + \qquad Bx^3 + B(n-3)x^3y + \ldots \\ \quad\quad\quad\quad + \qquad Cx^4 + \ldots \end{array}\right\} \ldots (3),$$

where A_1, B_1, etc., A_2, B_2, etc., are what A, B, etc. become when n is changed into $n-1$, $n-2$, etc. Now the series (2) and (3) must be identical; hence equating the coefficients of the same powers, or combinations of powers, of x and y, in both, we get

$$2A = n(n-1), \text{ and therefore } A = \frac{n(n-1)}{1.2}.$$

Hence, writing $n - 1$ for n, we get $A_1 = \dfrac{(n-1)(n-2)}{1.2}$; consequently the coefficients of xy^2 and x^2y in (3) are identical, each being $= \dfrac{n(n-1)(n-2)}{2}$. Comparing the coefficients of xy^2 and x^2y in both series, we get

$$3B = \frac{n(n-1)(n-2)}{1.2}, \text{ or } B = \frac{n(n-1)(n-2)}{1.2.3}.$$

Finding now the values of B_1 and A_2, we can find the coefficients of xy^3, x^2y^2, and x^3y in (3) to be respectively

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3},\quad \frac{n(n-1)(n-2)(n-3)}{1.2.2}, \text{ and}$$
$$\frac{n(n-1)(n-2)(n-3)}{1.2.3}.$$

Comparing either of these with the coefficients 4 c, 6 c, 4 c of the corresponding powers of x and y in (2), we get

$$C = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.$$

In this manner the law of the coefficients can be obtained, and we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots (4);$$

$$\therefore \left(1+\frac{x}{a}\right)^n = 1 + n\frac{x}{a} + \frac{n(n-1)}{1.2}\cdot\frac{x^2}{a^2} + \frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{x^3}{a^3} + \ldots;$$

$$\text{and } (a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x$$
$$+ \frac{n(n-1)(n-2)}{2.3}a^{n-3}x^3 + \ldots . (5).$$

Writing $-x$ for x in the last series, we have

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2$$
$$- \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \ldots . (6).$$

By a little consideration, it will be found that the p^{th} term of the expansion is

$$\pm \frac{n(n-1)(n-2)(n-3)\ldots\{n-(p-2)\}}{1.2.3.4\ldots(p-1)}a^{n-(p-1)}x^{p-1}\ldots(7).$$

EXAMPLES.

1. Expand $\sqrt{(a-x)}$ in a series by the binomial theorem.

Here $\quad (a-x)^{\frac{1}{2}} = \left\{a\left(1-\frac{x}{a}\right)\right\}^{\frac{1}{2}} = a^{\frac{1}{2}}\left(1-\frac{x}{a}\right)^{\frac{1}{2}}$, and

comparing $\left(1-\frac{x}{a}\right)^{\frac{1}{2}}$ with $(1+x)^n$, we have $x = -\frac{x}{a}$ and $n = \frac{1}{2}$;

therefore $\left(1-\frac{x}{a}\right)^{\frac{1}{2}}$

$$= 1 - \frac{1}{2}\cdot\frac{x}{a} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}\cdot\frac{x^2}{a^2} - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}\cdot\frac{x^3}{a^3} + \text{etc.}$$
$$= 1 - \frac{1}{2}\cdot\frac{x}{a} - \frac{1}{2.4}\cdot\frac{x^2}{a^2} - \frac{1.3}{2.4.6}\cdot\frac{x^3}{a^3} - \text{etc.};$$
$$\therefore (a-x)^{\frac{1}{2}} = a^{\frac{1}{2}}\left\{1 - \frac{1}{2}\cdot\frac{x}{a} - \frac{1}{2.4}\cdot\frac{x^2}{a^2} - \frac{1.3}{2.4.6}\cdot\frac{x^3}{a^3} - \frac{1.3.5}{2.4.6.8}\cdot\frac{x^4}{a^4} - \text{etc.}\right\}$$

2. Let it be required to expand $(a^2+x^2)^{\frac{3}{2}}$ in a series.

Ans. $a^3 + \frac{3}{2}ax^2 + \frac{3.1}{2.4}\cdot\frac{x^4}{a} - \frac{3.1.1}{2.4.6}\cdot\frac{x^6}{a^3} + \frac{3.1.1.3}{2.4.6.8}\cdot\frac{x^8}{a^5} - \text{etc.}$

3. Expand $\dfrac{a^3}{(a-x)^3}$ in a series.

$$Ans.\ 1+\frac{3x}{a}+\frac{3.4}{1.2}.\frac{x^2}{a^2}+\frac{3.4.5}{1.2.3}.\frac{x^3}{a^3}+\text{etc.}$$

4. Find the fifth term of the expansion of $(a^2-x^3)^{-\frac{3}{4}}$.

$$Ans.\ \frac{1155}{2048}.\frac{x^{12}}{a^{\frac{19}{2}}}.$$

5. Expand $(1+x)^{-\frac{1}{2}}$ in a series.

$$Ans.\ 1-\frac{1}{2}x+\frac{1.3}{2.4}x^2-\frac{1.3.5}{2.4.6}x^3+\text{etc.}$$

6. Expand $(3-7x)^{-\frac{1}{3}}$ in a series.

$$Ans.\ \frac{1}{\sqrt[3]{3}}\left\{1+\frac{7}{9}x+\frac{1.4}{3.6}\left(\frac{7}{3}\right)^2x^2+\frac{1.4.7}{3.6.9}\left(\frac{7}{3}\right)^3x^3+\text{etc.}\right\}$$

7. Expand $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$ in a series.

$$Ans.\ 1+\frac{x}{a}+\frac{1}{2}\frac{x^2}{a^2}+\frac{1}{2}.\frac{x^3}{a^3}+\frac{1.3}{2.4}.\frac{x^4}{a^4}+\frac{1.3}{2.4}.\frac{x^5}{a^5}+\text{etc.}$$

8. Expand $(a^2-ax)^{-\frac{3}{10}}$ in a series.

$$Ans.\ \frac{1}{a^{\frac{3}{5}}}\left\{1+\frac{3}{10}.\frac{x}{a}+\frac{3.13}{10.20}.\frac{x^2}{a^2}+\frac{3.13.23}{10.20.30}.\frac{x^3}{a^3}+\text{etc.}\right\}$$

EXPONENTIAL THEOREM.

137. Let it be required to develop a^x in a series.

Let $a=1+b$; then $b=a-1$, and we have $a^x=(1+b)^x$;

but $(1+b)^x=1+xb+\dfrac{x(x-1)}{1.2}b^2+\dfrac{x(x-1)(x-2)}{1.2.3}b^3+\ldots..$

$$=1+xb+\left(\frac{x^2}{2}-\frac{x}{2}\right)b^2+\left(\frac{x^3}{6}-\frac{x^2}{2}+\frac{x}{3}\right)b^3+\ldots..$$

$$=1+x\left(b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\ldots.\right)+Bx^2+Cx^3+Dx^4+\ldots..;$$

or $\quad a^x=1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots..;$

where $A=b-\dfrac{1}{2}b^2+\dfrac{1}{3}b^3-\dfrac{1}{4}b^4+\dfrac{1}{5}b^5-\dfrac{1}{6}b^6+\ldots..$

$$=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\frac{1}{5}(a-1)^5-\ldots$$

Now $\quad a^x=1+Ax+Bx^2+Cx^3+Dx^4+\ldots.,$

and writing $x+z$ for x gives

$$a^{x+z}=1+A(x+z)+B(x+z)^2+C(x+z)^3+\text{etc.}$$

$$\left.\begin{aligned}=1&+Ax+Bx^2+Cx^3+Dx^4+\text{etc.}\\&+Az+2Bxz+3Cx^2z+4Dx^3z+\text{etc.}\\&+Bz^2+3Cxz^2+6Dx^2z^2+\text{etc.}\end{aligned}\right\}\ldots.(1).$$
$$\text{etc.}$$

But $a^{x+z}=a^x\times a^z$

$$=(1+Ax+Bx^2+Cx^3+\ldots)\times(1+Az+Bz^2+Cz^3+\ldots),$$

$$\left.\begin{aligned}\text{or}\quad a^{x+z} = 1 &+ Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}\\ &+ Az + A^2xz + ABx^2z + ACx^3z + \text{etc.}\\ &+ Bz^2 + \ldots\ldots\ldots\ldots\end{aligned}\right\}\ldots\ldots(2).$$

Consequently, equating corresponding terms and coefficients in these two identical series, we get

$$2B = A^2,\ 3C = AB,\ 4D = AC,\ 5E = AD,\ \text{etc.}:$$

$$\text{hence}\quad B = \frac{A^2}{2},\ C = \frac{A^3}{2.3},\ D = \frac{A^4}{2.3.4},\ E = \frac{A^5}{2.3.4.5},\ \text{etc.};$$

$$\therefore a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \frac{A^4x^4}{1.2.3.4} + \text{etc.}\ldots\ldots(3);$$

$$\text{where } A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}$$

If e be that value of a which renders $A = 1$, then we have

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots\ldots$$

$$\text{and if } x = 1, \text{ then } e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots\ldots$$

$$= 2{\cdot}718281828\ldots$$

The theorem (3) will find its application presently in the calculation of logarithms.

NATURE AND PROPERTIES OF LOGARITHMS.

138. We have already seen in the Arithmetic that *logarithms are a series of numbers in arithmetical progression corresponding to another series in geometrical progression.* Thus in the two series

0,	1,	2,	3,	4,	5,	6,	etc.
1,	10,	100,	1000,	10000,	100000,	1000000,	etc.

the logarithm of 1 is 0, of 10 is 1, of 100 is 2, of 1000 is 3, and so on. But the best method of considering logarithms is derived from the following definition:—

A *logarithm* of a number is the index of the power to which a given quantity must be raised so as to be equal to that number. Thus in the equation $a^x = n$, x is the logarithm of n to the base a, and is usually written $x = \log_a n$.

A *system* of logarithms is a series of values of x corresponding to *different* values of n, the base a remaining the *same;* but since a may have *different* values, it is obvious that there may be as many systems of logarithms as we please.

In every system of logarithms the logarithm of its base is 1; for $a^1 = a$.

In every system of logarithms the logarithm of 1 is 0; for $a^0 = 1$.

Properties of Logarithms.

1. *The logarithm of the product of any number of factors is equal to the sum of the logarithms of those factors.*

Let $a^x = P$, $a^y = Q$, $a^z = R$ and $a^v = S$; then we have

$x = \log_a P$, $y = \log_a Q$, $z = \log_a R$, $v = \log_a S$.

But $a^x \times a^y \times a^z \times a^v = a^{x+y+z+v} = PQRS$;

$\therefore \log_a PQRS = x + y + z + v = \log_a P + \log_a Q + \log_a R + \log_a S.$

2. *The logarithm of a fractional quantity is equal to the excess of the logarithm of the numerator above the logarithm of the denominator.*

Let $a^x = N$ and $a^y = N'$; then $x = \log_a N$, $y = \log_a N'$.

But $\frac{N}{N'} = \frac{a^x}{a^y} = a^{x-y}$; consequently

$$\log_a \frac{N}{N'} = x - y = \log_a N - \log_a N'.$$

3. *The logarithm of a power of a quantity is equal to the logarithm of the quantity multiplied by the index of the power.*

Let $a^x = N$, then $x = \log_a N$, and $a^{nx} = N^n$; therefore

$$\log N^n = n\,x = n \log_a N.$$

4. *The logarithm of a root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

Let $a^x = N$, then $x = \log_a N$, and $a^{\frac{x}{n}} = N^{\frac{1}{n}}$; therefore

$$\log N^{\frac{1}{n}} = \frac{x}{n} = \frac{1}{n} \log_a N.$$

5. *The logarithms of any quantity in two different systems are inversely as the logarithms of their bases in any system.*

Let $a^x = N$, and $b^y = N$; then $x = \log_a N$ and $y = \log_b N$;
but since $a^x = b^y$, we have by (3), $x \log a = y \log b$;
$\therefore x : y :: \log b : \log a$, or $\log_a N : \log_b N :: \log b : \log a$;
where the two latter logs may be taken to any base whatever.

These properties of logarithms are of the highest importance, and the latter property will enable us to convert logarithms from one system to another with great facility.

Computation of Logarithms.

By the exponential theorem, we have

$$a^x = 1 + A\,x + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} + \frac{A^4 x^4}{1.2.3.4} + \ldots \quad . \quad . \quad (1),$$

$$a^{xz} = 1 + A\,x\,z + \frac{A^2 x^2 z^2}{1.2} + \frac{A^3 x^3 z^3}{1.2.3} + \frac{A^4 x^4 z^4}{1.2.3.4} + \ldots, \quad (2),$$

$$n^z = 1 + B\,z + \frac{B^2 z^2}{1.2} + \frac{B^3 z^3}{1.2.3} + \frac{B^4 z^4}{1.2.3.4} + \ldots; \quad . \quad . \quad (3);$$

where $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \ldots, \quad (4),$

and $\quad B = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \ldots \quad (5).$

Let $a^x = n$, then will $a^{xz} = n^z$; therefore the series in (2) and (3) ought to be identical; and hence $B = A\,x$, throughout the entire extent of the series. Consequently

$$x = \frac{B}{A} = \frac{1}{A}\left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \ldots\right\} (6).$$

In series (1) let $x = \frac{1}{A}$, then will

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots = e,$$

$$\therefore a^{\frac{1}{A}} = e \text{ or } a = e^A; \text{ hence } \frac{1}{A} = \log_a e, \text{ and } A = \log_e a,$$

$$\text{consequently } \log a \,.\, \log_a e = A \,.\, \frac{1}{A} = 1.$$

But since $a^x = n$, therefore also $x = \log_a n$, and (6) becomes

$$\log_a n = \frac{1}{\log_e a}\left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \ldots\right\} \quad (7).$$

The factor $\frac{1}{\log_e a}$ by which the series (7) is multiplied is independent of n, and is called the *modulus* of the system of logarithms whose base is a. In the same manner the respective moduli of the systems whose bases are b, e, and 10, are

$$\frac{1}{\log_e b},\ \frac{1}{\log_e e} = 1,\ \frac{1}{\log_e 10}.$$

The system whose base is e or 2·718281828 is called the Napierian system of logarithms, from the name of the inventor, Lord Napier, of Merchiston, and the system whose base is 10 is called the *common* system, because it is the system which is commonly, though not exclusively, employed in calculations.

If in (7) we write $x + 1$ for n, and put the modulus $\frac{1}{\log_e a} = M_a$, then

$$\log_a(1+x) = M_a\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \text{etc.}\right) \ldots (8).$$

$$\text{Sim}^y, \log_a(1-x) = M_a\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \ldots\right) \ldots (9).$$

Subtracting (9) from (8), and recollecting that

$$\log(1 + x) - \log(1 - x) = \log\frac{1+x}{1-x};$$

$$\therefore \log_a \frac{1+x}{1-x} = 2M_a\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \ldots\right) \ldots (10).$$

Let $\frac{1+x}{1-x} = \frac{n+1}{n}$, then we get $x = \frac{1}{2n+1}$, and (10) becomes

$$\log_a \frac{n+1}{n} = 2M_a\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \ldots\right\} \ldots (11).$$

But $\log\frac{n+1}{n} = \log(n+1) - \log n$; therefore

$$\log_a(n+1) = \log_a n + 2M_a\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \ldots\right\} \ldots (12).$$

Since $M_e = \frac{1}{\log_e e} = 1$, it is obvious that

$$\log_e(n+1) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \ldots\right\} \ldots (13).$$

If $n = 1$, $n = 2$, $n = 3$, etc., we shall have by (13)

$$\log_e 2 = \quad 2\left\{\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \frac{1}{7}\left(\frac{1}{3}\right)^7 + \ldots\right\} = \cdot 6931472$$

$$\log_e 3 = \log_e 2 + 2\left\{\frac{1}{5} + \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 + \frac{1}{7}\left(\frac{1}{5}\right)^7 + \ldots\right\} = 1\cdot 0986123$$

$$\log_e 4 = \log_e 2^2 = 2\log_e 2 = \quad . \quad . \quad . \quad . \quad = 1\cdot 3862944$$

$$\log_e 5 = \log_e 4 + 2\left\{\frac{1}{9} + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \frac{1}{5}\left(\frac{1}{9}\right)^5 + \frac{1}{7}\left(\frac{1}{9}\right)^7 + \ldots\right\} = 1\cdot 6094379$$

$$\log_e 6 = \log_e(2 \times 3) = \log_e 2 + \log_e 3 = \quad . \quad . \quad = 1\cdot 7917595$$

$$\log_e 7 = \log_e 6 + 2\left\{\frac{1}{13} + \frac{1}{3}\left(\frac{1}{13}\right)^3 + \frac{1}{5}\left(\frac{1}{13}\right)^5 + \frac{1}{7}\left(\frac{1}{13}\right)^7 + \ldots\right\} = 1\cdot9459101$$

$$\log_e 8 = \log_e 2^3 = 3\log_e 2 = \quad . \quad . \quad . \quad . \quad = 2\cdot0794415$$

$$\log_e 9 = \log_e 3^2 = 2\log_e 3 = \quad . \quad . \quad . \quad . \quad = 2\cdot1972246$$

$$\log_e 10 = \log_e (2 \times 5) = \log_e 2 + \log_e 5 = \quad . \quad . \quad = 2\cdot3025851$$

In this manner the Napierian logarithms of all numbers may be computed, and since $M_{10} = \dfrac{1}{\log_e 10} = \dfrac{1}{2\cdot3025851} = \cdot43429448$, the common logarithms of all numbers may be readily found.

139. The connexion between common and Napierian logarithms is seen from (138), where it has been shown (5) that if N be any number then

$$\log_a \text{N} : \log_b \text{N} :: \log_e b : \log_e a.$$

Let $a = 10$, and $b = e$; then will

$$\log_{10} \text{N} : \log_e \text{N} :: \log_e e : \log_e 10 :: 1 : 2\cdot3025851;$$

$$\therefore \log_{10} \text{N} = \frac{1}{2\cdot3025851} \log_e \text{N} = \cdot43429448 \log_e \text{N};$$

that is, the common $\log \text{N} = \cdot43429448 \times$ Napierian $\log \text{N}$,
the Napierian $\log \text{N} = 2\cdot3025851 \times$ common $\log \text{N}$.

We shall now work one example at length as a pattern for the computation of logarithms to any base.

Ex. Find the common logarithm of 2.

By the series (12) we have, when $n = 1$ and $a = 10$,

$$\log_{10} 2 = \cdot86858896\left\{\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \frac{1}{7}\left(\frac{1}{3}\right)^7 + \ldots\right\}$$

$$\begin{array}{lll}
3)\underline{\cdot86858896} & & \\
9)\underline{\cdot28952965} & \ldots\ldots\ldots\ldots & = \cdot28952965 \\
9)\underline{\cdot03216996} \div 3 & \ldots\ldots & = \cdot01072332 \\
9)\underline{\cdot00357444} \div 5 & \ldots\ldots & = \cdot00071489 \\
9)\underline{\cdot00039716} \div 7 & \ldots\ldots & = \cdot00005674 \\
9)\underline{\cdot00004413} \div 9 & \ldots\ldots & = \cdot00000490 \\
9)\underline{\cdot00000490} \div 11 & \ldots\ldots & = \cdot00000045 \\
\quad \cdot00000054 \div 13 & \ldots\ldots & = \underline{\cdot00000004} \\
 & \therefore \text{common logarithm of } 2 & = \cdot30102999
\end{array}$$

EXAMPLES.

1. Find the common logarithms of 5, 12, 18, 22·5 and ·00072, having given $\log_{10} 2 = \cdot3010300$ and $\log_{10} 3 = \cdot4771213$.

Ans. ·6989700, 1·0791812, 1·2552725, 1·3521825, and $\bar{4}\cdot8573325$.

2. Given $\log_{10} 3 = \cdot4771213$ and $\log_{10} 7 = \cdot8450980$, to find the logarithms of ·021 and 1470. *Ans.* $\bar{2}\cdot3222193$ and $3\cdot1673173$.

3. Calculate the common logarithm of 11. *Ans.* 1·0413927.

4. Given the logarithms of 2 and 3, as in Example 1, to find the logarithms of $\dfrac{9}{16}$ and $\dfrac{4}{375}$. *Ans.* $2\log 3 - 4\log 2 = \bar{1}\cdot7501225$;
and $5\log 2 - \log 3 - 3 = \bar{2}\cdot0280287$.

5. Find the logarithm of 180 to the base 6, or solve the equation $6^x = 180$. *Ans.* $x = \dfrac{1 + \log 2 + 2\log 3}{\log 2 + \log 3} = 2\cdot8982444$,

6. Solve the equation $\left(\frac{5}{7}\right)^x = \frac{2}{3}$. *Ans.* $x = 1 \cdot 2050476$.

7. Find the value of x in the equation $a b^x = c$.

$$Ans.\ x = \frac{\log c - \log a}{\log b}.$$

8. Find the value of x in the equation $a^{b^x} = c$.

$$Ans.\ x = \frac{\log d}{\log b};\ \text{where } d = \frac{\log c}{\log a}.$$

9. Given $a^x . b^y = c$, and $m y = n x$, to find x and y.

$$Ans.\ x = \frac{m \log c}{m \log a + n \log b},\ y = \frac{n \log c}{m \log a + n \log b}.$$

10. Given $2^{4x} . 3^{2x-5} = 4^{x-1} . 5^{2x}$, to find the value of x.

Ans. $x = 11 \cdot 26242$.

11. Given $(a^2 - b^2)^{2(x-1)} = (a - b)^{2x}$, to find the value of x.

$$Ans.\ x = 1 + \frac{\log(a-b)}{\log(a+b)}.$$

12. Given $2^x 3^y = 2000$, and $3 y = 5 x$, to find the values of x and y.

$$Ans.\ x = \frac{3\,(3 + \log 2)}{3 \log 2 + 5 \log 3},\ y = \frac{5\,(3 + \log 2)}{3 \log 2 + 5 \log 3}.$$

13. Given $3^{2x} + 3^x = 6$ and $4^{2xy} - 2 \times 4^{xy} = 8$ to find x and y.

$$Ans.\ x = \frac{\log 2}{\log 3},\ y = \frac{\log 3}{\log 2}.$$

PERMUTATIONS AND COMBINATIONS.

140. The *permutations* of any number of quantities are the different arrangements which can be made with them, when taken *two* together, *three* together, etc., or *all* together: thus the permutations of a, b, c, when taken *three* together, are $a b c$, $a c b$, $b a c$, $b c a$, $c a b$, $c b a$. The *combinations* of any number of quantities are the different collections which can be made of any assigned number of them, without reference to the order of their arrangement: thus, $a b c$, $a b d$, $a c d$, and $b c d$ are different combinations of the four letters a, b, c, d, taken *three* together.

141. *The number of permutations of n different quantities, taken r together, is*

$$n\,(n-1)\,(n-2)\ .\ .\ .\ .\ .\ \{n - (r-1)\}.$$

For if a, b, c, d, etc., be the n quantities, then, if taken *two* together, a may be placed before each of the $(n-1)$ quantities, and thus we have $(n-1)$ permutations. If b be placed first, we shall have also $(n-1)$ permutations, and c placed first will give $(n-1)$ permutations, and so on throughout the n different quantities: hence the total number of permutations, two together, is $n\,(n-1)$.

If the n quantities be taken *three* together, then, omitting any one of the n quantities, as a, the number of permutations of the remaining $(n-1)$ quantities is $(n-1)\,(n-2)$ by the preceding case when taken *two* together. Place a before each of these permutations, two together; then we shall have $(n-1)\,(n-2)$ permutations taken *three* together, in which a stands first; and there must be the same number of permutations where b, c, d, etc., successively occupy the first place in each; hence the entire number of permutations of the n different quantities must be n times the number $(n-1)\,(n-2)$, or $n\,(n-1)\,(n-2)$.

In a similar manner, the number of permutations of n quantities taken *four* together is $n\ (n-1)\ (n-2)\ (n-3)$, and generally when taken r together, the number is

$$n\,(n-1)\,(n-2)\,(n-3)\ \ldots\ \{n-(r-1)\}.$$

If the quantities are taken *all* together, then $r = n$, and the number of permutations is

$$n\,(n-1)\,(n-2)\ \ldots\ 3.2.1, \text{ or } 1.2.3.4.\ \ldots\ n.$$

142. *The number of permutations of n quantities taken all together, of which p quantities are alike, q other quantities are alike, r other quantities are alike, and so on, is*

$$\frac{1.2.3.4.\ \ldots\ n}{(1.2.3.\ \ldots\ p)\ (1.2.3.\ \ldots\ q)\ (1.2.3.\ \ldots\ r), \text{ etc.}}$$

For if the p quantities were unlike, they would form $1.2.3.\ \ldots\ p$ permutations, instead of only *one* when they are alike; therefore the whole number of permutations must be diminished $1.2.3.\ \ldots\ p$ times, if p quantities are alike, hence the number of permutations, if p quantities are alike, is

$$\frac{1.2.3.4.\ \ldots\ n}{1.2.3.4.\ \ldots\ p}.$$

For a similar reason, the whole number of permutations, if q other quantities are alike, must be diminished $1.2.3.\ \ldots\ q$ times, and the expression for the number of permutations becomes

$$\frac{1.2.3.4.\ \ldots\ n}{(1.2.3.\ \ldots\ p)\ (1.2.3.\ \ldots\ q)}.$$

If r other quantities are alike, the expression is

$$\frac{1.2.3.4.\ \ldots\ n}{(1.2.3.\ \ldots\ p)\ (1.2.3.\ \ldots\ q)\ (1.2.3\ \ldots\ r)},$$

and so on; hence the truth of the proposition is established.

143. *The number of combinations of n different quantities, taken r together, is*

$$\frac{n\,(n-1)\,(n-2)\ \ldots\ \{n-(r-1)\}}{1.2.3.4.\ \ldots\ r}.$$

For the number of permutations of n quantities, taken *two* together, is $n\,(n-1)$, and there are two permutations, $a\,b$, $b\,a$, corresponding to one combination, $a\,b$; hence there are twice as many permutations as combinations; and therefore the number of combinations is $\dfrac{n\,(n-1)}{1.2}$.

If the quantities are taken *three* together, the number of permutations is $n\,(n-1)\,(n-2)$, and there are $1.2.3$ permutations for each combination; therefore there are $1.2.3$ times as many permutations as combinations, and the number of combinations is $\dfrac{n\,(n-1)\,(n-2)}{1.2.3}$.

In a similar manner it may be shown that the number of combinations of n quantities, taken r together, is

$$\frac{n\,(n-1)\,(n-2)\ \ldots\ \{n-(r-1)\}}{1.2.3.\ \ldots\ r}\ \ldots\ (a).$$

144. *The number of combinations of n things, taken r together, is equal to the number of combinations taken n − r together.*

For writing $n-r$ for r in the last result, the number of combinations of n quantities taken $n-r$ together is

$$\frac{n(n-1)\ (n-2)\ .\ .\ .\ .\ (r+1)}{1.2.3.4.\ .\ .\ .\ .\ (n-r)}; \quad . \quad . \quad (\beta);$$

but these results (α) and (β) are equal to each other, for

$$\frac{n(n-1)\ (n-2)\ .\ .\ .\ (n-r+1)}{1.2.3.\ .\ .\ .\ .\ r}$$

$$= \frac{(r+1)\ (r+2)\ .\ .\ .\ .\ (n-2)\ (n-1)\,n}{(n-r)\ (n-r-1)\ .\ .\ .\ .\ 3.2.1},$$

as is evident by clearing the equation of fractions; for then both sides are identical, but written in reverse order. Thus if $n = 8$, and $r = 3$, then $\frac{8.7.6}{1.2.3} = \frac{4.5.6.7.8}{5.4.3.2.1}$, or the number of combinations of 8 quantities, taken 3 together, is equal to the number of combinations taken 5 together, as is evident by bringing these fractions to a common denominator.

These combinations of n quantities, when taken r together, and $n - r$ together, are said to be *supplementary* to each other.

145. Also if C_r denote the number of combinations of n quantities taken r together, then the sum of all the combinations that can be made of n quantities, taken *one*, *two*, *three*, ... n together is

$$C_1 + C_2 + C_3 + \dots C_n = n + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} + \text{etc.}$$
$$= (1+1)^n - 1 = 2^n - 1.$$

EXAMPLES.

1. Find the number of permutations of the letters in the word *change*. *Ans.* 720.

2. Find the number of permutations of the letters in the word *application*. *Ans.* 4989600.

3. In how many ways may 7 persons seat themselves at table? *Ans.* 5040.

4. How many changes may be rung with 5 bells out of 8, and how many with the whole peal? *Ans.* 6720 and 40320.

5. The number of permutations of n quantities *three* together, is to the number of permutations *five* together, as 1 to 42; find n. *Ans.* $n = 10$.

6. The number of combinations of n quantities *four* together, is to the number *two* together, as 15 to 2; find n. *Ans.* $n = 12$.

7. On how many nights may a *different* guard be posted of 4 men out of a company of 36? and on how many of these will any particular soldier be on guard? *Ans.* 58905 and 6545.

8. If a company consisting of 30 men are drawn up in column, with how many different fronts can that be done when 5 men are always in front? *Ans.* 142506.

9. A captain, who had been successful in war, was asked what reward he expected for his meritorious services; he replied that he would be satisfied with a farthing for every different file of 6 men he could form with his company, which consisted of 100 men; what was the amount of his request? *Ans.* 1241721*l.* 5*s.*

INTEREST AND ANNUITIES.

146. *To find the amount of a given sum of money in any number of years, at simple interest.*

Let P denote the *principal* or sum at interest in pounds, r the rate of interest of $1l.$ for one year, t the number of years the principal is at interest, and A the amount of principal and interest at the end of t years; then since

$$r = \text{the interest of } 1l. \text{ for } one \text{ year},$$
$$t\,r = \text{the interest of } 1l. \text{ for } t \text{ years},$$
$$P\,t\,r = \text{the interest of } Pl. \text{ for } t \text{ years},$$
$$\therefore\ A = P + P\,t\,r = P\,(1 + t\,r) \quad . \; . \; . \; . \quad (1).$$

From this equation, any three of the quantities P, r, t, A being given, the fourth may be found.

147. *To find the amount of a given sum of money in any number of years, at compound interest.*

In addition to the notation in the former proposition, we shall make use of R to denote the amount of $1l.$ for one year; then $R = 1 + r$. Now P in *one* year will amount to $P + P\,r = P\,(1 + r) = P\,R$, and this being the principal for the *second* year, the corresponding amount will be $P\,R + P\,R\,r = P\,R\ (1 + r) = P\,R^2$. In a similar manner $P\ R^3$ is the amount in *three* years, and consequently in t years the amount will be

$$A = P\,R^t \quad . \; . \; . \; . \; . \quad (2),$$
$$\text{or, } \log A = \log P + t \log R \quad . \; . \; . \quad (2').$$

Cor. If the interest is paid *half-yearly*, then $2\,t$ will be the number of payments, and $\frac{r}{2}$ the rate of interest; hence we have in this case

$$A = P\left(1 + \frac{r}{2}\right)^{2t} \quad . \; . \; . \; . \; . \quad (3).$$

$$\text{If paid quarterly, } A = P\left(1 + \frac{r}{4}\right)^{4t} \quad . \; . \; . \; . \; . \quad (4).$$

Hence we can find the time in which any sum at compound interest will amount to *twice*, *thrice*, or m times itself.

Thus if $A = 2\,P$; then by (2) we have $2 = R^t$, and $t \log R = \log 2$;
if $A = 3\,P$; then $3 = R^t$, and $t \log R = \log 3$;
or, if $A = m\,P$; then $m = R^t$, and $t \log R = \log m$.

Examples.

1. If $500l.$ be allowed to accumulate at compound interest at the rate of $5l.$ per cent. per annum, what will be the amount at the end of 21 years? *Ans.* $1392l.\ 19s.\ 7\tfrac{1}{2}d.$

2. In what time will any sum of money double itself at the rate of $4\tfrac{1}{2}$ per cent. per annum, compound interest being allowed? *Ans.* 15·7473 years.

3. If the population of a city contain one million of inhabitants, and increase at the average annual rate of 3 per cent., what will the population amount to at the end of 10 years? *Ans.* 1343915 inhabitants.

ANNUITIES CERTAIN.

148. An *annuity* is a sum of money which is payable at equal intervals of time.

When the possession of an annuity is not to be entered upon until the

expiration of a certain period, it is called a *reversionary* or *deferred* annuity; and when the time of possession is not deferred, the annuity is said to be *in possession.*

An *annuity certain* is one which is limited to a certain number of years; a *life annuity* is one which terminates with the life of any person, and a *perpetuity,* or *perpetual* annuity, is one which is entirely unlimited in its duration.

149. *To find the amount of an annuity in any number of years, at compound interest.*

Let a denote the annuity, A the amount, $R = 1 + r$; r and n the same as in the former investigations.

The first payment becomes due at the end of one year, and if unpaid during the remaining $n - 1$ years, it will amount in that time, at compound interest, to $a R^{n-1}$ pounds. The second payment becomes due at the end of two years, and, unpaid for $n - 2$ years, will amount to $a R^{n-2}$. In a similar manner, the third payment will amount in $n - 3$ years to $a R^{n-3}$, and so on until the last payment which, unburdened with interest, is simply a. Hence the entire amount is the sum of a geometrical series, and therefore

$$A = a + a R + a R^2 + a R^3 + \ldots + a R^{n-2} + a R^{n-1};$$
$$= a \cdot \frac{R^n - 1}{R - 1} = a \frac{(1+r)^n - 1}{r} \quad \ldots \ldots (5).$$

If the annuity is to be received in *half-yearly* instalments; then we have

$$A = \frac{a}{2} \cdot \frac{(1 + \frac{1}{2} r)^{2n} - 1}{\frac{1}{2} r} = a \cdot \frac{(1 + \frac{1}{2} r)^{2n} - 1}{r} \quad \ldots \ldots (5').$$

If *quarterly,* $A = \frac{1}{4} a \cdot \frac{(1 + \frac{1}{4} r)^{4n} - 1}{\frac{1}{4} r} = a \frac{(1 + \frac{1}{4} r)^{4n} - 1}{r} \quad . . (5'').$

Cor. If a pounds are placed out annually for n successive years, and the whole be allowed to accumulate at compound interest, then will

$$A = a R + a R^2 + a R^3 + \ldots + a R^n$$
$$= a R (1 + R + R^2 + \ldots + R^{n-1}) = a R \cdot \frac{R^n - 1}{R - 1} \quad \ldots (6).$$

150. *To find the present value of an annuity to be paid n years, at compound interest.*

Let P denote the present value of the annuity a; then the amount of P pounds in n years $= P R^n$ (147), and the amount of the annuity a in the same time is (149) $a \frac{R^n - 1}{R - 1}$; but these two amounts must be equal to each other; hence we get

$$P R^n = a \cdot \frac{R^n - 1}{R - 1}, \text{and } P = a \cdot \frac{R^n - 1}{R^n (R - 1)} = \frac{a}{R - 1}\left(1 - \frac{1}{R^n}\right). \ (7).$$

Cor. In the case of a perpetuity, n is infinite, and therefore we get

$$P = \frac{a}{R - 1} = \frac{a}{r} \quad \ldots . (8).$$

151. *To find the present value of an annuity in reversion, commencing at the end of p years, and to continue q years.*

It is evident that if an annuity be deferred p years, and then continue q years, its present value will be less than that of an annuity to be

received $p + q$ years, by the present value of the annuity for p years; hence we have (150)

$$P = \frac{a}{R-1}\left(1 - \frac{1}{R^{p+q}}\right) - \frac{a}{R-1}\left(1 - \frac{1}{R^p}\right)$$

$$= \frac{a}{R-1}\left(\frac{1}{R^p} - \frac{1}{R^{p+q}}\right) = \frac{a}{rR^p}\left(1 - \frac{1}{R^q}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

If the annuity is payable *for ever* after the expiration of p years, then the value of the *reversion of the perpetuity* is (since q is infinite)

$$P = \frac{a}{rR^p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9').$$

Examples.

1. What is the present value of 350*l.*, due at the end of 10 years, allowing 5 per cent. compound interest? *Ans.* 214*l.* 17*s.* 5*d.*

2. What is the value of a freehold estate, yielding an annual income of 250*l.*, allowing 3½ per cent. compound interest? *Ans.* 7142*l.* 17*s.* 1*d.*

3. What will an annuity of 75*l.* amount to in 6 years, at 4 per cent.? *Ans.* 497*l.* 9*s.* 4*d.*

4. A person has in perpetuity a property worth 525*l.* which he sells for 11666*l.* 13*s.* 4*d.*; what per cent. does the purchaser get for his money? *Ans.* 4½ per cent.

5. What is the present value of an annuity of 112*l.* 10*s.*, to commence at the end of 10 years, and to continue 20 years, at 4 per cent.? *Ans.* 1032*l.* 17*s.* 6½*d.*

6. If an annuity of 50*l.* be purchased for 613*l.* 18*s.* 3½*d.*, at 5 per cent. compound interest, what period must expire before the annuity is entered upon? *Ans.* 10 years.

PROBABILITIES.

152. The *probability* of an event is the ratio of the number of chances for its happening to the number of chances both for its happening and failing. Thus if a expresses the number of favourable events and b the number of unfavourable events, the probability of its happening is

$$\frac{a}{a+b} = \frac{\text{the number of favourable events}}{\text{the whole number of events}};$$

whilst the probability of its failure is expressed by

$$\frac{b}{a+b} = \frac{\text{the number of unfavourable events}}{\text{the whole number of events}}.$$

From this mode of representation it follows that *certainty* will be expressed by 1, and all probabilities will be expressed by fractions less than unity. The ratio of the probability of success to that of failure, or the ratio of the *odds for or against*, will be that of a to b, or of b to a.

Hence, if 14 white and 10 black balls be thrown into an urn, the probability of drawing a white ball out of it, at one trial, is $\frac{14}{24}$, and the probability of failing or of drawing a black ball is $\frac{10}{24}$.

153. *If a, a_1, be the number of ways in which two independent events may respectively happen, and b, b_1, the number of ways in which*

they may fail, then the probability that they will both happen is the product of the probabilities of the separate events, or

$$\frac{a}{a+b} \times \frac{a_1}{a_1+b_1} = \frac{a\,a_1}{(a+b)\ (a_1+b_1)}.$$

For every case in $a+b$ may be combined with every case in a_1+b_1, and thus form $(a+b)\ (a_1+b_1)$ combinations of cases altogether; and each of the a cases in which the first event can happen may be combined with each of the a_1 cases in which the second event can happen, and thus form $a\,a_1$ combinations of cases favourable to the compound event; hence the probability that both will happen is

$$= \frac{a\,a_1}{(a+b)\ (a_1+b_1)}.$$

Hence the probability that both events will fail is $= \dfrac{b\,b_1}{(a+b)\ (a_1+b_1)}$;

that the first will happen and the second fail is $= \dfrac{a\,b_1}{(a+b)\ (a_1+b_1)}$;

that the second will happen and the first fail is $= \dfrac{a_1\,b}{(a+b)\ (a_1+b_1)}$;

and the probability that both do not happen though one may, is

$$1 - \frac{a\,a_1}{(a+b)\ (a_1+b_1)} = \frac{a\,b_1 + a_1\,b + b\,b_1}{(a+b)\ (a_1+b_1)}.$$

The probability that the one will happen and the other fail, without specifying which event, is evidently the sum of the probabilities that the first will happen and the second fail, and that the second will happen and the first fail; hence it is

$$\frac{a\,b_1}{(a+b)\ (a_1+b_1)} + \frac{a_1\,b}{(a+b)\ (a_1+b_1)} = \frac{a\,b_1 + a_1\,b}{(a+b)\ (a_1+b_1)}.$$

154. *If a, a_1, a_2 be the number of ways in which three independent events may respectively happen, and b, b_1, b_2 the number of ways in which they may fail, then the probability that they will all happen is the continued product of the probabilities of the separate events, or*

$$\frac{a}{a+b} \times \frac{a_1}{a_1+b_1} \times \frac{a_2}{a_2+b_2} = \frac{a\,a_1\,a_2}{(a+b)\,(a_1+b_1)\,(a_2+b_2)}.$$

The several combinations of all the cases in the first two chances, which by the last article are $(a+b)\ (a_1+b_1)$ in number, may be severally combined with the a_2+b_2 different cases of the third chance, and thus form $(a+b)\ (a_1+b_1)\ (a_2+b_2)$ combinations altogether.

The favourable cases in the first two chances, which are $a\,a_1$ in number, may be combined severally with the a_2 favourable cases of the third chance, and thus form $a\,a_1\,a_2$ favourable cases, and therefore the probability that all the three events will happen is the continued product of the simple chances, viz.,

$$\frac{a\,a_1\,a_2}{(a+b)\ (a_1+b_1)\ (a_2+b_2)}.$$

The proposition may be extended to any number, n, of events, and the proof is similar to the preceding.

Hence the probability that any number of events will all fail is equal to the product of all the separate probabilities of failing.

155. If a always denote the probability of the happening of an event, and b the probability of its failing, then the fraction $\left(\frac{a}{a+b}\right)$ will express the probability of its happening n times in succession, and $\left(\frac{b}{a+b}\right)^n$ the probability of its failing n times successively.

EXAMPLES.

1. An urn contains 15 white and 11 black balls, what is the probability of drawing first a white and then two black balls.

Ans. $\frac{11}{104}$.

2. In how many trials may a person undertake, for an even bet, to throw an ace with a single die? *Ans.* 3·776.

3. Among 32 counters, 14 are red and 18 white, what is the probability of drawing 4 red ones in succession, and also the probability of drawing 6 white ones successively?

Ans. $\frac{1001}{35960}$ and $\frac{221}{10788}$.

PROBABILITY OF LIFE.

156. *To find the probability of a person of a given age* (m) *living any number of years* (n).

Let $p_{m,n}$ denote the probability required, l_m the number of persons living at the age m according to the Tables of Mortality, and l_{m+n} the number living at the age $m+n$; then l_{m+n} is the number of chances of living n years, and l_m is the whole number of chances; hence (152) the probability of a person A, aged m years, living n years is

$$p_{m,n} = \frac{l_{m+n}}{l_m} = \lambda_n \text{ suppose.}$$

The probability that A will be dead at the end of n years is $1 - \lambda_n$. Thus, by the Northampton Tables, of 11650 persons born, 2936 survive 49 years, and 2612 survive 53 years; hence the probability that a person aged 49 years will complete the age of 53 is

$$p_{m,n} = p_{49,4} = \frac{l_{53}}{l_{49}} = \frac{2612}{2936} = \cdot 8896.$$

By the Carlisle Tables, the probability is $p_{49,4} = \frac{4211}{4458} = \cdot 9446$.

157. *To find the probability that two persons, A and B, will live n years.*

Let m_1 denote the age of B, and, as in the last proposition, find

$$p_{m_1,n} = \frac{l_{m_1+n}}{l_{m_1}} = \lambda'_n \text{ suppose.}$$

Then the probability that both A and B will be alive at the end of n years is the product of the separate probabilities of A and B being alive at the end of n years; hence it is

$$p_{(m,m1),n} = \frac{l_{m+n}}{l_m} \times \frac{l_{m_1+n}}{l_{m_1}} = \lambda_n \lambda'_n.$$

The probability that both A and B are dead, is $(1-\lambda_n)(1-\lambda'_n)$.
A is alive and B dead, is $\lambda_n(1-\lambda'^n)$.
B is alive and A dead, is $\lambda'_n(1-\lambda_n)$.

If there be more lives than two, as A, B, C, it may be shown, in a similar manner, that the probability of all three being alive at the end of n years is $\lambda_n \lambda'_n \lambda''_n$, and the probability of their joint existence failing in n years is $1 - \lambda_n \lambda'_n \lambda''_n$, whilst the probability of their being all dead, is $(1 - \lambda_n)(1 - \lambda'_n)(1 - \lambda''_n)$.

The probability of two, at least, out of three persons, A, B, C, being alive at the end of n years will be found in the following manner:

The probability that all are alive is $\lambda_n \lambda'_n \lambda''_n$.
A and B are alive and C dead, is $\lambda_n \lambda'_n (1 - \lambda''_n)$.
A and C are alive and B dead, is $\lambda_n \lambda''_n (1 - \lambda'_n)$.
B and C are alive and A dead, is $\lambda'_n \lambda''_n (1 - \lambda_n)$.

Hence the probability of two, at least, of these three persons surviving n years is the sum of these four probabilities, and it is therefore

$$\lambda_n \lambda'_n + \lambda_n \lambda''_n + \lambda'_n \lambda''_n - 2\lambda_n \lambda'_n \lambda''_n.$$

The probability that one, at least, of these three persons will be alive at the end of n years is evidently

$$1 - (1 - \lambda_n)(1 - \lambda'_n)(1 - \lambda''_n).$$

158. *To find the probability of a life failing in any particular year.*

Let m denote the age of the person, and $q_{m,n}$ the probability that the person will die in the n^{th} year from the present time; then, by the Tables, the number now aged m who survive $n - 1$ years, or enter upon their $(m + n)^{\text{th}}$ year, is l_{m+n-1}, and the number who complete their $(m + n)^{\text{th}}$ year is l_{m+n}; the difference between these is the number who die in the n^{th} year, and this difference, divided by the number living at the age of m years, gives the probability of a person aged m dying in the n^{th} year from this time. Hence we have

$$q_{m,n} = \frac{l_{m+n-1} - l_{m+n}}{l_m} = p_{m,n-1} - p_{m,n} = \lambda_{n-1} - \lambda_n.$$

EXPECTATION OF LIFE

159. The number of years' *expectation of life* of a person whose prospect of longevity is the same with that of persons of the same age is the average number of years enjoyed by each person. Thus, in the Northampton Tables, opposite to the age 48, we find 19, and this signifies that persons aged 48 may expect to live 19 years longer. Suppose that l_m persons are alive at the age m, l_{m+1} are alive at the age $m + 1$, then $l_m - l_{m+1}$ die in their $(m + 1)^{\text{th}}$ year, and l_{m+1} survive their $(m + 1)^{\text{th}}$ year. Now, if we suppose those who complete their m^{th} year, but die before completing their $(m + 1)^{\text{th}}$ year, to die at equal intervals therein, so that for every one who dies before the expiration of a half of the year some other will survive so much more than the half-year, then each person who dies in the year survives, upon an average, one-half of that year. Hence $\frac{l_m - l_{m+1}}{2}$ is the number of years enjoyed by all those who die in the $(m + 1)^{\text{th}}$ year; add to this the number l_{m+1}, who complete their $(m + 1)^{\text{th}}$ year, and we have $l_{m+1} + \frac{1}{2}(l_m - l_{m+1})$ $= \frac{1}{2}(l_m + l_{m+1})$ = the number of years enjoyed in the first year by these l_m persons or the survivors. In a similar manner, it appears that

$$\frac{1}{2}(l_{m+1}+l_{m+2})\,;\ \frac{1}{2}(l_{m+2}+l_{m+3})\,;\ \frac{1}{2}(l_{m+3}+l_{m+4}),\ \text{etc.},$$

is the number of years that will be enjoyed in the 2nd, 3rd, 4th, etc., years by these l_m survivors. Hence the total number of years enjoyed by these l_m persons until they all cease to exist is

$$\frac{l_m+l_{m+1}}{2}+\frac{l_{m+1}+l_{m+2}}{2}+\ .\ .\ .\ .\ +\frac{l_{m+n+0}}{2}=$$

$$\frac{l_m}{2}+l_{m+1}+l_{m+2}+\ .\ .\ .\ .\ +l_{m+n}.$$

This expression, divided by l_m, the number of persons amongst whom this quantity of existence is divided, gives e_m, the *expectation of life* of a person aged m; hence

$$e_m=\frac{1}{2}+\frac{l_{m+1}+l_{m+2}+l_{m+3}+l_{m+4}+\ .\ .\ .\ .\ +l_{m+n}}{l_m},$$

or $e_m=\frac{1}{2}+\lambda_1+\lambda_2+\lambda_3+\lambda_4+\ .\ .\ .\ .$ to the end of the tables;

where $\lambda_1=\frac{l_{m+1}}{l_m}$, $\lambda_2=\frac{l_{m+2}}{l_m}$, etc.

Hence, to find the expectation of life, *divide the sum of the number of those who complete each age above the given one by the number living at the given age, and to the quotient add half unity.*

The following is the method of calculating a table of expectations, according to the Northampton rate of mortality:

$$\begin{array}{ll}
l_{96}=1 & \\
l_{95}=\underline{4} & \frac{1}{4}+\cdot 5=\ \cdot 75=e_{95}. \\
\quad\ \ \ \ 5 & \frac{5}{9}+\cdot 5=1\cdot 05=e_{94}. \\
l_{94}=\underline{9} & \\
\quad\ \ \ 14 & \frac{14}{16}+\cdot 5=1\cdot 37=e_{93}. \\
l_{93}=\underline{16} & \\
\quad\ \ \ 30 & \frac{30}{24}+\cdot 5=1\cdot 75=e_{92}. \\
l_{92}=\underline{24} &
\end{array}$$

The duration of life that a person has the present expectation of enjoying after a given period (t years) is found by multiplying the expectation at the advanced age by the probability the person has of attaining that age. The expression is, therefore,

$$e_{m+t}\frac{l_{m+t}}{l_m},\ \text{or}\ e_{m+t}\,\lambda_t.$$

LIFE ANNUITIES.

160. *To find the present value of an annuity of* 1*l. payable at the end of every year during the existence of a single life.*

Let m denote the age of the person during whose life the annuity is to be continued, and a_m the present value of the annuity; let also $v=\frac{1}{1+r}=\frac{1}{R}$ be the present value of 1*l.* due at the end of one year; then the present value of the first year's payment of the annuity is found by multiplying v by the probability of the person living one year; the present value of the second year's payment by multiplying the

present value of $1l.$ due at the end of two years by the propability of the person living two years, and so on. Hence the whole value required is $a_m = vp_{m,1} + v^2 p_{m,2} + v^3 p_{m,3} + v^4 p_{m,4} + \ldots$ to the end of the tables

$$= \frac{v l_{m+1} + v^2 l_{m+2} + v^3 l_{m+3} + v^4 l_{m+4} + \ldots}{l_m} \ldots \text{(1)}.$$

$$\text{Hence } a_m l_m = v l_{m+1} + v^2 l_{m+2} + v^3 l_{m+3} + v^4 l_{m+4} + \ldots \text{(1')}.$$

Also, the present value of the annuity on a life one year older is

$$a_{m+1} = \frac{v l_{m+2} + v^2 l_{m+3} + v^3 l_{m+4} + v^4 l_{m+5} + \ldots}{l_{m+1}};$$

$$\therefore v a_{m+1} l_{m+1} = v^2 l_{m+2} + v^3 l_{m+3} + v^4 l_{m+4} + v^5 l_{m+} + \ldots$$
$$= a_m l_m - v l_{m+1} \text{ by (1')};$$

consequently we have

$$a_m l_m = v l_{m+1} (1 + a_{m+1}), \text{ or } a_m = v \lambda_1 (1 + a_{m+1}) \ldots \text{(2)}.$$

Hence it appears that the present value of an annuity at any age may be deduced from the value at the age one year older; and if we commence at the oldest age in the table at which the value of the annuity is 0, and proceed through all the other ages to the time of birth, a "Table of the present values of Annuities" will be formed. Thus by the Northampton rate of mortality, of 11650 persons born, 1 survives 96 years, 4 survive 95 years, 9 survive 94 years, 16 survive 93 years, and so on; hence, calculating at 3 per cent., we get

$$a_{95} = \frac{l_{96}}{l_{95}} \cdot \frac{1}{1\cdot03} \cdot (1 + a_{96}) = \frac{1}{4} \times \cdot970874 \times (1 + 0) = \cdot2427.$$

$$a_{94} = \frac{l_{95}}{l_{94}} \cdot \frac{1}{1\cdot03} \cdot (1 + a_{95}) = \frac{4}{9} \times \cdot970874 \times 1\cdot2427 = \cdot5362.$$

In a similar manner the present value of an annuity which depends on the joint existence of two persons aged m and m_1 respectively is

$$a_{m,m_1} = \frac{v l_{m+1} l_{m_1+1} + v^2 l_{m+2} l_{m_1+2} + v^3 l_{m+3} l_{m_1+3} + \ldots}{l_m \cdot l_{m_1}} \ldots \text{(3)};$$

or, as before, $a_{m,m_1} = v \lambda_1 \lambda'_1 (1 + a_{m+1, m_1+1}) \ldots \text{(4)}.$

Also the value dependent on the joint existence of three lives is

$$a_{m, m_1, m_2} = v\lambda_1 \lambda'_1 \lambda''_1 (1 + a_{m+1, m_1+1, m_2+1}) \ldots \text{(5)}.$$

ASSURANCES ON LIVES.

161. When an engagement is entered into to secure the payment of a sum on the death of a person, in consideration of a single or annual payment, the transaction is called an *assurance* on the life of that person, and the stipulated payment is called the *premium*.

162. *To find the present value (P) of 1l. to be paid at the end of the year in which a person shall die.*

The probability of his dying in the first year is $1 - \lambda_1$, by (156), and the value of the first year's expectation is $v(1-\lambda_1)$. Again, the probability of his dying in the second year is $\lambda_1 - \lambda_2$ by (158), and the value of the second year's expectation is $v^2(\lambda_1 - \lambda_2)$, and so on. But the whole value of the expectation is the sum of all these contingent values; therefore the present value is

$$P = v(1 - \lambda_1) + v^2(\lambda_1 - \lambda_2) + v^3(\lambda_2 - \lambda_3) + \ldots$$
$$= v + v(v\lambda_1 + v^2\lambda_2 + v^3\lambda_3 + \ldots) - (v\lambda_1 + v^2\lambda_2 + v^3\lambda_3 + \ldots)$$
$$= v - (1 - v)\, a_m \text{ by equation (1) Art. (160)} \ldots \text{(1)},$$

or $P = 1 - (1 - v)(1 + a_m)$, which is well adapted for computation.

Similarly, the single premium, or present value of an assurance on the failure of the joint existence of two lives is $v - (1 - v)\, a_{m,\, m_1}$, and so on for any number of lives.

163. *To find the annual premium* (p) *that must be paid to secure* 1*l. at death.*

Here we must find the present value of 1*l.* by the preceding proposition, and then find how much must be paid annually to form an equivalent to the present value. Now since the annual payment is made at the *beginning* of each year, there will be *one* payment more than for an annuity; hence (160) we have

$$1 + a_m = 1 + v\lambda_1 + v^2\lambda_2 + v^3\lambda_3 + v^4\lambda_4 + \ldots\ldots,$$

and consequently $p\,(1 + a_m)$ is the present value of 1*l.* at the end of the year in which the assured dies. But by the last proposition that value is $P = v - (1 - v)\, a_m$; whence

$$p\,(1 + a_m) = P, \text{ or } p = \frac{P}{1 + a_m} = \frac{v - (1 - v)\, a_m}{1 + a_m};$$

$$\text{hence } p = \frac{v + v\,a_m - 1 + 1 - a_m}{1 + a_m} = \frac{1 - (1 - v)\,(1 + a_m)}{1 + a_m}$$

$$= \frac{1}{1 + a_m} - (1 - v).$$

Similarly the annual premium on two lives will be found to be

$$\frac{1}{1 + a_{m,\, m_1}} - (1 - v),$$

and so on for any number of lives.

The annual premium is equal to the single premium divided by the annuity on the given life increased by unity.

164. *To find the single and annual premiums of a temporary assurance of a single life.*

Let $a_{m,\,t}$ denote the value of an annuity of 1*l.* payable at the end of every year during t years only, instead of the whole life; then (160) we have

$a_{m,\,t} = v\lambda_1 + v^2\lambda_2 + v^3\lambda_3 + \ldots + v^t\lambda_t$. Now by (162) we get

$$P = v\,(1 - \lambda_1) + v^2(\lambda_1 - \lambda_2) + v^3(\lambda_2 - \lambda_3) + \ldots + v^t(\lambda_{t-1} - \lambda_t)$$

$$= v + v\,(v\lambda_1 + v^2\lambda_2 + \ldots\ldots + v^{t-1}\lambda_{t-1})$$

$$- (v\lambda_1 + v^2\lambda_2 + v^3\lambda_3 + \ldots\ldots + v^t\lambda_t).$$

Hence $P = v + v\,(a_{m,\,t} - v^t\lambda_t) - a_{m,\,t} = v\,(1 - v^t\lambda_t) - (1 - v)\, a_{m,\,t}$.

Now since the number of annual payments will be t, consisting of an immediate payment, and of a temporary annuity for $t - 1$ years, the single premium must therefore be divided by

$$1 + a_{m,\,t-1} = 1 - v^t\lambda_t + a_{m,\,t}.$$

$$\text{Hence } p = \frac{v\,(1 - v^t\lambda_t) - (1 - v)\, a_{m,\,t}}{1 - v^t\lambda_t + a_{m,\,t}} = \frac{1 - v^t\lambda_t}{1 - v^t\lambda_t + a_{m,\,t}} - (1 - v).$$

In exactly the same manner we can find the single and annual premiums to secure a sum payable at the end of the year in which any number of joint lives shall fail, provided the event happen within t years.

For further information on this important subject we must refer to the valuable Treatise on Annuities in the Library of Useful Knowledge.

EXAMPLE.

Find the *single* and *annual premiums* that would be required to

secure the payment of 500*l.* at the end of the year, in which the existence of a person now aged 49 shall fail, Carlisle 3 per cent.

By continuing the process described in (160), we find

$$a_{49} = \frac{l_{50}}{l_{49}} \cdot \frac{1 + a_{50}}{1\cdot 03} = 14\cdot 654 \text{; whence (162).}$$

$$P = 1 - (1 - v)(1 + a_{n}) = 1 - \cdot 029126 \times 15\cdot 654,$$

or $P = \cdot 54407 l.$ = the single premium to secure 1*l.*;

$\therefore \cdot 54407 \times 500 = 272l.\ 0s.\ 8d.$, the single premium to secure 500*l.*

Hence the annual premium $= \dfrac{272\cdot 035}{15\cdot 654} = 17\cdot 378l. = 17l.\ 7s.\ 6d.$

ON SERIES.

165. Let a, b, c, d, e, f etc. be a series of quantities progressing according to any regular law; then if each term be subtracted from that which succeeds it, the several remainders will form a new series, progressing by another regular law of a simpler form. If this new series, which is called the *first order of differences*, be a_1, b_1, c_1, d_1, e_1, etc., and if the same process be repeated on this series, there will arise another series a_2, b_2, c_2, d_2, etc., called the *second order of differences*, and so on, as in the following scheme:

Given series . .	a	b	c	d	e	f	g	etc.
		$\underline{a}$	$\underline{b}$	$\underline{c}$	$\underline{d}$	$\underline{e}$	$\underline{f}$	
1st order of diff. . .		a_1	b_1	c_1	d_1	e_1	f_1	
			$\underline{a_1}$	$\underline{b_1}$	$\underline{c_1}$	$\underline{d_1}$	$\underline{e_1}$	
2nd order of diff.			a_2	b_2	c_2	d_2	e_2	
				$\underline{a_2}$	$\underline{b_2}$	$\underline{c_2}$	$\underline{d_2}$	
3rd order of diff.				a_3	b_3	c_3	d_3	

166. *To find the first term of the n^{th} order of differences.*

Let a, b, c, d, e, etc. be the series, and d_1, d_2, d_3, d_4, etc. the first terms of the several orders of differences; then writing the series and the differences vertically, we have

$$\begin{array}{c|llll}
a & d_1 & d_2 & d_3 & d_4 \\
b & -a+b, & & & \\
c & -b+c, & a-2b+c, & & \\
d & -c+d, & b-2c+d, & -a+3b-3c+d, & \\
e & -d+e, & c-2d+e, & -b+3c-3d+e, & a-4b+6c-4d+e.
\end{array}$$

Hence generally the first term of the n^{th} order of differences is

$$d_n = \pm\left\{a - nb + \frac{n(n-1)}{1.2}c - \frac{n(n-1)(n-2)}{1.2.3}d + \dots\right\} \dots (1)$$

where the sign + corresponds to an even order of differences, and the sign — to an odd order, as the 1st, 3rd, 5th, etc.

167. *To find the n^{th} term of a series of quantities.*

Let a, b, c, d, etc., be the given series, and d_1, d_2, d_3, d_4, etc. the first terms of the successive orders of differences; then from the preceding Article we have

$$\begin{array}{lll}
d_1 = -a+b, & \therefore\ b = a+d_1 & = a+d_1 \\
d_2 = a-2b+c, & c = -a+2b+d_2 & = a+2d_1+d_2 \\
d_3 = -a+3b-3c+d, & d = a-3b+3c+d_3 & = a+3d_1+3d_2+d_3
\end{array}$$

And generally the n^{th} term is

$$= a + (n-1)\,d_1 + \frac{(n-1)\,(n-2)}{1.2}\,d_2 + \,.\,.\,.\,.\,.\,.\,.\,.\,.\; (2),$$

or the $(n+1)^{\text{th}}$ term

$$= a + n\,d_1 + \frac{n\,(n-1)}{1.2}\,d_2 + \frac{n\,(n-1)\,(n-2)}{1.2.3}\,d_3 + .\,.\,.\; (2').$$

168. *To find the sum of n terms of a series of quantities.*

Let us take the series,

$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,$ etc. . . . . (3),

then the first order of differences is the series

$a,\ b,\ c,\ d,\ e,\ f,\ g,$ etc. . . . . (4).

Now it is obvious that the sum of n terms of series (4) is the $(n+1)^{\text{th}}$ term of series (3), and therefore the $(n+1)^{\text{th}}$ term of series (3), or the sum (s) of n terms of series (4), is by the last Article

$$s = n\,a + \frac{n\,(n-1)}{1.2}\,d_1 + \frac{n\,(n-1)\,(n-2)}{1.2.3}\,d_2 + \,.\,.\,.\,.\,.\,.\; (5).$$

Examples.

1. Find the sum of n terms of the series $1.2^2,\ 2.3^2,\ 3.4^2,$ etc.

Squaring the latter factor in each term, and multiplying out, gives the series

series	4,	18,	48,	100,	180,	294,	etc.
1st diff.		14,	30,	52,	80,	114,	etc.
2nd diff.			16,	22,	28,	34,	etc.
3rd diff.				6,	6,	6,	etc.

Hence $a = 4,\ d_1 = 14,\ d_2 = 16,\ d_3 = 6,\ d_4 = 0$; therefore

$$s = n\,a + \frac{n\,(n-1)}{1.2}\,d_1 + \frac{n\,(n-1)\,(n-2)}{1.2.3}\,d_2 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{1.2.3.4}\,d_3$$

$$= 4\,n + 7\,n\,(n-1) + \frac{8}{3}\,n\,(n-1)\,(n-2) + \frac{1}{4}\,n\,(n-1)\,(n-2)\,(n-3)$$

$$= \frac{n^4}{4} + \frac{7\,n^3}{6} + \frac{7\,n^2}{4} + \frac{5\,n}{6}.$$

2. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \,.\,.\, + n^2.$ *Ans.* $\dfrac{n\,(n+1)\,(2\,n+1)}{1.2.3}.$

3. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \,.\,.\, + n^3.$ *Ans.* $\left\{\dfrac{n\,(n+1)}{2}\right\}^2.$

4. $2+6+12+20+30+$ . . to n terms. *Ans.* $\dfrac{n\,(n+1)\,(n+2)}{3}.$

5. $1+4+10+20+35+$ . . to n terms *Ans.* $\dfrac{n\,(n+1)\,(n+2)\,(n+3)}{1.2.3.4}.$

6. $1.2.3 + 2.3.4 + 3.4.5 +$ . . . . to 18 terms. *Ans.* 35910.

7. Find an expression for the number of shot in a rectangular pile of n courses, having $m+1$ shot in the top row.

Ans. $\dfrac{n\,(n+1)\,(2\,n+1+3\,m)}{1.2.3}.$

INTERPOLATION OF SERIES.

169. In calculating the various tables employed in the different departments of physical science, it would be very laborious to repeat the process of computation for each particular number. This labour may often be

avoided by the method of differences, when the *law of the series* is known, and when several terms of a series are given other terms may be introduced between them, or the series continued in such a manner that the law of the series shall not be changed.

In most cases the law of the series is not given, but only the numerical values of certain terms of the series at stated intervals, and then we can only approximate either to the law of the series or to the value of any term of the series.

To construct a table of squares by differences.

Let n^2, $(n+1)^2$, $(n+2)^2$, etc. be a series of squares, and let them be expanded and their differences be taken as below:

$$n^2,\ n^2+2n+1,\ n^2+4n+4,\ n^2+6n+9, \text{ etc.}$$
$$2n+1,\quad 2n+3,\quad 2n+5, \text{ etc.}$$
$$2,\quad 2, \text{ etc.}$$

The second differences are constant, and a table of squares may be formed in the following manner. Let us commence with $781^2 = 609961$ and $782^2 = 611524$, whose difference is 1563; then since the second differences are constant and equal to 2, the difference between the squares of 782 and 783 will be 1565; and this added to 611524 gives 613089, which is the square of 783; and so on as in the following scheme:

$609961 = 781^2$	$614656 = 784^2$	$619369 = 787^2$
1563	1569	1575
$611524 = 782^2$	$616225 = 785^2$	$620944 = 788^2$
1565	1571	1577
$613089 = 783^2$	$617796 = 786^2$	$622521 = 789^2$
1567	1573	1579
$614656 = 784^2$	$619369 = 787^2$	$624100 = 790^2$

In a similar manner a table of cube numbers may be computed.

170. When the 4th order of differences of any given series of quantities vanish, or approximately vanish, then (166) we have the equation $a - 4b + 6c - 4d + e = 0$, and any one of the quantities a, b, c, d, or e may be found when the other four are given. Similarly, if the third differences vanish, then $a - 3b + 3c - d = 0$.

Ex. Let it be required to find the logarithm of 104, having given the logarithms of 101, 102, 103, and 105.

Here four quantities are given to find a fifth; therefore supposing the fourth order of differences to vanish, we have

$$a - 4b + 6c - 4d + e = 0,$$

where d is the term to be interpolated; hence we have

$$\begin{aligned} 4d &= a + 6c + e - 4b \\ &= \log 101 + 6 \log 103 + \log 105 - 4 \log 102 \\ &= 8\cdot0681331; \text{ hence } d = 2\cdot0170333 = \log 104. \end{aligned}$$

171. When the terms are equidistant, and it is required to interpolate a term intermediate to any two of them, we may consider the term to be interpolated as the n^{th} from the first term, and then by (2′) Article 167 we have the formula,

$$n^{\text{th}} \text{ term} = a + n d_1 + \frac{n(n-1)}{1.2} d_2 + \frac{n(n-1)(n-2)}{1.2.3} d_3 + \dots.$$

Ex. Let it be required to determine the logarithm of 103·55, having given the logarithms of 101, 102, 103, 104, and 105.

Series.	Logarithms.	1st Diff.	2nd Diff.	3rd Diff.
101	2·0043214			
		42788		
102	2·0086002		− 418	
		42370		+ 9
103	2·0128372		− 409	
		41961		+ 8
104	2·0170333		− 401	
		41560		
105	2·0211893			

Now $n = 2{\cdot}55$, $n-1 = 1{\cdot}55$, $n-2 = {\cdot}55$; hence we have

$$a \quad = \quad \ldots\ldots \quad = +2{\cdot}0043214$$

$$n\,d_1 \quad = \quad 2{\cdot}55 \times 42788 \quad = +0{\cdot}0109109$$

$$\frac{n(n-1)}{2}d_2 \quad = \quad \frac{2{\cdot}55 \times 1{\cdot}55}{2} \times -418 = -0\ 0000826$$

$$\frac{n(n-1)(n-2)}{2{\cdot}3}d_3 = \frac{2{\cdot}55 \times 1{\cdot}55 \times {\cdot}55}{2{\cdot}3} \times 9 = +0{\cdot}0000003$$

$$\log 103{\cdot}55 = \quad 2{\cdot}0151500.$$

Note.—This is equivalent to having given the logarithms of 10100, 10200, etc., to find log 10355, or the 255[th] term from the first.

172. In most instances it will be sufficient to make use of first and second differences only, and the correction to be applied to the first term is

$$n\,d_1 + \frac{n(n-1)}{2}d_2 = n\left(d_1 + \frac{n-1}{2}d_2\right).$$

Thus in the preceding example, taking the terms 103, 104, 105, we have $a = 2{\cdot}0128372$, $d_1 = 41961$, $d_2 = -401$, and $n = {\cdot}55$; therefore

$$n\left(d_1 + \frac{n-1}{2}d_2\right) = {\cdot}55\left(41961 + \frac{{\cdot}45}{2} \times 401\right)$$

$$= {\cdot}55 \times 42051 \quad = \quad 23128$$

$$\log 103 \quad = 2{\cdot}0128372$$

$$\log 103{\cdot}55 = 2{\cdot}0151500.$$

But the result may be obtained to a greater degree of accuracy, by taking the two terms of the series which immediately precede and the two terms which follow the term required; then finding the three first differences and the two second differences. Let d_1 be the mean of the three first differences, d_2 the mean of the two second differences, and employ these values of d_1 and d_2 in the preceding formula for correction. Thus:

Series.	Logarithms.	1st Diff.	2nd Diff.	
102	2·0086002			
		42370		
103	2·0128372		− 409	sum = − 810
		41961		
104	2·0170333		− 401	½ sum = − 405
		41560		
105	2·0211893			

Then if $d_1 = 41961$, and $d_2 = -405$, we get

$$n\left(d_1 + \frac{n-1}{2}d_2\right) = {\cdot}55\left(41961 + \frac{{\cdot}45}{2} \times 405\right) \quad = \quad 23129$$

$$\log 103 \quad = 2{\cdot}0128372$$

$$\log 103{\cdot}55 = 2{\cdot}0151501$$

Examples.

1. Given the cube roots of 43, 44, 46, and 47 to find the cube root of 45. *Ans.* 3·5568934.

2. Given the logarithmic sines of 1°, $1^\circ\ 1'$, $1^\circ\ 2'$, and $1^\circ\ 3'$ to find the log sine of $1^\circ\ 1'\ 40''$. *Ans.* 8·2537534.

3. Given the sun's declination at 12 o'clock on the 19th, 20th, 21st, and 22nd of June, 1849, viz.,

$$23^\circ\ 26'\ 28{\cdot}4'',\ 23^\circ\ 27'\ 7{\cdot}8'',\ 23^\circ\ 27'\ 22{\cdot}5'',\ 23^\circ\ 27'\ 12{\cdot}4'',$$

respectively, to find the declination on the 20th at $6^h\ 40^m$.

Ans. $23^\circ\ 27'\ 8{\cdot}9''$.

4. Find the right ascension of the moon on Friday, the 11th of May, 1849, at $8^h\ 25^m$ Greenwich mean time, from the following data:—

Moon's R. A. at 3^h	. . .	18^h	34^m	$15{\cdot}61^s$
,, 6	. . .	18	40	31·57
,, 9	. . .	18	46	47·65
,, 12	. . .	18	53	3·83

Ans. $18^h\ 45^m\ 34{\cdot}54^s$.

5. Find the moon's true longitude on July 17th, 1849, at $15^h\ 20^m$ from the following data:—

Long. on 17th at 0^h	. . .	81°	9′	37·6″
,, 12	. . .	88	40	10·0
Long. on 18th at 0	. . .	96	11	46·7
,, 12	. . .	103	43	17·2
Long. on 19th at 0	. . .	111	13	29·4
,, 12	. . .	118	41	14·3

Ans. $91^\circ\ 48'\ 5{\cdot}3''$.

APPLICATION OF ALGEBRA TO GEOMETRY.

The subject of algebra is the general science of reasoning by means of arbitrary symbols, whilst that of geometry is the science of reasoning by means of magnitudes. In applying algebra, then, to the resolution of a problem of geometry, the symbols employed must be adequate to represent all the affections and relations of which the magnitudes represented by them are susceptible. Such symbols, therefore, must be regarded as the representatives of quantities admitting of arithmetical interpretation only in the case in which the magnitudes of geometry admit of such interpretation.

When, however, the geometrical magnitudes in any problem are all related to some other magnitude of the same kind, in the same manner that certain abstract numbers are related to unity, these numbers which may be taken to represent the magnitudes must no longer be considered as abstract but concrete quantities. Thus if A, B, C, D be straight lines related to a straight line U in the same manner that the abstract numbers a, b, c, d are related to unity; so that

$$\begin{aligned} A : U :: a : 1, &\text{ or } A = a\,U, \\ B : U :: b : 1, &\text{ or } B = b\,U, \\ C : U :: c : 1, &\text{ or } C = c\,U, \\ \text{etc.} \quad \text{etc.} &\quad \text{etc.} \end{aligned}$$

Then because A contains U, a times, B contains the same, b times, etc. if A, B, C be denoted by a, b, c, these symbols will now be concrete numbers; and hence in the solution of a problem of this kind particular care must be taken to reduce all the terms to the same denomination.

The quantity U, which is called the *linear unit*, may be of any length: if it be a *yard*, a mile will be represented by 1760; if it be an *inch*, a foot will be represented by 12, and so on with other quantities.

If upon the linear unit a square be described, that figure is called the *square unit*.

It will moreover be obvious, that every equation of the problem under discussion will be *homogeneous*, that is, the sum of the exponents of each term will be the same unless one of the quantities under consideration be taken for the unit.

What has been stated refers principally to the *magnitude* of lines; their *position* is to be determined by the principles laid down in the *Geometry of Coordinates*.

In the solution of a particular problem a figure must be constructed to represent the conditions of the problem, and such other lines drawn as may seem to facilitate the solution. Then denoting as usual known quantities by a, b, c, etc., and unknown ones by z, y, x, etc., as many independent equations must be formed as there are unknown quantities to be determined. The entire number of conditions is never given,

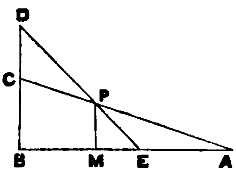
except implicitly as geometrical properties of the figure to which the problem relates. These implicit ones are geometrical theorems which must be sought for to suit the case, or investigated independently by geometrical methods.

The following solutions are given for illustration :—

Problem I.

Through a given point P in the hypothenuse A C of a right-angled triangle A B C to draw a straight line D P E, meeting A B in E, and B C produced in D, so that the triangles A P E and C P D may be equal in area.

From P draw P M perpendicular to A B and put A M $= a$, M B $= b$, M P $= c$ and M E $= x$. Then by similar triangles,



$$\text{A M} : \text{M P} :: \text{A B} : \text{B C}, \text{ or } a : c :: a + b : \text{B C} = \frac{c\,(a+b)}{a};$$

$$\text{and E M} : \text{M P} :: \text{E B} : \text{B D}, \text{ or } x : c :: x + b : \text{B D} = \frac{c\,(x+b)}{x}.$$

$$\text{Hence, B D} - \text{B C} = \frac{c\,(x+b)}{x} - \frac{c\,(a+b)}{a} = \frac{b\,c\,(a-x)}{a\,x} = \text{C D}.$$

Now by Euc. i. 41, and the conditions of the problem,

$$\text{C D} . \text{B M} = \text{A E} . \text{M P}, \text{ or } \frac{c\,b^2\,(a-x)}{a\,x} = c\,(a-x)\text{; whence}$$

$$x = \frac{b^2}{a} \ldots\ldots (1),$$

which gives the position of the required line D P E.

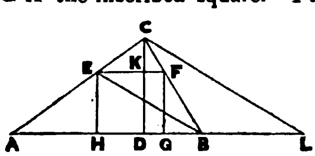
The formula (1) points out a simple geometrical construction. For $a\,x = b^2$, or A M . M E = B M^2, hence *M E is a third proportional to A M and M B.*

Problem II.

Given the base and perpendicular of a triangle, to find the side of the inscribed square.

Let A B C be the triangle and E F G H the inscribed square. Put A B $= c$, the perpendicular C D $= p$, the side of the square E F $= x$; then C K = C D − D K = C D − E F $= p - x$.

Now E F being parallel to A B, the triangles A B C, C E F are similar, hence,



$$\text{A B} : \text{C D} :: \text{E F} : \text{C K}, \text{ or } c : p :: x : p - x.$$

Equating the rectangle of the extremes to that of the means (Euc. vi. 16), we readily get—

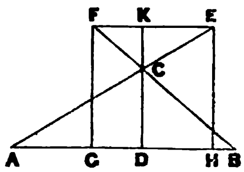
$$x = \frac{cp}{c+p} \ldots . (1),$$

the side of the required square.

Cor. 1. When $c = p$, or when the base and perpendicular are equal, $x = \frac{1}{2}p$ or $\frac{1}{2}c$, that is, the side of the inscribed square is equal to half the base or half the perpendicular.

Let us next inquire whether the angular points E, F may not be in A C and B C produced.

In this case,—



$c : p :: x : x - p$, or $x = \dfrac{cp}{c-p} \ldots . (2)$,

which is possible and finite unless $c = p$, and then x is infinite.

Hence two squares can be described on A B which have two of their angular points in A C and B C, unless the base and perpendicular are equal, and then one only can be described.

Cor. 2. Since by (1), $c + p : p :: c : x$

$:: \text{A B} : \text{E F} :: \text{A C} : \text{C E}$,

that is, $\text{A B} + \text{C D} : \text{C D} :: \text{A C} : \text{C E}$;

hence the following geometrical construction:—

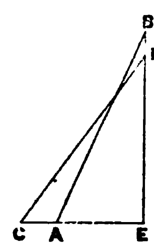
Produce the base A B till the produced part B L is equal to the perpendicular C D, and join the points L and C; then draw B E parallel to L C to meet A C in the point E, and this will be an angular point of the inscribed square.

The formula (2) is constructed in a similar way.*

Problem III.

The front wall of a house is of such height, that if a ladder of certain length be placed at the distance of 12 *feet from it, the top of the ladder will just reach to the top of the wall; but if it be placed at a distance of* 20 *feet from it, its top will be* 4 *feet below the top of the wall. Find the height of the wall and length of the ladder.*

Let B E be the height of the wall, and A B, D C the first and second positions, respectively, of the ladder. Let the length of the ladder A B = C D = x, the height of the wall B E = y; then D E = $y - 4$. Hence (Euc. i. 47), $x^2 = (12)^2 + y^2 \ldots . (1)$, $x^2 = (20)^2 + (y-4)^2 \ldots . (2)$.



Taking (2) from (1) we readily get

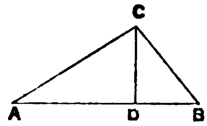
$8y = 272$, or $y = 34$ ft., the height of the wall; and $x = \sqrt{\{(12)^2 + y^2\}} = \sqrt{1300} = 36{\cdot}0555$ ft., the length of the ladder.

* All the results that are obtained algebraically by the solution of a quadratic, may be constructed in like manner, but this part of the subject must necessarily be very brief in a work like the present.

PROBLEM IV.

Given the base of a plane triangle, the perpendicular, and the rectangle of the two sides, to determine the sides.

Let A B C be the plane triangle. Put A D + D B * = A B = $2m$, A D − D B = $2x$, A C . C B = a^2, and C D = p; then A D = $m + x$, B D = $m - x$. A $C^2 = (m+x)^2 + p^2$, B $C^2 = (m-x)^2 + p^2$. Hence the condition



$$\{(m+x)^2 + p^2\}\,\{(m-x)^2 + p^2\} = a^4.$$

This equation readily reduces to the following

$$x^4 + 2\,(p^2 - m^2)\,x^2 = a^4 - (m^2 + p^2)^2,$$

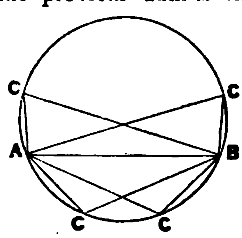
which gives for x the four values

$$x = \pm\,\{(m^2 - p^2) \pm \sqrt{(a^4 - 4\,p^2\,m^2)}\}^{\frac{1}{2}}$$

$$= \pm\,\{(m+p)\,(m-p) \pm \sqrt{(a^2 + 2p\,m)\,(a^2 - 2p\,m)}\}^{\frac{1}{2}};$$

from which the sides are easily determined.

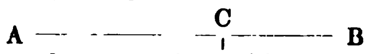
The four resulting values of x show that the problem admits in general of ***four solutions***, as indeed it might be expected from Euc. vi. C; for as the perpendicular and rectangle of the sides are given, the diameter of the circumscribed circle is also given by that proposition. Hence the problem is the same as that of inscribing in a given circle a triangle of given base and altitude, as in the annexed diagram.



PROBLEM V.

To divide a given straight line in extreme and mean ratio.

Let the given line A B = a, and the greater part A C = x, then the other part B C = $a - x$. Hence (Euc. vi. 17),



$$x^2 = a\,(a - x) = a^2 - a\,x, \text{ or } x^2 + a\,x = a^2 \;\ldots.\; (1).$$

The solution of this equation gives the two values

$$x = \frac{a}{2}\,(-1 \pm \sqrt{5}) \;\ldots.\; (2),$$

the former being positive and the latter negative. It will be obvious that the positive value of x determines the point C; so that

$$x = \text{A C} = \frac{\sqrt{5} - 1}{2}.\ \text{A B, and B C} = a - x = \frac{3 - \sqrt{5}}{2}\,\text{A B} \;\ldots.\; (3).$$

The negative value of x gives a point *to the left of* A (Geo. of coordinates), and the property to which it belongs may be determined by writing $-x$ for x in the equation (1). Hence we have $x^2 = a\,(a + x)$, which, when translated into words, gives the following problem:

To produce a given straight line so that the square of the produced

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* This method will be found, in similar cases, very effective in the solution of a problem. Should the *difference* of two quantities be given, instead of the sum, denote the difference by twice a given quantity and the sum by twice an unknown quantity.

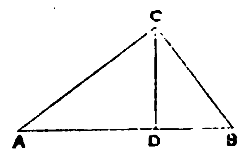
part shall be equal to the rectangle contained by the given line, and the line made up of the whole and the part produced.

This is merely the given problem enunciated more generally, as might be expected.

PROBLEM VI.

Given the perimeter ($77\frac{1}{4}$) *of a right-angled triangle, to find the sides, so that a perpendicular from the right angle may cut the hypothenuse in extreme and mean ratio.*

Let A B C be the right-angled triangle, the hypothenuse A B of which is divided in extreme and mean ratio in D.



Put A B $= x$, then by (3) of last problem,

$$A\,D = \frac{x}{2}(\sqrt{5} - 1) = m\,x,$$

$$B\,D = \frac{x}{2}(3 - \sqrt{5}) = n\,x,$$

of which

$$m = \tfrac{1}{2}(\sqrt{5} - 1) = \cdot 618034,$$
$$n = \tfrac{1}{2}(3 - \sqrt{5}) = \cdot 381966.$$

But (Euc. vi. 8),

$A\,C^2 = B\,A.A\,D = m\,x^2$, and $B\,C^2 = B\,A\cdot B\,D = n\,x^2$; hence by the condition of the problem we have

$$x + x\sqrt{m} + x\sqrt{n} = 77\tfrac{1}{4},$$

$$\text{or } x = \frac{77\frac{1}{4}}{1 + \sqrt{m} + \sqrt{n}} = 32.36, \text{ the side A B.}$$

Consequently, $A\,C = x\sqrt{m} = 25.44$, and $B\,C = x\sqrt{n} = 20$.

PROBLEM VII.

Given the sides of a plane triangle, to find—

(1). *The area of the triangle;*

(2). *The radius of the inscribed circle;*

(3). *The radius of the circumscribed circle.*

In this and some other discussions relative to a plane triangle, the following notation will be used:

Δ will denote the area of the triangle; r, R, the radii of the inscribed and circumscribed circles; a, b, c, the sides opposite, respectively, to the angles A, B, C; $2\,s$, the perimeter $(a + b + c)$; and, consequently, $b + c - a = 2\,(s - a)$, $a + c - b = 2\,(s - b)$, $a + b - c = 2\,(s - c)$.

Area of the Triangle.—Put the perpendicular C D $= x$, and B D $= y$ (fig. to Problem VI.). Now (Euc. i. 41), $\Delta = \frac{1}{2}\,c\,x$, or

$$\Delta^2 = \tfrac{1}{4}c^2 x^2 = \tfrac{1}{4}\,c^2\,(a^2 - y^2) = \tfrac{1}{4}\,c^2\,(a + y)\,(a - y) \ldots . (1).$$

But (Euc. ii. 13),

$$b^2 = a^2 + c^2 - 2\,c\,y, \text{ or } y = \frac{a^2 + c^2 - b^2}{2\,c}; \text{ hence (1) becomes}$$

$$\Delta^2 = \tfrac{1}{4}\,c^2\left\{a + \frac{a^2 + c^2 - b^2}{2\,c}\right\}\left\{a - \frac{a^2 + c^2 - b^2}{2\,c}\right\}$$

$$= \tfrac{1}{16}\{(a+c)^2 - b^2\}\{b^2 - (a-c)^2\}$$
$$= \tfrac{1}{16}(a+b+c)(a+c-b)(b+a-c)(b+c-a)$$
$$= s(s-b)(s-c)(s-a);$$

or $\Delta = \sqrt{s(s-a)(s-b)(s-c)} \ldots . (2)$,

the expression for the area in terms of the sides.

Cor. 1. If p_1, p_2, p_3, be the perpendiculars on the sides a, b, c, respectively, from the opposite angles, then

$$p_1 = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)},$$

$$p_2 = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)},$$

$$p_3 = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

The last is deduced from (1) and (2), and the other two are got in a similar way.

Radius of the Inscribed Circle.—Let O be the centre of the circle inscribed in the triangle A B C, and F, D, E the points of contact with the sides A B, B C, C A. Then because the whole triangle A B C is made up of the triangles A O B, B O C, and A O C, we have

$$(AB + BC + AC)\,OF = 2\,ABC,$$

$$\text{or } (a+b+c)\,r = 2\,\Delta\text{ ; hence}$$

$$r = \frac{2\,\Delta}{a+b+c} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \quad \ldots\ldots (3),$$

the required radius.

If r_1, r_2, r_3 be the radii of the circles which touch, respectively, the sides a, b, c, *exteriorly*, then

$$\left.\begin{aligned} r_1 &= \sqrt{\frac{s(s-b)(s-c)}{s-a}} \\ r_2 &= \sqrt{\frac{s(s-a)(s-c)}{s-b}} \\ r_3 &= \sqrt{\frac{s(s-a)(s-b)}{s-c}} \end{aligned}\right\} \quad \ldots\ldots\ldots (4).$$

The investigation of these, which is exactly similar to that for r, is left for the student's exercise.

Radius of the Circumscribed Circle.—If p_1 be the perpendicular from C on A B, then (Euc. vi. C.),

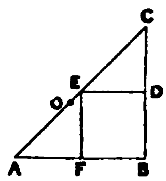
$$2\,R\,p_1 = a\,b, \text{ or } R = \frac{a\,b}{2\,p_1} = \frac{a\,b\,c}{4\sqrt{s(s-a)(s-b)(s-c)}}. \quad . (5).$$

Cor. 2. By (3), $r = \frac{\Delta}{s}$, and by (5), $R = \frac{a\,b\,c}{4\,\Delta}$; hence

$$2\,r\,R = \frac{a\,b\,c}{a+b+c}.$$

PROBLEM VIII.

Given the hypothenuse of a right-angled triangle, and the side of the inscribed square, to find the base and perpendicular, when one of the angles of the square is coincident with the right angle of the triangle.



Let A B C be the right-angled triangle and E D B F the inscribed square, so that two of its sides lie on the base and perpendicular of the triangle. Bisect the given hypothenuse A C in O, and put A E + E C = A C = $2m$, A E − E C = $2x$, the side of the square E D = a; then A E = $m + x$, E C = $m - x$ and O E = x.

Hence by similar triangles

$$\text{A E} : \text{E F} :: \text{A C} : \text{C B}, \text{ or } m + x : a :: 2m : \text{C B} = \frac{2ma}{m+x},$$

$$\text{and E C} : \text{E D} :: \text{A C} : \text{A B}, \text{ or } m - x : a :: 2m : \text{A B} = \frac{2ma}{m-x}.$$

Consequently (Euc. i. 47),

$$\frac{4m^2a^2}{(m+x)^2} + \frac{4m^2a^2}{(m-x)^2} = 4m^2,$$

$$\text{or } a^2\{(m-x)^2 + (m+x)^2\} = (m+x)^2(m-x)^2 = (m^2-x^2)^2.$$

This equation reduces to

$$x^4 - 2(m^2 + a^2)x^2 = m^2(2a^2 - m^2); \text{ hence}$$

$$x = \pm\{(m^2+a^2) \pm a\sqrt{(4m^2+a^2)}\}^{\frac{1}{2}}.$$

Let us now ascertain how many of these values of x fulfil the conditions of the problem geometrically; and first in respect of the double sign preceding the radical expression.

The expressions for C B and B A show at once that the triangle is the same (except that A B and B C interchange places), whether the plus or minus sign is used. Thus if the value of x be denoted by $\pm\,\alpha$, then

$$\text{C B} = \frac{2ma}{m+\alpha}, \text{ and A B} = \frac{2ma}{m-\alpha},$$

when we use the plus sign, and

$$\text{C B} = \frac{2ma}{m-\alpha}, \text{ and A B} = \frac{2ma}{m+\alpha},$$

when the negative sign is used.

Again, since $m^2 + a^2 > m^2$, the ***negative sign*** within the radical must be taken, otherwise x will be greater than m, and consequently A B will be negative. The only value of x, then, which fulfils the conditions of the problem geometrically, is

$$x = \{(m^2+a^2) - a\sqrt{(4m^2+a^2)}\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad (1).$$

Scholium.—The value of x, when the plus sign within the brackets is used, does not, as we have seen, belong to the figure as constructed; it corresponds in fact to the case in which the square E B is *exterior* to the triangle A B C, that is, when the point C is between D and B, and A in A B produced, the angular point E being in the hypothenuse A C produced. This case is evidently comprehended in the general investi-

gation that has been given. The student can verify this result for himself. He should also be able to point out the *limits* of the problem when the negative sign is used, for the solution in some cases is impossible.

Exercises for Practice.

1. Given one side of a right-angled triangle $= 5$, and the sum of the hypothenuse and the other side $= 25$, to find the hypothenuse and the remaining side. *Ans.* 13 and 12.

2. The hypothenuse of a right-angled triangle is 13, and the excess of the perpendicular above the base is 7; find the sides of the triangle. *Ans.* Base $BC = 5$, $AB = 12$.

3. The hypothenuse of a right-angled triangle is 13, and the sum of the base and perpendicular is 17; find the sides. *Ans.* 5, 12.

4. The product of the base and perpendicular of a triangle is 60 and the hypothenuse 13; what are the base and perpendicular? *Ans.* 12 and 5.

5. Find the side of a square inscribed in a given semicircle whose diameter is d. *Ans.* $\frac{1}{5} d \sqrt{5}$.

6. Find a point in the diagonal of a given square (side a) from which if a perpendicular to the diagonal be drawn, the part of it intercepted between the diagonal and the side of the square may be equal to the difference of the diagonal and side.
Ans. Distance from one extremity of diagonal $= a$.

7. Having given two contiguous sides (m, n) of a parallelogram, and one of its diagonals (d), to find the other diagonal.
Ans. $\sqrt{(2m^2 + 2n^2 - d^2)}$.

8. Given the two sides (a, b) and the line (l) which bisects the vertical angle of a triangle, to find the base. *Ans.* $(a + b)\left\{\frac{ab - l^2}{ab}\right\}^{\frac{1}{2}}$.

9. In a triangle, having given the segments (p, q) of the base made by a perpendicular from the vertical angle, and the ratio $(m : 1)$ of the two sides, to find the sides.

$$Ans.\ m\left\{\frac{(p+q)(p-q)}{(m+1)(m-1)}\right\}^{\frac{1}{2}}, \text{ and } \left\{\frac{(p+q)(p-q)}{(m+1)(m-1)}\right\}^{\frac{1}{2}}.$$

10. In a right-angled triangle the lengths of the lines drawn from the acute angles to the points of bisection of the opposite sides are p and q, to find the sides of the triangle.

$$Ans.\ \text{Base} = 2\sqrt{\left(\frac{4q^2 - p^2}{15}\right)},\ \text{perp.} = 2\sqrt{\left(\frac{4p^2 - q^2}{15}\right)}.$$

11. The radii of two circles which intersect one another are r and r'; and the distance of their centres is c; find the length of their common chord. *Ans.* $\frac{1}{c}\sqrt{(r' + c + r)(r' + c - r)(r + r' - c)(r + c - r')}$.

12. Given the lengths (a, b) of two chords which cut each other at right angles in a circle, and the distance (d) of their point of intersection from the centre, to find the diameter of the circle.
Ans. $\sqrt{\{\frac{1}{2}(a^2 + b^2) + 2d^2\}}$.

13. Find the side of an equilateral triangle inscribed in a circle whose diameter is d, and that of another circumscribed about the same circle.

Ans. $\frac{1}{2} d \sqrt{3}$ and $d \sqrt{3}$.

14. Given one side $AB = a$ of a triangle ABC, to find BC and CA, so that AC, CB, BA and a perpendicular BD on AC may be in continued geometrical progression.

Ans. $BC = a\sqrt{\dfrac{1+\sqrt{5}}{2}}$, $AC = \dfrac{a}{2}(1+\sqrt{5})$.

15. A gentleman has a garden 60 feet long and 40 feet broad, and a walk is to be made of an equal width half round it so as to occupy half the garden; find the breadth of the walk. *Ans* 13·944487 feet.

16. Draw the line D P E (solution of Problem I.) so that the circles described about the triangles A E P and D C P may be equal.

Ans. M E is a fourth proportional to A M, B M, and P M.

17. The lengths of two lines that bisect the acute angles of a right-angled triangle are 40 and 50, it is required to find the sides.

Ans. 35·80737, 47·40728, and 59·41143.

18. Find the side of a regular pentagon inscribed in a circle whose diameter is d, and that of another circumscribed about the same circle.

Ans. $\frac{1}{4} d \sqrt{(10 - 2\sqrt{5})}$ and $d\sqrt{(5 - 2\sqrt{5})}$.

19. A statue eighty feet high stands on a pedestal fifty feet high, and to a spectator on the horizontal plane they subtend equal angles; find the distance of the observer from the base, the height of the eye being five feet. *Ans.* 99·874922 feet.

20. If from one of the angles of a rectangle a perpendicular be drawn to its diagonal, and from the point of intersection with the diagonal lines be drawn perpendicular to the sides which contain the opposite angle; then if p, p' be the lengths of the perpendiculars last drawn, and d the diagonal of the rectangle,

$$p^{\frac{2}{3}} + p'^{\frac{2}{3}} = d^{\frac{2}{3}}.$$

PLANE TRIGONOMETRY.

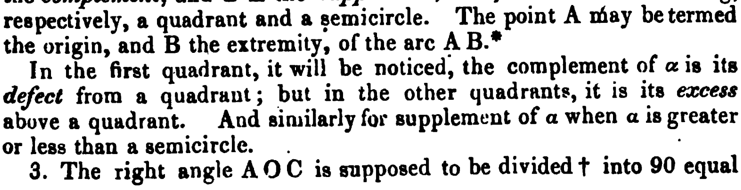
DEFINITIONS AND FIRST PRINCIPLES.

Art. 1. Plane Trigonometry was restricted originally to the science by which the relations between the parts of plane triangles are established. In its extended signification, however, it is understood to comprehend the general relations of arcs and angles, and the properties which connect the sides and angles of any plane rectilineal figures.

In the varied applications of mathematics to physics, trigonometry enables us to combine the practical exactness of numerical calculations with the graphic constructions of geometry. Hence the great importance of this branch of mathematics.

The mode of measuring an angle by means of an arc of the circle whose centre is the angular point will first be pointed out, and then it will be shown how an angle may also be measured by straight lines drawn in and about the circle.

2. Let A O B be any angle; with the *linear unit of length as radius*, and with O as centre, describe a circle, meeting O A, O B in A and B. Produce A O to meet the circle again in E, and draw the diameter C D perpendicular to A E. Then denoting the arc A B by α, B C is called



the *complement*, and B E the *supplement*, of α; A C and A C E being, respectively, a quadrant and a semicircle. The point A may be termed the origin, and B the extremity, of the arc A B.*

In the first quadrant, it will be noticed, the complement of α is its *defect* from a quadrant; but in the other quadrants, it is its *excess* above a quadrant. And similarly for supplement of α when α is greater or less than a semicircle.

3. The right angle A O C is supposed to be divided † into 90 equal

* An angle, as defined by *geometry*, is less than two right angles, and hence the subtending arc of a circle whose centre is the angular point is less than a semicircumference. In trigonometry, however, the terms angle and arc have a more extended meaning. Thus the whole angular space described by the radius O B in revolving about the point O is called an angle, which may therefore be of any magnitude whatever; and, consequently, the subtending arc may consist of any number of circumferences, or parts of these.

† This division of the right angle into 90 equal parts, which is called the *sexagesimal* division, was that adopted by the ancients. Attempts have been made by

parts or degrees; each degree again into 60 equal parts called minutes; each minute into 60 equal parts called seconds; and so on. The notation for degrees, minutes, etc., is $^\circ$, $'$, $''$, etc., written similarly to indices in Algebra. Thus 21 degrees 2 minutes 17 seconds are written, $21^\circ\ 2'\ 17''$.

4. By the preceding Art. and Euc. vi. 33 (fig. to Art. 2),

$$90^\circ : \angle AOB :: \text{quadrant } AC : \text{arc } AB, \text{ or } \angle AOB = \frac{90^\circ}{AC}\cdot AB.$$

Hence if $\frac{90}{AC}$ (that is, 90° divided by the quadrant AC) be taken for the *unit of angular measure*, then the arc $AB = \angle AOB$. In this sense, therefore, the arc AB is the measure of the angle AOB, and either the one or the other is employed indifferently to express the inclination of the lines AO, BO. And, moreover, if the arc AB be given, the angle AOB can be found by this relation between the arc and the angle, and *vice versâ.*

Cor. The angle in a segment of a circle whose radius is the linear unit is measured by half the arc of the opposite segment.

For as the arc AB or α measures the angle AOB, and as AOB (Euc. iii. 20) is double the angle in the segment BCDA, hence the angle in this segment is measured by $\frac{1}{2}\alpha$.

5. The semicircle ACE is represented by ϖ, and hence the quadrant AC by $\frac{1}{2}\varpi$, and the whole circumference ACEDA by 2ϖ. Whence $\frac{1}{2}\varpi-\alpha$, and $\varpi-\alpha$, are respectively the complement and supplement (Art. 2) of α.

It will be shown in a subsequent part of the Course, that $\varpi = 3\cdot14159265......$, the value of this commonly used being $\varpi = 3\cdot1416$.*

6. Two right angles or 180° are represented by π; and therefore 90° by $\frac{1}{2}\pi$. In many works of science, π represents 3·1416, as well as 180°. In this Treatise, two right angles will always be denoted by π or 180°, and 3·1416 by ϖ (the *round* letter), as in Art. 5.

In the preceding articles it has been shown how angles can be expressed numerically by the subtending arcs of the circle whose radius is the linear unit. It will now be shown how arcs, and consequently their corresponding angles, can be measured by means of certain *straight lines* drawn in and about the circle, called the *trigonometrical*

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some eminent mathematicians to introduce the *centesimal* division, by which the right angle is divided into 100 equal parts or degrees, and each degree again into 100 equal parts or minutes, and so on. This method, though partially adopted in France, is not given in any English work. Its adoption in this country would have this advantage, that arithmetical operations could be performed on angles in the same manner as on any other decimal fractions; but it would require a complete transformation of all our tables which depend upon the sexagesimal division.

* In the *measurement* of an angle by *geometry*, the right angle is assumed as the primitive angle, or angular unit, with which all other angles are compared. By the *algebraic* method, however, the angular unit (as in Art. 4) is 90° divided by the length of the quadrant to radius unity. Hence to convert the *measure* of an angle into degrees it is necessary to multiply it by this unit; thus, if it be required to find the number of degrees (x°) in an angle whose measure is $\frac{2}{5}$, we have by Arts. 4 and 5,

$$x^\circ = \frac{90^\circ}{\frac{1}{2}\varpi}\cdot\frac{2}{5} = \frac{180^\circ}{\varpi}\times .4 = 22^\circ.9183118 = 22^\circ\ 55'\ 5''.9.$$

functions of an arc. These lines are named Sines ~~sines~~, tangents, etc., and they have been calculated (by methods which will be explained hereafter) for small divisions of the quadrant. By these functions of the angle or arc, it will be seen, we are enabled to compare the sides and angles of triangles.

We will first describe these lines, and then deduce their geometrical properties.

The lines B F, G A, referred to in the subsequent definitions, are drawn perpendicular to the diameter A E, and B K, C H, to C D (see fig. to Art. 2).

7. The *sine* of an arc, or of the angle of which the arc is the measure, is the perpendicular let fall from the extremity of the arc upon the diameter passing through the centre and the origin of the arc: thus B F is the sine of α, or of the angle A O B, and is written for brevity $\sin \alpha$.*

Cor. The chord of an arc is equal to twice the sine of half that arc. (*By chord of an arc* is meant the straight line passing through the origin and extremity of the arc.)

For if we produce B F to meet the circle again in b, then (Euc. iii., 3, 28), the arc B A = ½ arc B A b, and B b = 2 B F, or the chord of B A $b = 2 \sin \alpha$.

8. The *tangent* of an arc is the line drawn perpendicular to the diameter at the origin of the arc, and limited by the intersection of a line passing through the centre and the extremity of the arc: thus A G is the tangent of α, and is written $\tan \alpha$.

9. The *secant* of an arc is the line passing through or from the centre, through the extremity of the arc, and limited by the centre and the tangent: thus O G is the secant of α, and is written $\sec \alpha$.

10. The *versed sine* of an arc is the line or distance between the origin of the arc and the sine: thus A F is the versed sine of α, and is written $\text{vers}\ \alpha$.

11. The *cosine*, *cotangent*, *cosecant*, and *coversed sine*, are the *sine*, *tangent*, *secant*, and *versed sine* of the complemental arc: thus B K, C H, O H, and C K are, respectively, the cosine, cotangent, cosecant, and coversed sine of α, and are written $\cos \alpha$, $\cot \alpha$, $\text{cosec}\ \alpha$, $\text{covers}\ \alpha$. Hence $\cot \alpha = \tan (\frac{1}{2} \varpi - \alpha)$, $\cos \alpha = \sin (\frac{1}{2} \varpi - \alpha)$, etc.

Cor. Since $\cos \alpha$ = B K = O F, hence the cosine of an arc is equal to that part of the radius which is intercepted between the sine and the centre.

12. The *algebraical signs* (+ or −) of the preceding functions of an arc ($\sin \alpha$, $\tan \alpha$, etc.) are determined by the *geometry of co-ordinates*. Thus the sine and tangent are positive or negative according as they are above or below the diameter A E, and the cosine and cotangent are positive or negative according as they are to the right or left of the diameter C D. Hence,

The sine is positive (+) in the first and second quadrants, and negative (−) in the third and fourth.

* It must be kept constantly in mind that in this and the subsequent definitions, reference is made to a *particular circle*, viz., that whose radius is the linear unit. It would not be correct to say that B F is the sine of the angle A O B to *any* radius. As will be seen presently, the sine in such case is expressed by a ratio. (See *Functions of an Angle*.)

The tangent is positive in the first and third, and negative in the second and fourth, quadrants.

The cosine is positive in the first and fourth, and negative in the second and third, quadrants.

The cotangent is positive in the first and third, and negative in the second and fourth, quadrants.

The secant and cosecant, being *revolving lines*, are not drawn in the same direction as any of the other functions, and hence their algebraic signs are not determined by the same principles.* They are, however, positive or negative according as they pass from, or through, the centre. Hence,

The secant (like the cosine) is positive in the first and fourth, and negative in the second and third, quadrants.

The cosecant (like the sine) is positive in the first and second quadrants, and negative in the third and fourth.

The versed sine and coversed sine are positive in all the quadrants, as they are always measured in the same direction: the former increases from 0 to 2 in the first and second quadrants, and then decreases from 2 to 0 in the third and fourth; the latter from 1 to 0 in the first quadrant, from 0 to 2 in the second and third, and from 2 to 1 in the fourth.

We might proceed in a similar way to the fifth, sixth, etc., quadrants, but the functions would be merely repeated.

13. The following values are readily deduced from the preceding definitions :—

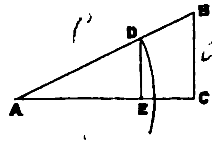
Angle.	Sin.	Cos.	Tan.	Cot.	Sec.	Cosec.	Vers.	Covers.
0	0	1	0	$\frac{1}{0}$	1	$\frac{1}{0}$	0	1
$\frac{1}{2}\pi$	1	0	$\frac{1}{0}$	0	$\frac{1}{0}$	1	1	0
π	0	-1	0	$-\frac{1}{0}$	-1	$\frac{1}{0}$	2	1
$\frac{3}{2}\pi$	-1	0	$-\frac{1}{0}$	0	$\frac{1}{0}$	-1	1	2
2π	0	1	0	$\frac{1}{0}$	1	$\frac{1}{0}$	0	1

The symbol $\frac{1}{0}$ denotes a quantity whose value is infinitely great.

* The following definitions of the secant and cosecant, which were suggested to me by my friend and colleague, Mr. Heather, are not liable to this objection :—

The secant of an arc is the straight line drawn from the centre through the origin of the arc, and limited by the centre and the line which touches the circle at the extremity of the arc. Thus O L is the secant of a. Also the cosecant of a is the line O M.

TRIGONOMETRICAL FUNCTIONS OF AN ANGLE.



14. Let A B C be a triangle, right-angled at C, and A D the radius or linear unit, as in the preceding definitions. Draw D E perpendicular to A C, and denote the sides of the triangle A B C opposite to the angles A, B, C, by a, b, c, respectively. Then by similar triangles, and Art. 7,

$$c : a :: 1 : \sin A, \text{ or } \sin A = \frac{a}{c}.$$

Similarly, $\tan A = \frac{a}{b}$, $\cos A = \frac{b}{c}$, $\cot A = \frac{b}{a}$, $\sec A = \frac{c}{b}$, etc.

The ratios $\frac{a}{c}$, $\frac{a}{b}$, etc., are termed the *trigonometrical functions of the angle* B A C; they are obviously of the same value, whatever be the lengths of the lines A B, A C. These are the ratios to which reference is made in Art. 7, *note.*

Cor. Let α be the measure of an angle as in Art. 4, and a the length of the corresponding arc to radius r; then,

$$1 : \alpha :: r : a, \text{ or } \alpha = \frac{a}{r}.$$

Hence the magnitude of an angle is determined by the expression $\frac{\text{arc}}{\text{radius}}$.

Ex. Given the hypothenuse A B = 11, of a right-angled triangle A B C, and the angle A = 60°, to find the other parts, a and b.

If (fig. to Art. 2) A O B = 60°, and O A = O B, then if the line A B be drawn, the triangle A O B will be equilateral; and hence O F = cos 60° = ½ A O = ½. Whence sin 60° = B F = $\sqrt{}$ (B O² − O F²) = ½ $\sqrt{3}$. Consequently by the preceding formula (fig. of this Art.),

$$a = c \sin A = 11 \sin 60^\circ = \frac{11}{2}\sqrt{3} = 9 \cdot 526279.$$

Also, $b = c \cos A = 11 \cos 60^\circ = 5 \cdot 5.$

FUNCTIONS OF ONE ARC OR ANGLE.

15. *Relations amongst the trigonometrical functions of one arc or angle, and deductions from these.*

In the following investigations fig. 1 of Art. 2 is referred to, and the angle A O B is denoted by A.

By the preceding definitions, and Euc. i., 47,

$$BF^2 + FO^2 = OB^2, \text{ or } \sin {}^2A + \cos {}^2A = 1 \quad \ldots\ldots \quad (1^*).$$

Cor. $\sin A = \sqrt{(1 - \cos {}^2A)}$, and $\cos A = \sqrt{(1 - \sin {}^2A)}$.

Also by the properties of similar triangles (Euc. vi., 4, 8),

$$OF : FB :: OA : AG = \frac{FB.OA}{OF},$$

* The expression sin ²A denotes the second power of sin A. This power is sometimes written (sin A)²; and similarly for the second and higher powers of other functions. The former notation is adopted in this Treatise.

$$\text{or } \tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{(1-\sin^2 A)}} \quad \ldots\ldots \quad (2),$$

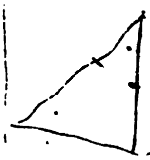
$$OK : KB :: OC : CH = \frac{KB.OC}{OK},$$

$$\text{or } \cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{(1-\sin^2 A)}}{\sin A} \quad \ldots\ldots \quad (3),$$

$$OF : OB :: OB : OG = \frac{OB^2}{OF},$$

$$\text{or } \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{(1-\sin^2 A)}} \quad \ldots\ldots \quad (4),$$

$$KO : OB :: OB : OH = \frac{OB^2}{KO},$$



$$\text{or } \operatorname{cosec} A = \frac{1}{\sin A} \quad \ldots\ldots \quad (5),$$

$$GA : AO :: OC : CH = \frac{AO.OC}{GA},$$

$$\text{or } \cot A = \frac{1}{\tan A} = \frac{\sqrt{(1-\sin^2 A)}}{\sin A} \quad \ldots\ldots \quad (6).$$

Hence also the following relations:—

$$OG^2 = OA^2 + AG^2, \text{ or } \sec^2 A = 1 + \tan^2 A = 1 + \frac{\sin^2 A}{1-\sin^2 A} \quad \ldots \quad (7),$$

$$OH^2 = OC^2 + CH^2, \text{ or } \operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{1-\sin^2 A}{\sin^2 A} \quad \ldots \quad (8),$$

$$AF = OA - OF, \text{ or vers } A = 1 - \cos A = 1 - \sqrt{(1-\sin^2 A)} \quad \ldots \quad (9),$$

$$CK = OC - OK, \text{ or covers } A = 1 - \sin A \quad \ldots\ldots \quad (10).$$

These properties are equally true for an angle greater than a right angle, Art. 12 being kept in mind.

Scholium.—It will be seen from the preceding relations that each of the trigonometrical functions of A can be expressed in terms of sin A; and similarly each might be expressed in terms of any other function. Hence we might have restricted ourselves to *one function* only in the definitions. This method, however, would have been much less effective in the complete development of trigonometry than that which has been adopted. The ancients, it would seem, confined themselves to one function (the chord), to which the Arabians added the sine.

Again, at the points C and E make the angles C O P, E O Q, each equal to the angle A O B, and draw the perpendiculars P N, Q R. Then (Euc. i. 26), the triangles, F O B, P N O, Q R O, are equal, and F O = R O = P N, B F = Q R = N O. Hence (Arts. 7, 11, 12),

$$\sin(\tfrac{1}{2}\pi + A) = \sin AOP = PN = FO = \cos A \quad \ldots \quad (11),$$

$$\cos(\tfrac{1}{2}\pi + A) = \cos AOP = -ON = -BF = -\sin A \quad \ldots \quad (12),$$

$$\sin(\pi - A) = \sin AOQ = QR = BF = \sin A \quad \ldots \quad (13),$$

$$\cos(\pi - A) = \cos AOQ = -OR = -OF = -\cos A \quad \ldots \quad (14).$$

Whence by the expressions (1) .... (5),

$$\tan(\tfrac{1}{2}\pi + A) = \frac{\sin(\frac{1}{2}\pi + A)}{\cos(\frac{1}{2}\pi + A)} = -\frac{\cos A}{\sin A} = -\cot A \quad \ldots \quad (15),$$

$$\tan(\pi - A) = \frac{\sin(\pi - A)}{\cos(\pi - A)} = -\frac{\sin A}{\cos A} = -\tan A \quad . \quad (16),$$

$$\cot(\pi - A) = \frac{\cos(\pi - A)}{\sin(\pi - A)} = -\frac{\cos A}{\sin A} = -\cot A \quad . \quad (17),$$

$$\sec(\pi - A) = \frac{1}{\cos(\pi - A)} = -\frac{1}{\cos A} = -\sec A \quad . \quad (18),$$

$$\operatorname{cosec}(\pi - A) = \frac{1}{\sin(\pi - A)} = \frac{1}{\sin A} = \operatorname{cosec} A \quad . \quad (19).$$

Consequently *the sine and cosecant of an angle are the same as those of its supplement; the cosine, tangent, cotangent, and secant, are equal in magnitude with contrary signs.*

The following are left as exercises for the student:—

$\cot(\frac{1}{2}\pi + A) = -\tan A \ldots (20)$, $\sec(\frac{1}{2}\pi + A) = -\operatorname{cosec} A$. (21),
$\operatorname{covers}(\frac{1}{2}\pi + A) = 1 - \cos A \ldots (22)$, $\operatorname{vers}(\pi - A) = 1 + \cos A$. (23),
$\sec(\pi + A) = -\sec A \ldots (24)$, $\tan(\pi + A) = \tan A$ . . . (25),
$\cot(\pi + A) = \cot A \ldots (26)$, $\operatorname{vers}(\pi + A) = 1 + \cos A$. (27),
$\sin(\pi + A) = -\sin A \ldots (28)$, $\cos(\pi + A) = -\cos A$. (29),
$\sec(2\pi - A) = \sec A \ldots (30)$, $\operatorname{vers}(2\pi - A) = \operatorname{vers} A$ . . (31).

If, moreover, the angle A O B be *positive* when measured towards C, it will be negative when measured in the contrary direction (Art. 12). Hence the angle A O b will be denoted by $-A$, and consequently by Arts. 7, 8, 11, 12,

$\operatorname{Sin}(-A) = -b\,F = -BF = -\sin A$. (32), $\cos(-A) = OF = \cos A$ . . (33).

$$\text{Whence } \tan(-A) = \frac{\sin(-A)}{\cos(-A)} = -\tan A \quad . \ . \ . \ . \ . \quad (34).$$

And similarly for other functions of $-$ A.

The numerical values of the trigonometrical functions of 30°, 60°, 45° may now be found.

By Art. 14,

$$\cos 60^\circ = \tfrac{1}{2}, \ \sin 60^\circ = \tfrac{1}{2}\sqrt{3}\,; \text{ hence } \tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3} \ . \ . \ (35).$$

Also because 30° is the complement of 60°,

30°. to the angle

$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$, $\cos 30^\circ = \sin 60^\circ = \frac{1}{2}\sqrt{3}$, etc. . . . . (36).

Again (fig. to Art. 2), if the angle A O B = 45°, then O B F = 45°, and O F = F B, or sin 45° = cos 45°. But $\sin^2 45^\circ + \cos^2 45^\circ = 1$,

$$\text{whence } \sin 45^\circ = \cos 45^\circ = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}, \ \tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1 \ \ldots \ (37).$$

The values of other functions of 30°, 45°, and 60° may be found in a similar way.

Exercises.

1. The tangent of an angle is $\frac{1}{2}$; find the surd expressions for the sine, cosine, secant, and cosecant of the same.

Ans. If α be the angle,

$\sin\alpha = \frac{1}{5}\sqrt{5}$, $\cos\alpha = \frac{2}{5}\sqrt{5}$, $\sec\alpha = \frac{1}{2}\sqrt{5}$, $\operatorname{cosec}\alpha = \sqrt{5}$.

2. Find the complements of the angles

26° 7′ 8″·21, 98° 16′ 30″, and 218° 5′ 6″·67.

3. Find the supplements of

$$155^\circ\ 7'\ 8''\cdot 9, \text{ and } 224^\circ\ 5'\ 8''.$$

4. Find the circular measures of the angles

$$61^\circ,\ 72^\circ\ 5'\ 20'', \text{ and } 88^\circ\ 19'\ 30''.$$

Ans. 1·0646, 1·2582, 1·5415.

5. Find the angles whose circular measures are $\frac{3}{4}$ and $\frac{2}{5}$.

Ans. 43° nearly, and 22°·9.

6. Required the arc whose supplement is to its complement as four to one. *Ans.* $\theta = 60^\circ$.

7. Required the arc the sum of whose supplement and complement is to their difference as two to one. *Ans.* $\theta = 45^\circ$.

Prove the following relations :—

8. $\text{Sin}^2\alpha \sin^2\beta + \cos^2\alpha \cos^2\beta + \sin^2\alpha \cos^2\beta + \sin^2\beta \cos^2\alpha = 1.$

9. $(\text{Sec}\,\alpha + \tan\alpha)^2 = \dfrac{1+\sin\alpha}{1-\sin\alpha}$, and $(\sec\alpha - \tan\alpha)^2 = \dfrac{1-\sin\alpha}{1+\sin\alpha}$.

10. $\text{Vers}\,(\frac{1}{2}\pi + \theta)\ \text{vers}\,(\frac{1}{2}\pi - \theta) + \text{vers}\,\theta\ \text{vers}\,(\pi - \theta) = 1.$

11. $\text{Sec}^2\alpha \tan^2\alpha (\sec^2\beta - 1)(\tan^2\beta + 1) - \sec^2\beta \tan^2\beta (\sec^2\alpha - 1) \times (\tan^2\alpha + 1) = 0.$

12. $(\text{Sin}^2\alpha - \sin^2\alpha \sin^2\beta)(\cos^2\alpha - \cos^2\alpha \cos^2\beta) - (\sin^2\beta - \sin^2\alpha \sin^2\beta)(\cos^2\beta - \cos^2\alpha \cos^2\beta) = 0.$

13. $\text{Tan}^2(\frac{1}{2}\pi + \theta) - \cot^2(\frac{1}{2}\pi + \theta) = \text{cosec}^2\theta - \sec^2\theta.$

14. $\text{Cot}^2\alpha \cos^2\alpha = \cot^2\alpha - \cos^2\alpha$, $\sec^2\alpha\, \text{cosec}^2\alpha = \sec^2\alpha + \text{cosec}^2\alpha$.

15. $\text{Tan}^2\theta = \dfrac{\sec\theta}{\text{cosec}\,\theta \cot\theta}$, and $\sec^2\theta + \tan^2\theta = \dfrac{\cot^2\theta + \cos^2\theta}{\cot^2\theta - \cos^2\theta}$.

16. $\text{Sin}^2\theta - \sin^2\phi = \sin^2\theta \cos^2\phi - \cos^2\theta \sin^2\phi$, and $\cos^2\theta - \sin^2\phi = \cos^2\theta \cos^2\phi - \sin^2\theta \sin^2\phi$.

17. $\text{Sin}^2\theta \tan^2\theta = \tan^2\theta - \sin^2\theta$, and $\tan^2\theta - \tan^2\phi = \dfrac{\cos^2\phi - \cos^2\theta}{\cos^2\theta \cos^2\phi}$.

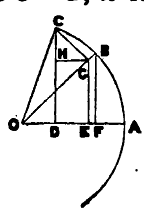
18. $\text{Cosec}\,\theta \sec\theta = \dfrac{\sec\theta \cot\theta - \text{cosec}\,\theta \tan\theta}{\cos\theta - \sin\theta}$, and

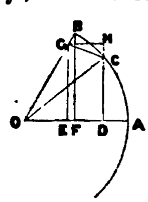
$$\sin\theta = \frac{\text{chd}^2(\frac{1}{2}\pi + \theta) - \text{chd}^2(\frac{1}{2}\pi - \theta)}{\text{chd}^2(\frac{1}{2}\pi + \theta) + \text{chd}^2(\frac{1}{2}\pi - \theta)}.$$

FUNCTIONS OF TWO OR MORE ANGLES.

16. *Expressions for the sine and cosine of the sum and difference of any two angles, and formulæ deducible from these.*

Let the angle A O B = A, B O C = B, to radius unity; then will A O C = A + B in fig. 1, and A O C = A − B in fig. 2. Draw C G perpendicular to the radius O B; B F, G E, C D perpendicular to O A, and H G to C D.





In both figures we have by the similar triangles O E G, O F B, C H G,

$GE(=HD) : GO :: BF : BO$, or $HD = BF.GO = \sin A \cos B$,
$HC \quad : CG :: FO : BO$, or $HC = GC.FO = \sin B \cos A$,
$EO \quad : OG :: FO : BO$, or $EO = FO.GO = \cos A \cos B$,
$HG(=DE) : GC :: FB : BO$, or $DE = FB.GC = \sin A \sin B$.

Hence, taking fig. 1 and fig. 2 alternately,

$\sin (A+B) = CD = HD + HC = \sin A \cos B + \sin B \cos A \ldots\ldots (1),$*
$\sin (A-B) = CD = HD - HC = \sin A \cos B - \sin B \cos A \ldots\ldots (2),$
$\cos (A+B) = OD = OE - DE = \cos A \cos B - \sin A \sin B \ldots\ldots (3),$
$\cos (A-B) = OD = OE + DE = \cos A \cos B + \sin A \sin B \ldots\ldots (4).$

It is assumed in this investigation that $A + B < \frac{1}{2}\pi$, and $A > B$; it therefore remains to be proved that the formulæ (1, 2, 3, 4) hold for every positive or negative value of A and B.

For this purpose assume $A = \frac{1}{2}\pi - x$, $B = \frac{1}{2}\pi - y$; and let these values be such that $A + B = \pi - (x + y)$, may be positive, and less than $\frac{1}{2}\pi$: then it will be obvious that $x + y > \frac{1}{2}\pi$. Now

$\sin (A + B) = \sin \{(\frac{1}{2}\pi - x) + (\frac{1}{2}\pi - y)\} = \sin\{\pi - (x + y)\}$,

since $A + B$ is equal to each of these expressions within the brackets.

But by (1), $\sin \{(\frac{1}{2}\pi - x) + (\frac{1}{2}\pi - y)\} = \sin (\frac{1}{2}\pi - x) \cos (\frac{1}{2}\pi - y)$
$+ \sin (\frac{1}{2}\pi - y) \cos (\frac{1}{2}\pi - x) = \cos x \sin y + \cos y \sin x$;

and $\quad \sin \{\pi - (x + y)\} = \sin (x + y)$.

Hence $\quad \sin (x + y) = \sin x \cos y + \sin y \cos x$.

And in general, if we put (m and n being any positive or negative integers)

$$A = m\pi - x, \text{ and } B = n\pi - y,$$

it might be shown in a similar way that

$$\sin (x + y) = \sin x \cos y + \sin y \cos x.$$

In (1) and (3) let $B = A$; then keeping in mind that $\sin^2 A + \cos^2 A = 1$,

$\sin 2A = 2 \sin A \cos A$, or $\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A \; . \; . \; . \; . \; . \; . \; (5),$

$$\left.\begin{array}{l}\text{and } \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A \\ \text{or } \cos A = \cos^2 \tfrac{1}{2} A - \sin^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2} A - 1 = 1 - 2 \sin^2 \tfrac{1}{2} A\end{array}\right\} \ldots (6).$$

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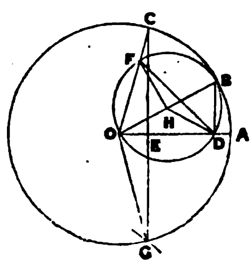
* The following is another very simple method of proving the formulæ (1, 2, 3, 4).

Let AOB, BOC be two angles of the circle ACG whose radius is the linear unit. From B draw BD, BF perpendicular respectively to the radii OA, OC, and from C draw CE perpendicular to OA, meeting the circle again in G. Bisect OB in H; join FH, HD, OG; and denote the angles AOB, BOC by A and B.



Then because each of the angles ODB, OFB is a right angle, a circle described from H, with $\frac{1}{2}$OB as radius will pass through the points O, D, B, F. Hence the angle $FHD = 2COA = COG$; the triangles FHD, COG, are consequently similar, and therefore $FD = \frac{1}{2}GC = EC$, since $FH = \frac{1}{2}CO$. Also the perpendicular from H on $FD = \frac{1}{2}OE$; whence the difference of the perpendiculars from O and B on FD, is equal to EO, as each of these quantities (Euc. i. Ex.) is double of the perpendicular from H on DF. Hence (Euc. vi. D.C.),

$OB.EC = BD.OF + BF.OD$, and $OB.OE = OD.OF - BD.BF$;

or $\sin (A+B) = \sin A \cos B + \sin B \cos A$, $\cos (A+B) = \cos A \cos B - \sin A \sin B$.

If we interchange the perpendiculars BD, CE, so that $CE = \sin (A - B)$, we shall get in a similar way,

$\sin (A-B) = \sin A \cos B - \sin B \cos A$, $\cos (A-B) = \cos A \cos B + \sin A \sin B$.

In this case, OE is equal to the *sum* of the perpendiculars from O and B on FD.

From (6),

$$\sin^2 \tfrac{1}{2}A = \frac{1 - \cos A}{2}, \text{ and } \cos^2 \tfrac{1}{2}A = \frac{1 + \cos A}{2} \quad \ldots\ldots (7).$$

$$\text{Hence } \tan^2 \tfrac{1}{2}A = \frac{\sin^2 \frac{1}{2}A}{\cos^2 \frac{1}{2}A} = \frac{1 - \cos A}{1 + \cos A} \quad \ldots\ldots\ldots\ldots (8).$$

By (5) also and the formula $\sin^2 A + \cos^2 A = 1$, we have

$$\sin^2 A + 2 \sin A \cos A + \cos^2 A = 1 + \sin 2A,$$
$$\sin^2 A - 2 \sin A \cos A + \cos^2 A = 1 - \sin 2A;$$

which give the values

$$\sin A = \tfrac{1}{2} \{ \pm \sqrt{(1 + \sin 2A)} \pm \sqrt{(1 - \sin 2A)} \} \ldots (9),$$
$$\cos A = \tfrac{1}{2} \{ \pm \sqrt{(1 + \sin 2A)} \mp \sqrt{(1 - \sin 2A)} \} \ldots (10)^*.$$

Again, by addition and subtraction of the expressions (1, 2, 3, 4),

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B \ldots\ldots (11),$$
$$\sin (A + B) - \sin (A - B) = 2 \cos A \sin B \ldots\ldots (12),$$
$$\cos (A + B) + \cos (A - B) = 2 \cos A \cos B \ldots\ldots (13),$$
$$\cos (A - B) - \cos (A + B) = 2 \sin A \sin B \ldots\ldots (14).$$

In (11) and (13), respectively, let $A = nB$; then we have the general expressions,

$$\sin (n + 1) B = 2 \sin nB \cos B - \sin (n - 1) B \ldots (15),$$
$$\cos (n + 1) B = 2 \cos nB \cos B - \cos (n - 1) B \ldots (16).$$

Next, to find the tangents of the sum and difference of two angles in terms of the tangents of the angles, we have by (1) and (3) of this Art., and (2) of Art. 15,

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B}$$
$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \ldots\ldots (17).$$

The last form is got by dividing both numerator and denominator of the preceding, by $\cos A \cos B$.

$$\text{Similarly, } \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \ldots (18).$$

$$\text{Hence by (17), } \tan (A + B + C) = \frac{\tan (A + B) + \tan C}{1 - \tan (A + B) \tan C}$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C} \ldots (19).$$

---

* The signs of these formulæ will be best interpreted by special cases.

For values of A between 0 and 45°, sin A and cos A will both be positive, and $\cos A > \sin A$; $\sin A + \cos A$, or its equal $\sqrt{(1 + \sin 2A)}$, will therefore be positive, and $\sin A - \cos A$, or its equal $\sqrt{(1 - \sin 2A)}$, will be negative. Hence in this case (9) and (10) become, by means of the preceding expressions from which they are derived,

$$\sin A = \tfrac{1}{2}\{ \sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\},$$
$$\cos A = \tfrac{1}{2}\{ \sqrt{(1 + \sin 2A)} + \sqrt{(1 - \sin 2A)}\}.$$

Again, for values of A between 45° and 90° (the values of the trigonometrical functions of 45° have already been determined in Art. 15), $\sin A + \cos A$ and $\sin A - \cos A$ will both be positive, hence

$$\sin A = \tfrac{1}{2}\{ \sqrt{(1 + \sin 2A)} + \sqrt{(1 - \sin 2A)}\},$$
$$\cos A = \tfrac{1}{2}\{ \sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\}.$$

And in a similar way for other cases.

The following are of frequent use :—

$$\frac{\sin(A+B)}{\sin(A-B)} = \frac{\sin A \cos B + \sin B \cos A}{\sin A \cos B - \sin B \cos A} = \frac{\tan A + \tan B}{\tan A - \tan B} \ldots (20),$$

$$\sin(A+B)\sin(A-B) = (\sin A \cos B + \sin B \cos A)$$
$$\times (\sin A \cos B - \sin B \cos A) = \sin^2 A \cos^2 B - \sin^2 B \cos^2 A$$
$$= \sin^2 A - \sin^2 B = \sin^2 A \sin^2 B (\operatorname{cosec}^2 B - \operatorname{cosec}^2 A) \ldots\ldots (21).$$

Again, if $A > B$, then

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B),$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B);$$

hence by (1, 2, 3, 4), and addition and subtraction,

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \ldots\ldots (22),$$
$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) \ldots\ldots (23),$$
$$\cos B + \cos A = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \ldots\ldots (24),$$
$$\cos B - \cos A = 2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) \ldots\ldots (25).$$

Whence,

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \ldots\ldots (26),$$

$$\frac{\cos B + \cos A}{\cos B - \cos A} = \frac{\cot \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \ldots\ldots (27).$$

By different combinations of the preceding, a great number of other formulæ might be deduced ; but in an elementary work like the present, those only can be given that are of frequent occurrence. The following are left as exercises for the student :—

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A} \ldots\ldots (28),$$

$$\sec(A+B) = \frac{\sec A \sec B}{1 - \tan A \tan B} \ldots\ldots (29),$$

$$\cot A + \tan A = 2 \operatorname{cosec} 2A \ldots\ldots\ldots (30),$$
$$\cot A - \tan A = 2 \cot 2A \ldots\ldots\ldots (31),$$

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A+B) \ldots\ldots (32),$$

$$\frac{\sin A - \sin B}{\cos B - \cos A} = \cot \tfrac{1}{2}(A+B) \ldots\ldots (33),$$

$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$$
$$= \cos^2 A \sin^2 B (\operatorname{cosec}^2 B - \sec^2 A) \ldots\ldots (34).$$

Additional Exercises.

Prove the following formulæ :—

1. $\operatorname{Sin} A = \sin B \cos(A-B) + \cos B \sin(A-B)$.
2. $\operatorname{Cos} A = \sin B \sin(A+B) + \cos B \cos(A+B)$.
3. $\dfrac{1+\sin A}{1+\cos A} = \tfrac{1}{2}(1+\tan\tfrac{1}{2}A)^2$, and $\dfrac{1+\sin A}{1-\cos A} = \tfrac{1}{2}(1+\cot\tfrac{1}{2}A)^2$.
4. $\dfrac{1-\sin A}{1+\cos A} = \tfrac{1}{2}(1-\tan\tfrac{1}{2}A)^2$, and $\dfrac{1-\sin A}{1-\cos A} = \tfrac{1}{2}(\cot\tfrac{1}{2}A - 1)^2$.

5. Tan α tan $(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{\cot\alpha-\tan\beta}$, and

$$\tan\alpha\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{\cot\alpha+\tan\beta}.$$

6. (Cos $\alpha+\cos\beta)\{1-\cos(\alpha+\beta)\}=(\sin\alpha+\sin\beta)\sin(\alpha+\beta)$.

7. Sin $2\theta=\dfrac{\tan 2\theta\tan\theta}{\tan 2\theta-\tan\theta}$, and $\cos 2\theta=\dfrac{\tan\theta}{\tan 2\theta-\tan\theta}$.

8. Sec $2\theta=\dfrac{1+\tan^2\theta}{1-\tan^2\theta}$, and cosec $2\theta=\dfrac{\sec^2\theta}{2\tan\theta}$.

9. Sec $^2\theta=\dfrac{2\sec 2\theta}{\sec 2\theta+1}$, and cosec $^2\theta=\dfrac{2\sec 2\theta}{\sec 2\theta-1}$.

10. Vers $(\pi-\theta)=\sin\theta\cot\frac{1}{2}\theta$, chd $(\pi-\theta)=\dfrac{\text{chd } 2\theta}{\text{chd }\theta}$.

11. (Sec $\frac{1}{4}\pi+\sin 2\theta)(\sec\frac{1}{4}\pi-\sin 2\theta)=2(\sin^4\theta+\cos^4\theta)$.

12. $2\cot 2\theta=\cot(\frac{1}{4}\pi+\theta)-\tan(\frac{1}{4}\pi+\theta)$.

13. Sin $45^\circ=\sin 75^\circ-\sin 15^\circ$, and cosec $30^\circ=$ cosec $18^\circ-$ cosec 54°.

14. $1+\cos 2A\cos 2B=2(\sin^2 A\sin^2 B+\cos^2 A\cos^2 B)$.

INVERSE TRIGONOMETRICAL FUNCTIONS.

17. Let $\sin\alpha=s$, $\cos\alpha=c$, $\tan\alpha=t$, etc. . . . . (1); then α is an arc whose sine, cosine, tangent, are s, c, t, etc., and these are termed *direct* functions of the arc α. On the contrary, the arc α is termed an *inverse* function of s, c, or t, and is expressed by the index -1 written above the symbol which indicates the function. Thus, instead of saying that α is the arc whose sine is s, etc., this relation is expressed by the notation,

$$\alpha=\sin^{-1}s=\cos^{-1}c=\tan^{-1}t,\text{ etc. . . . . (2)},$$

in which -1 merely expresses the connexion between the arc and its function in accordance with the preceding explanation. The relations then,

$$\alpha=\sin^{-1}1,\ \beta=\cos^{-1}\tfrac{1}{2},\ \gamma=\tan^{-1}1,$$

are equivalent to

$$\alpha=90^\circ,\ \beta=60^\circ,\ \gamma=45^\circ;$$

for it has been shown that $\sin 90^\circ=1$, $\cos 60^\circ=\frac{1}{2}$, $\tan 45^\circ=1$.

Again by (2), $\sin\sin^{-1}s=\sin\alpha$, but by (1), $\sin\alpha=s$, hence $\sin.\sin^{-1}s=s$; consequently *sin* and *sin*$^{-1}$ indicate operations that mutually destroy each other, and so for other functions.

The student will see the value of this notation when he comes to the Integral Calculus.

CHANGING THE RADIUS IN TRIGONOMETRICAL EQUATIONS.

18. As has been noticed in a preceding part (the "Application,") when in applying algebra to the resolution of a problem of geometry or physics, we have not taken any of the quantities under consideration for the linear unit, the equation or equations which we obtain are necessarily *homogeneous*. In the preceding investigations, however, the radius has been assumed as the unit, and hence if we wish to transform the results into others in which the radius is some other quantity r, this

quantity must be introduced as a multiplier or divisor, as the case may be, to render all the terms of the equation or equations of the same degree. Thus, (1), (2), (3) of single angles, to radius 1, become

$$\sin^2 A + \cos^2 A = r^2, \frac{\tan A}{r} = \frac{\sin A}{\cos A}, \frac{\cot A}{r} = \frac{\cos A}{\sin A} \text{ to radius } r.$$

Again, in the equation $\sin 3A - 3\sin A + 4\sin^3 A = 0$, the first and second terms are of the *first* degree, but the remaining one is of the *third*; it is necessary, therefore, to render the equation homogeneous to multiply the first and second terms by r^2. Hence

$$r^2 \sin 3A - 3r^2 \sin A + 4\sin^3 A = 0.$$

This method applies to all equations which involve *direct* functions only of the arcs or angles. In the case of *inverse functions*, we thus proceed:—

Let α and β be the arcs which subtend an angle A° to radii 1 and r, then,

$$1 : r :: \alpha : \beta, \text{ hence } \beta = r\alpha, \text{ and } \alpha = \frac{\beta}{r}.$$

Consequently, if the expressions are to be transformed from radius r to radius 1, $r\alpha$ must be written for the arc β, and if they are to be transformed to radius r, $\frac{\beta}{r}$ must be written for the arc α.

Miscellaneous Exercises on Arcs and Angles.

1. Given $\operatorname{cosec} \alpha = 5\cdot6$, to find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$.
2. Prove the following formulæ:—

$$\frac{\sin 2A}{1+\cos 2A}\cdot\frac{\cos A}{1+\cos A} = \tan \tfrac{1}{2}A; \quad 2(\operatorname{cosec}^2 2A + \cot^2 2A) = \tan^2 A + \cot^2 A.$$

3. If $A + B + C = 90°$, then

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C,$$
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C + \sec A \sec B \sec C,$$
$$\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C.$$

4. If $A + B + C = 180°$, then

$$\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C,$$
$$\cos A + \cos B + \cos C = 4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C + 1,$$
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

5. Find x from each of the equations:

$$\sin(x+a) = \cos(x-a),$$
$$\sin(x+a) + \cos(x+a) = \sin(x-a) + \cos(x-a),$$
$$\sin a + \sin(x-a) + \sin(2x+a) = \sin(x+a) + \sin(2x-a).$$

6. Prove that

$$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = 90°, \text{ and } \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45°.$$

7. Eliminate θ and ϕ from the equations:

$$a\sin^2\theta + b\cos^2\theta = m,\ b\sin^2\phi + a\cos^2\phi = n,\ a\tan\theta = b\tan\phi.$$

8. Given $\sin\phi + \cos\theta = a$, and $\sin\theta + \cos\phi = b$, to find θ and ϕ.
9. Determine α from each of the equations

$$\tan\alpha + \cot\alpha = 4, \text{ and } \tan\alpha + 3\cot\alpha = 4.$$

10. Prove that if

$$\cos(A - C)\cos B = \cos(A - B + C);$$

$\tan A$, $\tan B$, and $\tan C$ are in harmonical progression.

11. Prove that if $\cos A = \cos B \cos C$, then

$$\tan^2 \tfrac{1}{2} C = \tan \tfrac{1}{2}(A + B) \tan \tfrac{1}{2}(A - B).$$

12. Prove that

$$\frac{chd^2\left(\frac{\pi}{2} + \theta\right) - chd^2\left(\frac{\pi}{2} - \theta\right)}{chd\left(\frac{\pi}{2} + \theta\right) chd\left(\frac{\pi}{2} - \theta\right)} = 2 \tan\theta, \text{ and } \frac{\text{ver}(\pi + \theta)}{\cos(2\pi - \theta)} = \frac{\tan\theta}{\tan\frac{1}{2}\theta}.$$

13. Show that

$$2\sin\theta = \frac{\sin(-2\theta) - 2\cos(-\theta)\tan(-\theta)}{\text{cosec}(-\theta)\sin(-\theta)\text{vers}(-\theta)}.$$

14. Prove that

$$chd\, 72^\circ = 2\cos 54^\circ, \text{ and } \sec 72^\circ = \sec 60^\circ + \sec 36^\circ.$$

15. Prove that vers 45° is an arithmetic mean between sec 60° and sec 225°, and that cosec 150° is a geometric mean between cosec 105° and cosec 165°.

ON THE USE OF THE TRIGONOMETRICAL TABLES.

19. It will now be necessary to give a description of the tables of sines, tangents, etc., the method of constructing such being reserved for a subsequent part.

In the Table of *Natural Sines*, etc., the radius is unity, and therefore the sines and cosines of all angles are either unity or decimal fractions less than unity. The decimal point, however, is sometimes omitted. The same is the case with the *tangents* of angles less than 45°.

The Table of *Log. Sines, etc.*, is formed by taking the logarithms of the numbers in the corresponding Table of Natural Sines, etc.; and, to avoid *negative indices*, 10 is added to each, so that log sin A = log nat sin of A + 10, and so on.

(A.) *To find the sine, cosine, etc., of any angle less than 90° expressed by degrees and minutes.*

If the angle be less than 45°, find the degrees at the *top*, and the minutes on the left-hand side, of the page. In the same line with the minutes, and under the proper name at the *top* (sine, cosine, etc.), take out the number required. Thus the *natural* sine and tangent of $27^\circ\ 16'$ are ·4581325 and ·5154019 respectively. Also the *log* sine and tangent of the same are 9·6609911 and 9·7121461.

For angles between 44° and 90°, find the degrees at the *bottom*, and the minutes on the right hand side, of the page. In the same line with the minutes, and above the proper name at the *bottom*, take out the number. Thus the natural cosine of $56^\circ\ 7'$ is ·5575036.

(B.) *To find the sine, cosine, etc., of any angle less than 90°, expressed by degrees, minutes, and seconds.*

Find the sine, etc., of the next less and next greater angles in the table, and take the difference of these; then because the difference of any two trigonometrical functions of two angles (except in extreme

cases) is as the difference of the angles themselves when the difference of the angles is less than one minute, we have—

60″ : given number of seconds :: diff. thus found : correction for seconds.

It must be kept in mind that the sine, tangent, secant, when the angle is under 90°, *increase* as the angle increases; but the cosine, cotangent, and cosecant *decrease* as the angle increases. Hence the correction must be added in the case of the sine, tangent, and secant, but subtracted in the case of the cosine, cotangent, and cosecant.

1. Let it be required to find the sine and cosine of 59° 14′ 16″.

sin 59° 14′ = ·8592576*	cos 59° 14′ = ·5115431
sin 59° 15′ = ·8594064	cos 59° 15′ = ·5112931
diff. for 1′ = 1488	diff. for 1′ = 2500
60″ : 16″ :: 1488 : 396·8	60″ : 16″ :: 2500 : 666·6
sin 59° 14′ = ·8592576	cos 59° 14′ = ·5115431
pro. pts. for 16″ = 397	pro. pts. for 16″ = 667
sin 59° 14′ 16″ = ·8592973	cos 59° 14′ 16″ = ·5114764

The log sines, cosines, etc., of angles expressed by degrees, minutes, and seconds, are found in the same way. The last figure must be taken to the *nearest unit;* and hence, if the decimal fraction omitted be greater than ·5, we must add 1 to the last figure, as in the preceding examples.

(C.) *To find the angle corresponding to any given sine, cosine, etc., to the nearest second.*

If the function be a sine, tangent, or secant, find the *next less* to the given one; but if a cosine, cotangent, or cosecant, the *next greater*, and take out the corresponding number of degrees and minutes. Then having found the difference between the next less or next greater, as the case may be, and the given one, we have this proportion :—

tabular diff. : diff. found :: 60″ : seconds, to be added to the degrees and minutes already found.

This proportion will be obvious from what has been stated. If the "tabular diff." be not given in the tables, it will be found by taking the next less from the next greater in the tables.

2. Given sin A = ·5432107, to find A.

sin A = ·5432107
sin 32° 54′ (next less) = ·5431744
diff. = 363

2443 (tab. diff.) : 363 :: 60″ : 8″·91.
Hence, A = 32° 54′ 8″·91.

3. Given cos A = ·6780174, to find A.

cos A = ·6780174
cos 47° 18′ (next greater) = ·6781597
diff. = 1423

* Hutton's Tables are referred to, but other tables are used in a similar way.

$$2138 \text{ (tab. diff.)} : 1423 :: 60'' : 40'' \text{ nearly.}$$

$$\text{Whence, } A = 47^\circ\ 18'\ 40''.$$

The method for log sines, etc., is exactly similar.

Scholium.—For angles greater than 90°, and less than 180°, the functions of $\pi - A$ may be substituted for them, subject to the ***changes of sign*** as indicated in Art. 15.

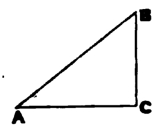
Thus $\sin A = \sin (180^\circ - A)$, $\cos A = -\cos (180^\circ - A)$, $\tan A = -\tan (180^\circ - A)$, etc.

Hence, $\sin 91^\circ\ 11'\ 12'' = \sin (180^\circ - 91^\circ\ 11'\ 12'') = \sin 88^\circ\ 48'\ 48''$, $\cos 91^\circ\ 11'\ 12'' = -\cos 88^\circ\ 48'\ 48''$, etc.

PROPERTIES OF PLANE TRIANGLES, WITH NUMERICAL EXAMPLES.*

The Right-angled Triangle.

20. Let A B C be a triangle, right-angled at C. Denote the angles by A, B, C, and the sides opposite to these, respectively, by a, b, c. Then by Art. 14,



$$\tan A = \cot B = \frac{a}{b}, \text{ similarly } \tan B = \cot A = \frac{b}{a} \ldots \text{(A)};$$

$$\text{also, } \sin A = \cos B = \frac{a}{c}, \sin B = \cos A = \frac{b}{c} \ldots\ldots\ldots\ldots \text{(B)};$$

$$\text{and, } \sec A = \operatorname{cosec} B = \frac{c}{b}, \sec B = \operatorname{cosec} A = \frac{c}{a} \ldots\ldots\ldots \text{(C)}.$$

These relations comprise every trigonometrical property of the right-angled triangle.

Numerical Examples.

1. In the triangle A B C, right-angled at C, there are given BC = 142, and the angle $A = 26^\circ\ 17'\ 19''$, to find the other parts.

The angle B being the complement of A, we have

$$B = 90^\circ - A = 63^\circ\ 42'\ 41'';$$

also, $$\frac{b}{a} = \tan B, \text{ or } b = a \tan B,$$

and $$c = a \operatorname{cosec} A,$$

by the preceding expressions (A) and (C).

Calculation of b and c, by Natural Tangents, etc.

(p. p. in the work means proportional parts.)

Nat tan $63^\circ\ 42' = 2 \cdot 0233462$	Nat cosec $26^\circ\ 17' = 2 \cdot 2583029$
p. p. for $41'' =$ 10131	p. p. for $19'' =$ 4209
$2 \cdot 0243593$	$2 \cdot 2578820$
This being multipled by 142, gives A C or $b = 287 \cdot 459$.	Multiply by 142, and we get A B or $c = 320 \cdot 6192$.

* A triangle consists of six parts, viz., three sides and three angles; and if three of these be given (one being a *side*), the triangle is completely defined.

2. Given $BA = 467{\cdot}817$, and $AC = 328{\cdot}914$, to find the other parts.

By (B) of the preceding formulæ,

$b = c \sin B$; this will find the angle B, as b and c are both given. Then $A = 90^\circ - B$, and $a = b \tan A$, or we may find a from the relation (Euc. i. 47), $a = \sqrt{(c^2 - b^2)}$.

*By Logarithms.**

To find the angle B, we have

$$\log \sin B = \log b - \log c + 10.$$

(Here 10 is added to the right-hand member of this equation, because in the *Tables of Log. Sines*, etc., 10 is added to each, *to avoid negative indices*, as explained in Art. 19.)

The work will hence stand thus:—

$$\begin{array}{rr|rr}
 & 10 & & \\
\log 328{\cdot}91 = & 2{\cdot}5170771 & \log.\ 467{\cdot}81 = & 2{\cdot}6700695 \\
\text{p. p. for } 4 = & 53 & \text{p. p. for } 7 = & 65 \\
\hline
 & 12{\cdot}5170824 & & 2{\cdot}6700760 \\
 & 2{\cdot}6700760 & & \\
\hline
\log \sin B = & 9{\cdot}8470064\dagger & & \\
\log \sin 44^\circ\ 40' = & 9{\cdot}8469436 & & \\
\hline
 & 628 & & \\
 & 60 & & \\
\hline
 & 1278)37680(29'' & &
\end{array}$$

Hence $B = 44^\circ\ 40'\ 29''$ and $A = 45^\circ\ 19'\ 31''$, the complement of B.

Again, $a = \sqrt{(c^2 - b^2)} = \sqrt{(c + b)\,(c - b)}$;

$$c = 467{\cdot}817$$
$$b = 328{\cdot}914$$

$$\log (c + b) = \log 796{\cdot}731 = 2{\cdot}9013117$$
$$\log (c - b) = \quad 138{\cdot}903 = 2{\cdot}1427116$$
$$2\,\underline{|\,5{\cdot}0440233}$$
$$\log a = 2{\cdot}5220116, \text{ or } a = 332{\cdot}668.$$

In similar cases the result may be verified by the solution of the equation $a = b \tan A$.

3. Given $c = 176{\cdot}21$ (C being again the right angle), and $B = 32^\circ\ 15'\ 26''$; to find A, b, and a.

The angle $A = 90^\circ - B = 57^\circ\ 44'\ 34''$, and $b = c \sin B$, or $\log b = \log c + \log \sin B - 10$. We take away 10 in this case, because the tables give $\log \sin B + 10$ for $\log \sin B$, as explained in the preceding solution.

$$\begin{array}{rr}
\text{Now, } \log\ 176{\cdot}21 = & 2{\cdot}2460306 \\
\log \sin 32^\circ\ 15' = & 9{\cdot}7272276 \\
\text{p. p. for } 26'' = & 868 \\
\hline
\text{(Taking 10 away)} & 1{\cdot}9733450
\end{array}$$

$$\log 94{\cdot}047 = 1{\cdot}9733449 \text{ or } b = 94{\cdot}047.$$

Again, $a = c \sin A$, which gives $a = 149{\cdot}014$.

---

* Logarithms are employed in trigonometry, as in other branches of mathematics, merely to facilitate the calculations when the numbers are large. The results are obtained in the preceding example without the use of logarithms.

† The same result would be obtained by *adding* the quantities if the *arithmetical complement* of the log of c were employed.

4. Given $AC = 56 \cdot 7816$, $B = 36° 7' 18''$, $C = 90°$, to find the other parts. *Ans.* $BC = 77 \cdot 8052$, $AB = 96 \cdot 3213$, $A = 53° 52' 42''$.

5. Given $AB = 12$, $BC = 15$, $B = 90°$, to find A C and the angles without logs.

Ans. $A = 51° 20' 24'' \cdot 68$, $C = 38° 39' 35'' \cdot 32$, $AC = 19 \cdot 2093727$.

6. Given $a = 101$, $b = 103$, $C = 90°$; to find c and the angles A and B. *Ans.* $c = 144 \cdot 257$, $A = 44° 26' 17'' \cdot 8$, $B = 45° 33' 42'' \cdot 2$.

7. Given $a = 379 \cdot 628$, $A = 39° 26' 15''$, $C = 90°$; to find the other parts. *Ans.* $B = 50° 33' 45''$, $b = 461 \cdot 5504$, $c = 597 \cdot 6171$.

8. In the triangle A B C right-angled at C, there are given $AB = 170 \cdot 235$, $A = 44° 1' 10''$; to find A C, C B, and the angle at B.

Ans. $AC = 122 \cdot 416$, $BC = 118 \cdot 297$, $B = 45° 58' 50''$.

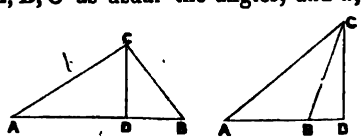
9. Given $BA = 402 \cdot 015$, $B = 56° 7' 18''$ and $C = 90°$ to find the other parts. *Ans.* $BC = 224 \cdot 0957$, $AC = 333 \cdot 7621$, $A = 33° 52' 42''$.

10. Given $b = 31 \cdot 76$, $A = 17° 12' 51''$ and $C = 90°$, to find a, c, and B. *Ans.* $a = 9 \cdot 8399$, $c = 33 \cdot 249$, $B = 72° 47' 9''$.

THE OBLIQUE-ANGLED TRIANGLE.

21. *To investigate a relation between the sides of a plane triangle and the angles opposite to them.*

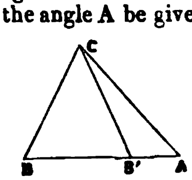
Let A B C be a plane triangle; A, B, C as usual the angles, and a, b, c, the sides respectively opposite to these. Draw C D perpendicular to A B or A B produced; then by (B) of Art. 20, and (13) of Art. 15,



$$CD = b \sin A = a \sin B, \text{ or } \frac{b}{a} = \frac{\sin B}{\sin A}.$$

$$\text{Similarly, } \frac{b}{c} = \frac{\sin B}{\sin C}, \text{ and } \frac{a}{c} = \frac{\sin A}{\sin C}.$$

These properties are employed in the solution of a plane triangle, when two angles and a side, or when two sides and an angle opposite to one of them, form the data of the problem. If in the latter case *the given angle be opposite to the less side*, there are two triangles which fulfil the conditions of the problem. Thus if C B, C A, and the angle A be given, then it will be obvious that when C B is less than C A, C B will meet A B in two points B and B′, *to the left of A*, so that the angle A will be common to the two triangles A B C, A B′ C. But if C B be greater than C A, B′ will be to the right of A, and hence the angle A will not be common to both triangles.



This double solution is also indicated by the formula for the particular case. For since two sides and an angle opposite to the less are given, we find the *sine* of the angle opposite to the greater side, and as the sine of an angle is equal to the sine of its supplement, there is no reason, without other considerations, to prefer the acute angle found by the tables to the supplement of this angle.

This double solution constitutes what is called the *ambiguous case* in the solution of a plane triangle.

22. *To find a relation between two sides of a plane triangle and their included angle.*

By the preceding Art., $a : b :: \sin A : \sin B$.
Hence by Euc. v. 17, 18, and (20) of Art. 16,

$$a + b : a - b :: \sin A + \sin B : \sin A - \sin B ;$$

or $$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

But $\frac{1}{2}(A+B+C) = 90^\circ$ or $\frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C$; hence $\tan \frac{1}{2}(A+B) = \cot \frac{1}{2} C$.

Whence $$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A-B)};$$

and similarly for any two sides and their included angle.

This is the formula of solution when two sides and the included angle of a plane triangle are given. It will be obvious that the opposite angles are first found, and then the third side by the preceding Article. Another method will be investigated presently, by which the third side can be found without the opposite angles.

23. *To find a relation between any angle of a plane triangle and its three sides.*

By Euc. ii. 12, 13, (see fig. of Art. 21),

$$b^2 = c^2 + a^2 \mp 2c . BD ;$$

and by (14) of Art. 15, and (B) of Art. 20,

$$BD = \pm\, a \cos B :$$

the upper or lower sign to be used according as the angle B is acute or obtuse.

Hence, Similarly,

$$\left.\begin{aligned} b^2 &= c^2 + a^2 - 2ac \cos B. \\ a^2 &= b^2 + c^2 - 2bc \cos A, \\ c^2 &= a^2 + b^2 - 2ab \cos C, \end{aligned}\right\} \ldots . (1).$$

These important formulæ may also be put in the form,

$$\left.\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{c^2 + a^2 - b^2}{2ac} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\} \ldots\ldots (1').$$

The expressions (1′), though not in a form for logarithms, are convenient for finding the angles of a triangle when the sides are given in *small numbers*. By (1) also we can find the third side of a triangle when two sides and the included angle are given. Other forms adapted to logarithmic use are deduced in the following manner, the *notation* being the same as in the *Application, Problem VII.*

By (7) of Art. 16, and the preceding expressions (1′),

$$2\sin^2 \tfrac{1}{2} A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a+c-b)}{2bc} = \frac{2(s-c)(s-b)}{bc};$$

or, $$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}} \ldots\ldots (2).$$

Also, $$2\cos^2 \tfrac{1}{2} A = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc} = \frac{2s(s-a)}{bc};$$

or, $$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}} \ldots\ldots\ldots (3).$$

Hence, $$\tan \tfrac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \ldots (4).$$

And so for any other angle.

Cor. By the preceding expressions for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, and the formula $\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$, we get

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly, $$\sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)},$$

$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}.$$

Scholium.—Each of the four expressions given in this Article for the determination of the angles of a triangle is convenient for the application of logarithms. In some cases, however, the value of an angle found from its log sine or log cosine cannot be trusted to seconds. This arises from the variation of the log sine or log cosine corresponding to a given variation of the angle being *very small*, and hence a large variation of the angle is attended by a small variation of the log sine or log cosine. The Astronomer Royal (Mr. Airy) has established the following rule in such cases:—

An angle cannot be determined accurately from its sine or cosecant when it is near 90°, from its cosine or secant when very small, or from its versed sine when near 180°; but from its tangent it can always be found with great accuracy.—*Trigonometry*, page 696.

Again, by Euc. i., 47 (see the figure of Art. 21),

$$CD^2 = AC^2 - AD^2 = BC^2 - BD^2;$$

hence $AC^2 - BC^2 = AD^2 - BD^2$, or $(AC + BC)\,AC - BC)$

$$= AB\,(AD - BD).$$

Whence denoting the difference of the segments AD and BD by d, and the sides of the triangle as usual, we have

$$d = \frac{(a+b)\,(b-a)}{c} \ldots\ldots (5).$$

By this formula, when the sides of a plane triangle are given, we can find the segments of the base made by a perpendicular from the opposite angle, and thence by the properties of the right-angled triangle, the angles of the triangle.

This method is convenient in some cases.

24. *To investigate a formula adapted to logarithms, by which the third side of a triangle (when two sides and the included angle are given) can be found without finding the angles which are opposite to the given sides.*

By (1) Art. 23,

$c^2 = a^2 + b^2 - 2ab\cos C = (a-b)^2 + 2ab(1 - \cos C);$

hence, since by (7) of Art. 16, $1 - \cos C = 2\sin^2\frac{1}{2}C$, we have

$$c^2 = (a-b)^2\left\{1 + \frac{4ab\sin^2\frac{1}{2}C}{(a-b)^2}\right\}:$$

Now put $$\frac{4ab\sin^2\frac{1}{2}C}{(a-b)^2} = \tan^2\theta \ldots\ldots (1);$$

then remembering that $\sec^2\theta = 1 + \tan^2\theta$, we get

$$c = (a-b)\sec\theta \ldots\ldots\ldots (2).$$

The *subsidiary** angle θ is found by (1).

Hence, if the third side only be required, it can be found with more facility by these formulæ than by that of Art. 22.

If the subsidiary angle (θ) be very large or very small, the following formulæ for determining the third side are to be preferred:—

$$c^2 = a^2 + b^2 - 2ab\cos C = (a+b)^2 - 2ab(1 + \cos C);$$

hence, since $1 + \cos C = 2\cos^2\frac{1}{2}C$ (Art. 18), we get

$$c^2 = (a+b)^2\left\{1 - \frac{4ab}{(a+b)^2}\cos^2\tfrac{1}{2}C\right\}.$$

Assume $\frac{4ab}{(a+b)^2}\cos^2\frac{1}{2}C = \sin^2\theta$; then because $1 - \sin^2\theta = \cos^2\theta$,

$$c = (a+b)\cos\theta.$$

Hence, to find θ and c we have the equations

$$2\log\sin\theta = \log 4 + \log a + \log b + 2\log\cos\tfrac{1}{2}C - 2\log(a+b)\ldots(3),$$

$$\log c = \log(a+b) + \log\cos\theta - 10\ldots(4).$$

NUMERICAL SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

Case I.

25. *When two angles and a side, or when two sides and an angle opposite to one of them, are given.*

The principles employed in the solution of the three following examples are developed in Art. 21:—

1. Given the angle B = 50° 17′ 26″, C = 38° 20′ 32″, and the side $a = 318\cdot541$ of a plane triangle A B C, to find the other parts.

* The term "*subsidiary*" is applied to angles or arcs that are employed either to facilitate the calculation of some quantity, or to modify the form of an expression so as to adapt it to logarithmic computation.

The angle $A = 180° - (B + C) = 91° 22' 2''$.

Also, $a : c :: \sin A : \sin C$, or $c = \dfrac{a \sin C}{\sin A}$.

Hence, $\log c = \log a + \log \sin C - \log \sin A$.

Similarly, $\log b = \log a + \log \sin B - \log \sin A$.

The work will hence stand thus, remembering that $\sin A = \sin (180° - A)$:—

log 318·54 =	2·5031640
p. p. 1 =	14
log sin 38° 20′ =	9·7925566
p. p. 32″ =	851
	12·2958071
log sin 88° 37′ =	9·9998734
p. p. 56″ =	29
	9·9998763
log c =	2·2959308
or c =	197·6654

log 318·541 =	2·5031654
log sin 50° 17′ =	9·8860470
p. p. 26″ =	454
	12·3892578
log sin 88° 37′ 56″ =	9·9998763
log b =	2·3893815
or b =	245·1216

2. Given $b = 152·67$, $c = 163·18$, and $C = 50° 18' 32''$, to find the other parts.

As the given angle is opposite to the greater side, the problem is not ambiguous.

Now $b : c :: \sin B : \sin C$, or $\sin B = \dfrac{b \sin C}{c}$;

and $b : a :: \sin B : \sin A$, or $a = \dfrac{b \sin A}{\sin B}$.

Hence, $\log \sin B = \log b + \log \sin C - \log c$; $\log a = \log b + \log \sin A - \log \sin B$:

log 152·67 =	2·1837537
log sin 50° 18′ =	9·8861519
p. p. 32″ =	559
	12·0699615
log 163·18 =	2·2126669
log sin B =	9·8572946
or B =	46° 2′ 57″;

log 152·67 =	2·1837537
log sin 83° 38′ =	9·9973132
pp. 31″ =	73
	12·1810742
log sin B =	9·8572946
log a =	2·3237796
or a =	210·756

Hence $A = 180° - (B + C) = 83° 38' 31''$.

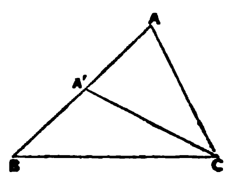
3. Given $a = 272·13$, $b = 252·69$, and $B = 64° 18' 20''$, to find A, C, and c.

Since the given angle is opposite to the less side, the solution is ambiguous. Now,

$a : b :: \sin A : \sin B$, or $\sin A = \dfrac{a \sin B}{b}$,

$a : c :: \sin A : \sin C$, or $c = \dfrac{a \sin C}{\sin A}$.

Hence, $\log \sin A = \log a + \log \sin B - \log b$,
$\log c = \log a + \log \sin C - \log \sin A$:



$$\begin{array}{rl}
\log 272\cdot13 = & 2\cdot4347764 \\
\log\sin 64^\circ\ 18'\ 20'' = & 9\cdot9547821 \\ \hline
 & 12\cdot3895585 \\
\log 252\cdot69 = & 2\cdot4025881 \\ \hline
\log\sin A = & 9\cdot9869704
\end{array}$$

Hence

$$\begin{array}{rrrr}
 & {}^\circ & ' & '' \\
A = & 76 & 2 & 6 \\
B = & 64 & 18 & 20 \\ \hline
A + B = & 140 & 20 & 26 \\
ACB = & 39 & 39 & 34
\end{array}$$

$$\begin{array}{rl}
\log 272\cdot13 = & 2\cdot4347764 \\
\log\sin 39^\circ\ 39'\ 34'' = & 9\cdot8049725 \\ \hline
 & 12\cdot2397489 \\
\log\sin 76^\circ\ 2'\ 6'' = & 9\cdot9869701 \\ \hline
\log c = & 2\cdot2527788
\end{array}$$

Hence A B = 178·9694

And

$$\begin{array}{rrrr}
 & {}^\circ & ' & '' \\
A' = & 103 & 57 & 54 \\
B = & 64 & 18 & 20 \\ \hline
A' + B = & 168 & 16 & 14 \\
A'CB = & 11 & 43 & 46
\end{array}$$

$$\begin{array}{rl}
\log 272\cdot13 = & 2\cdot4347764 \\
\log\sin 11^\circ 43' 46'' = & 9\cdot3081170 \\ \hline
 & 11\cdot7428934 \\
\log\sin 103^\circ 57' 54'' & \\
= \log\sin 76^\circ\ 2'\ 6'' = & 9\cdot9869701 \\ \hline
\log A'B = & 1\cdot7559233
\end{array}$$

Hence A′ B = 57·00635

4. Given $a = 305\ 296$, $B = 51^\circ\ 15'\ 35''$, and $C = 37^\circ 21'\ 25''$, to find b, c, and A. *Ans.* $b = 238\cdot1974$, $c = 185\cdot3011$, $A = 91^\circ\ 23'$.

5. Given $c = 195\cdot265$, $b = 203\cdot162$, and $B = 45^\circ\ 0'\ 55''$, to find the other parts. *Ans.* $A = 92^\circ\ 9'\ 23''$, $C = 42^\circ\ 49'\ 42''$, $a = 287\cdot035$.

6. Given $a = 350\cdot169$, $b = 236\cdot291$, and $B = 38^\circ\ 39'\ 15''$, to find A, C, and c. *Ans.* $A = 67^\circ\ 45'\ 58''$ or $112^\circ\ 14'\ 2''$, $C = 73^\circ\ 34'\ 47''$ or $29^\circ\ 6'\ 43''$, $c = 362\cdot8674$ or $184\cdot048$.

7. Given $BC = 145\cdot3$, $AC = 178\cdot3$, and $A = 41^\circ\ 10'$ to find the other parts. *Ans.* $B = 53^\circ\ 52'\ 36''$ or $126^\circ\ 7'\ 24''$, $C = 84^\circ\ 57'\ 24''$ or $12^\circ\ 42'\ 36''$, $c = 219\cdot882$ or $48\cdot5656$.

Case II.

26. *When two sides and the included angle are given.*

1. The sides B C, C A, of a plane triangle A B C, are 17·802 and 21·704, and the included angle C is $26^\circ\ 12'\ 16''$; what are the values of A B, A and B?

By Art. 22,

$$\frac{b+a}{b-a} = \frac{\tan\frac{1}{2}(B+A)}{\tan\frac{1}{2}(B-A)}, \text{ or } \tan\tfrac{1}{2}(B-A) = \frac{b-a}{b+a}\tan\tfrac{1}{2}(B+A);$$

hence $\log\tan\frac{1}{2}(B-A) = \log(b-a) + \log\tan\frac{1}{2}(B+A) - \log(b+a)$.

Now

$$\begin{array}{rl}
b = & 21\cdot704 \\
a = & 17\cdot802 \\ \hline
b + a = & 39\cdot506 \\
b - a = & 3\cdot902
\end{array}
\qquad
\begin{array}{rl}
B + A = 180^\circ - C & = 153^\circ\ 47'\ 44'' \\
\frac{1}{2}(B + A) & = 76^\circ\ 53'\ 52'';
\end{array}$$

$$\begin{array}{rl}
\log 3\cdot902 = & 0\cdot5912873 \\
\log\tan 76^\circ\ 53'\ 52'' = & 10\cdot6331137 \\ \hline
 & 11\cdot2244010 \\
\log 39\cdot506 = & 1\cdot5966631 \\ \hline
\log\tan\frac{1}{2}(B - A) = & 9\cdot6277379 \\
\text{or } \frac{1}{2}(B - A) = & 22^\circ\ 59'\ 41''.
\end{array}$$

Hence $B = \frac{1}{2}(B + A) + \frac{1}{2}(B - A) = 99^\circ\ 53'\ 33''$, $A = \frac{1}{2}(B + A) - \frac{1}{2}(B - A) = 53^\circ\ 54'\ 11''$.

To find AB or c we have, by Art. 21,

$$a : c :: \sin A : \sin C, \text{ or } c = \frac{a \sin C}{\sin A};$$

whence $\log c = \log a + \log \sin C - \log \sin A$:

$$\begin{aligned} \log 17\cdot802 &= 1\cdot2504688 \\ \log \sin 26^\circ\ 12'\ 16'' &= 9\cdot6450049 \\ \hline &\ 10\cdot8954737 \\ \log \sin 53^\circ\ 54'\ 11'' &= 9\cdot9074228 \\ \hline \log c &= 0\cdot9880509 \\ \text{or } c &= 9\cdot7286. \end{aligned}$$

As a verification of this result, and an illustration of (1) and (2), Art. 24, the value of c will now be found without finding the angles A and B.

By the properties just referred to,

$$2 \log \tan \theta = \log 4 + \log a + \log b + 2 \log \sin \tfrac{1}{2} C - 2 \log (b - a),$$
$$\log c = \log (b - a) + \log \sec \theta - 10.$$

Now,

$$\begin{aligned} \log 4 &= 0\cdot6020600 \\ \log 17\cdot802 &= 1\cdot2504688 \\ \log 21\cdot704 &= 1\cdot3365398 \\ 2 \log \sin 13^\circ\ 6'\ 8'' &= 18\cdot7108610 \\ \hline &\ 21\cdot8999296 \\ 2 \log 3\cdot902 &= 1\cdot1825746 \\ \hline &\ 20\cdot7173550 \\ \log \tan \theta &= 10\cdot3586775 \\ \text{or } \theta &= 66^\circ\ 21'\ 14''\cdot16. \end{aligned}$$

$$\begin{aligned} \log 3\cdot902 &= 0\cdot5912873 \\ \log \sec 66^\circ\ 21'\ 14''\cdot16 &= 10\cdot3967629 \\ \hline \log c &= 0\cdot9880502 \end{aligned}$$

Hence $c = 9\cdot7286$, the same as before nearly.

2. Given $a = 16\cdot9584$, $b = 11\cdot9613$, and $C = 60^\circ\ 43'\ 36''$, to find the other parts.

Ans. $A = 76^\circ\ 4'\ 12''\cdot23$, $B = 43^\circ\ 12'\ 11''\cdot77$, $c = 15\cdot24098$.

3. Given $a = 874\cdot56$, $b = 859\cdot56$, $C = 91^\circ\ 58'\ 10''$, to find A, B, and c. *Ans.* $A = 44^\circ\ 29'\ 39''$, $B = 43^\circ\ 32'\ 11''$, $c = 1247\cdot14$.

4. Given $a = 3754$, $b = 3277\cdot628$, and the included angle $C = 57^\circ\ 53'\ 16''\cdot8$, to find A, B, and c.

Ans. $A = 68^\circ\ 2'\ 24''$, $B = 54^\circ\ 4'\ 19\cdot2''$, $c = 3428\cdot426$.

5. Given $BA = 1786\cdot7$, $AC = 1921\cdot8$, and $A = 30^\circ\ 26'\ 20''$, to find BC without the angles B and C. *Ans.* $BC = 982\cdot24$.

Case III.

27. *When all the sides of a triangle are given.*

1. Given $a = 2$, $b = 4$, $c = 5$, to find the angles.

By (1′) of Art. 23,

$$\cos A = \frac{4^2 + 5^2 - 2^2}{2.4.5} = \frac{37}{40} = \cdot925 = \cos 22^\circ\ 19'\ 53''\cdot91,$$

$$\cos B = \frac{2^2 + 5^2 - 4^2}{2.5.2} = \frac{13}{20} = \cdot 65 = \cos 49^\circ 27' 30''\cdot 23,$$

$$\cos C = \frac{2^2 + 4^2 - 5^2}{2.2.4} = -\frac{5}{16} = -\cdot 3125 = \cos 108^\circ 12' 35''\cdot 86.$$

Hence A = $22^\circ 19' 53''\cdot 91$, B = $49^\circ 27' 30''\cdot 23$, C = $108^\circ 12' 35''\cdot 86$.

Since cos C is *negative*, $\cdot 3185 = \cos(\pi - C)$, by (14) of Art. 15; hence having found the angle whose cosine is $\cdot 3125$, it is taken from 180° for C.

As the angles A, B, C, together, make 180°, it may be inferred that the work is correct.

2. Given A B = 17862, B C = 19876, A C = 18304, to find the angles.

By (4) of Art. 23 (the symbol A.C is put for *arithmetical complement*), we have

$2 \log \tan \frac{1}{2} A = \log(s - b) + \log(s - c) + \text{A.C} \log s + \text{A.C} \log(s - a)$;

$2 \log \tan \frac{1}{2} B = \log(s - a) + \log(s - c) + \text{A.C} \log s + \text{A.C} \log(s - b)$;

$2 \log \tan \frac{1}{2} C = \log(s - a) + \log(s - b) + \text{A.C} \log s + \text{A.C} \log(s - c)$.

Now

$$\begin{array}{rl} a = & 19876 \\ b = & 18304 \\ c = & \underline{17862} \\ 2\,) & \overline{56042} \\ s = & 28021 \end{array} \qquad \begin{array}{rl} s - a = & 8145 \\ s - b = & 9717 \\ s - c = & 10159 \end{array}$$

$$\begin{array}{rl} \log(s - b) & = 3\cdot 9875322 \\ \log(s - c) & = 4\cdot 0068510 \\ \text{A.C} \log s & = 5\cdot 5525164 \\ \text{A.C} \log(s - a) & = \underline{6\cdot 0891089} \\ & 2\,)\,19\cdot 6360085 \\ \log.\tan \frac{1}{2} A & = 9\cdot 8180042 \\ \text{or } \frac{1}{2} A & = 33^\circ 19' 53''\cdot 36 \\ \text{or } A & = 66^\circ 39' 46''\cdot 72 \end{array} \qquad \begin{array}{rl} \log(s - a) & = 3\cdot 9108911 \\ \log(s - c) & = 4\cdot 0068510 \\ \text{A.C} \log s & = 5\cdot 5525164 \\ \text{A.C} \log(s - b) & = \underline{6\cdot 0124678} \\ & 2\,)\,19\cdot 4827263 \\ \log \tan \frac{1}{2} B & = 9\cdot 7413631 \\ \text{or } \frac{1}{2} B & = 28^\circ 51' 59''\cdot 62 \\ \text{or } B & = 57^\circ 43' 59''\cdot 24 \end{array}$$

$$\begin{array}{rl} \log(s - a) & = 3\cdot 9108911 \\ \log(s - b) & = 3\cdot 9875322 \\ \text{A C} \log s & = 5\cdot 5525164 \\ \text{A C} \log(s - c) & = \underline{5\cdot 9931490} \\ & 2\,)\,19\cdot 4440887 \\ \tan \frac{1}{2} C & = 9\cdot 7220443 \\ \text{or } \frac{1}{2} C & = 27^\circ 48'\ 7\cdot 02'' \\ \text{or } C & = 55^\circ 36' 14''\cdot 04 \end{array}$$

Since A + B + C = 180°, the work is consequently proved.

3. Given $a = 3$, $b = 5$, $c = 7$; to find A, B, and C.

Ans. A = $21^\circ 47' 12''\frac{1}{2}$, B = $38^\circ 12' 47''\frac{1}{2}$, C = 120°.

4. Given A B = 7, B C = 6, A C = 5, to find the angles.
Ans. A = 57° 7′ 17″·95, B = 44° 24′ 55″·12, C = 78° 27′ 46″·93.

5. Given a = 32986, b = 43628, c = 62984, to find A, B, and C.
Ans. A = 29° 31′ 11″, B = 40° 40′ 8″, C = 109° 48′ 41″.

6. Given A B = 567·18, B C = 610·91, C A = 714·23, to find the angles.
Ans. A = 55° 31′ 30″·28, B = 74° 32′ 7″·96, C = 49° 56′ 21″·76.

7. Given a = 24804, b = 57876, c = 74412, to find A, B, and C.
Ans. A = 16° 11′ 42″·44, B = 40° 36′ 3″·84, C = 123° 12′ 13″·72.

8. Given the sides A B = 13·219, B C = 11·76, A C = 14·507, of a triangle A B C, to find the angles.
Ans. A = 49° 55′ 43″·5, B = 70° 44′ 2″·32, C = 59° 20′ 14″·18.

9. The three sides of a triangle are 352·96, 628·54, and 569·16; find the angles.
Ans. 33° 49′ 6″·12, 82° 21′ 12″·60, and 63° 49′ 41″·28.

Miscellaneous Exercises on Triangles.

1. Express each side of a plane triangle in terms of the remaining sides and their opposite angles.
Ans. $a = b\cos C + c\cos B$, $b = a\cos C + c\cos A$, $c = b\cos A + a\cos B$.

2. Prove the following properties of a right-angled triangle, C being the right angle:—

$$\sin^2\tfrac{1}{2}A = \frac{c-b}{2c} \qquad \sin 2A = \frac{2ab}{b^2+a^2} \qquad \sin(A-B) = \frac{a^2-b^2}{c^2}$$

$$\cos^2\tfrac{1}{2}A = \frac{c+b}{2c} \qquad \cos 2A = \frac{b^2-a^2}{b^2+a^2} \qquad \cos(A-B) = \frac{2ab}{c^2}$$

$$\tan^2\tfrac{1}{2}A = \frac{c-b}{c+b} \qquad \tan 2A = \frac{2ab}{b^2-a^2} \qquad \tan(A-B) = \frac{a^2-b^2}{2ab}$$

3. In an isosceles triangle in which $a = b$, prove that

$$\cos A = \frac{c}{2a}, \text{ and vers } C = \frac{c^2}{2a^2}.$$

4. Prove that if $2\cos B \sin C = \sin A$, the triangle is isosceles; and if $\tan A \sin^2 B = \tan B \sin^2 A$, it is either isosceles or right-angled at C.

5. Prove that in any plane triangle A B C,

$$\sin^2\tfrac{1}{2}A : \sin^2\tfrac{1}{2}B :: a(s-b) : b(s-a).$$

6. Show that in any plane triangle,

$$\sin\tfrac{1}{2}(A-B) = \frac{a-b}{c}\cos\tfrac{1}{2}C, \text{ and } \cos\tfrac{1}{2}(A-B) = \frac{a+b}{c}\sin\tfrac{1}{2}C.$$

7. If A, B, C, be the angles of a plane triangle, then

$$\cot A \cot B + \cot B \cot C + \cot C \cot A = 1,$$
$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$$

8. Prove that the lengths of the three straight lines drawn from the angles A, B, C, of a triangle A B C, to the points of bisection of the opposite sides, are

$$(bc\cos A + \tfrac{1}{4}a^2)^{\frac{1}{2}}, \ (ac\cos B + \tfrac{1}{4}b^2)^{\frac{1}{2}}, \text{ and } (ab\cos C + \tfrac{1}{4}c^2)^{\frac{1}{2}}.$$

9. Prove that the lengths of the three perpendiculars on the opposite sides a, b, c, of a triangle, are

$$\frac{b^2 \sin C + c^2 \sin B}{b+c}, \quad \frac{a^2 \sin C + c^2 \sin A}{a+c}, \text{ and } \frac{a^2 \sin B + b^2 \sin A}{a+b}.$$

10. Prove that the lengths of the lines which bisect the angles A, B, C, of a triangle (being limited by their intersection with the opposite sides) are

$$\frac{2\,b\,c \cos \frac{1}{2} A}{b+c}, \quad \frac{2\,a\,c \cos \frac{1}{2} B}{a+c}, \text{ and } \frac{2\,a\,b \cos \frac{1}{2} C}{a+b}.$$

11. Two sides of a plane triangle and their included angle are, respectively, $\frac{1}{2}(\sqrt{3}-1)\sqrt{6}$, $\sqrt{3}-1$, and 75°; find the third side without the aid of tables. *Ans.* Third side $= 1$.

12. In any plane triangle it is required to prove that

$$c^2 \sin(A - B) = (a^2 - b^2) \sin(A + B),$$
$$c^2 \cos(A - B) = 2\,a\,b + (a^2 + b^2) \cos(A + B).$$

13. The cotangents of the semi-angles of a plane triangle are consecutive natural numbers; required the nature of the triangle and the proportion of its sides.

Ans. The triangle is right-angled, and its sides are as 3, 4, 5.

14. In every right-angled triangle (C being the right angle),

$$2 \operatorname{cosec} 2\,A \cot B = \frac{c^2}{b^2}, \text{ and } \tan 2\,A - \sec 2\,B = \frac{b+a}{b-a}.$$

15. Prove that if A, B, C, be the angles of a plane triangle,

$$\frac{(\cot A + \cot C)(\cot B + \cot C)}{\operatorname{cosec}^2 C} = 1.$$

16. Prove that if $\sec 2\,B + \tan 2\,A = \tan(\frac{1}{4}\pi + A)$, and $1 - \cos 2\,C$ $= \dfrac{\cos 2\,C - \cos 2\,A}{(2 \cos C + 1)(2 \cos C - 1)}$, the triangle is an isosceles one, having each of the angles at the base double of that at the vertex.

17. If in a plane triangle $\tan A \tan C = 3$, then $\cot A$, $\cot B$, and $\cot C$ are in harmonic progression.

18. The angles A, B, C of a plane triangle are in arithmetic progression, and

$$\frac{\sin A}{\sin C} = \frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}.$$

Show that x is the common difference of the angles.

19. The angles A, B, C, of a triangle are in arithmetic progression, and A D, which is perpendicular to B C, divides it in D, so that $BD : DC :: 1 : 3$. Find the angles. *Ans.* $A = 90^\circ$, $B = 60^\circ$, $C = 30^\circ$.

20. Prove that if A, B, C be the angles of a plane triangle,

$$(\sin A - \sin B)^2 + (\cos A + \cos B)^2 = 4 \sin^2 \tfrac{1}{2} C.$$

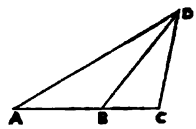
21. In every scalene triangle

$$\sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C < \tfrac{1}{8}, \text{ and } \cos A + \cos B + \cos C < \tfrac{3}{2}.$$

THE APPLICATION OF TRIGONOMETRY TO THE DETERMINATION OF THE HEIGHTS AND DISTANCES OF OBJECTS.

28. It is not intended to give in this place any explanations as to the *measuring* of lines or the *observing* of angles; these will be given in another part of the Course. The solutions of some examples are given to illustrate this mode of calculation, and other examples are left for solution.

PROBLEM I.—*To find the height and distance of an inaccessible object on a horizontal plane.*



Measure a straight line A B on the horizontal plane in the direction of the object C D, and let the angles of elevation C A D, C B D, at the points A and B, be observed.

Put $AB = a$, $< CAD = \alpha$, $CBD = \beta$; then $ADB = \beta - \alpha$. Now by Arts. 20, 21,

$$BD = a\frac{\sin\alpha}{\sin(\beta-\alpha)}, \text{ and } CD = BD \,.\, \sin\beta\text{; hence}$$

$$CD = a\frac{\sin\alpha \,.\, \sin\beta}{\sin(\beta-\alpha)} \ldots. (1),$$

the height of the object.

Also, $$BC = DB\cos\beta = a\,.\frac{\sin\alpha\,.\cos\beta}{\sin(\beta-\alpha)} \ldots. (2),$$

its distance from B.

As an example, let $\alpha = 19^\circ 17' 20''$, $\beta = 47^\circ 11' 12''$ and $a = 412\cdot78$. By (1) and (2),

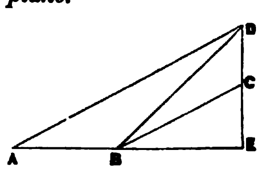
$$\log CD = \log a + \log\sin\alpha + \log\sin\beta - \log\sin(\beta-\alpha) - 10,$$
$$\log BC = \log a + \log\sin\alpha + \log\cos\beta - \log\sin(\beta-\alpha) - 10;$$

$\log 412\cdot78 = 2\cdot6157186$	$\log 412\cdot78 = 2\cdot6157186$
$\log\sin 19^\circ 17' 20'' = 9\cdot5189498$	$\log\sin 19^\circ 17' 20'' = 9\cdot5189498$
$\log\sin 47^\circ 11' 12'' = 9\cdot8654426$	$\log\cos 47^\circ 11' 12'' = 9\cdot8322610$
$22\cdot0001110$	$21\cdot9669294$
$\log\sin 27^\circ 53' 52'' = 9\cdot6701489$	$\log\sin 27^\circ 53' 52'' = 9\cdot6701489$
$\log CD = 2\cdot3299621$	$\log BC = 2\cdot2967805$
Hence $CD = 213\cdot7775.$	Hence $BC = 198\cdot0526.$

Scholium.—If the object C D be accessible, measure from C a distance $BC = b$ on the horizontal plane, and observe the angle of elevation $CBD = \beta$; then to determine the height of the object, we have

$$CD = b\tan\beta.$$

II. *To determine the height of an object situated on an eminence, and seen from a horizontal plane.*



Let C D be the object, and A B a given line in the horizontal plane, measured in a vertical plane passing through the object. At A take the angle of elevation of D, and at B the angles of elevation of D and C.

Put A B = a, $<$ D A E = α, D B E = β, C B E = γ; then by the last problem,

$$E D = a \frac{\sin \alpha \sin \beta}{\sin (\beta - \alpha)} \dots (1), \quad E B = a \frac{\sin \alpha \cos \beta}{\sin (\beta - \alpha)} \dots (2).$$

Hence $$C E = B E \tan \gamma = a \frac{\sin \alpha \cos \beta \tan \gamma}{\sin (\beta - \alpha)} \dots (3).$$

Whence by (1) and (3), we can find D C.

Ex. Let $\alpha = 33^\circ 45' 6''$, $\beta = 51^\circ 5' 12''$, $\gamma = 40^\circ 10' 20''$, $a = 200$.

To find E D and E C by (1) and (3):

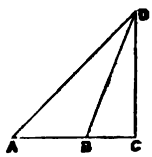
$\log E D = \log a + \log \sin \alpha + \log \sin \beta + \log \text{cosec} (\beta - \alpha) - 30$,

$\log E C = \log a + \log \sin \alpha + \log \cos \beta + \log \tan \gamma + \log \text{cosec} (\beta - \alpha) - 40$:

log 200 = 2·3010300	log 200 = 2·3010300
log sin 33° 45′ 6″ = 9·7447579	log sin 33° 45′ 6″ = 9·7447579
log sin 51° 5′ 12″ = 9·8910337	log cos 51° 5′ 12″ = 9·7980593
log cosec 17° 20′ 6″ = 10·5258449	log tan 40° 10′ 20″ = 9·9264632
	log cosec 17° 20′ 6″ = 10·5258449
log E D = 2·4626665	log E C = 2·2961553
or, E D = 290·1793	or, E C = 197·7677.

Hence, D C = E D − E C = 92·4116, the height of the object.

III. *To determine the height of an object as in the last problem, supposing the plane on which the observer is stationed is not horizontal.*



Let C D be the object, and A B a given part of the line in which a vertical plane passing through C D meets the plane of the observer. At A and B take the angles of elevation of D; let these be α, β. Also put A B = a and the angle of inclination of A B to the horizon = γ. Then it will be obvious that

$$< C B D = \beta - \gamma, \; C A D = \alpha - \gamma, \; A D B = C B D - C A D$$
$$= \beta - \alpha, \; D C B = 90^\circ + \gamma \text{ and } B D C = 90^\circ - \beta.$$

Now $$D C = D B \frac{\sin (\beta - \gamma)}{\cos \gamma}, \text{ and } D B = a \frac{\sin (\alpha - \gamma)}{\sin (\beta - \alpha)};$$

hence, $$\left.\begin{aligned} D C &= a \frac{\sin (\alpha - \gamma) \sin (\beta - \gamma)}{\cos \gamma \sin (\beta - \alpha)}; \\ \text{or, } \log D C &= \log a + \log \sin (\alpha - \gamma) + \log \sin (\beta - \gamma) \\ &+ \log \sec \gamma + \log \text{cosec} (\beta - \alpha) - 40. \end{aligned}\right\} \dots (1).$$

Ex. $a = 18·04$ feet, $\beta = 37^\circ 30' 15''$, $\alpha = 30^\circ 12' 12''$, $\gamma = 15^\circ 6' 9''$; then $\alpha - \gamma = 15^\circ 6' 3''$, $\beta - \gamma = 22^\circ 24' 6''$, $\beta - \alpha = 7^\circ 18' 3''$.

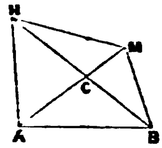
Hence,

log 18·04 =	1·2562365
log sin 15° 6′ 3″ =	9·4158386
log sin 22° 24′ 6″ =	9·5810358
log sec 15° 6′ 9″ =	10·0152651
log cosec 7° 18′ 3″ =	10·8959261
log D C =	1·1643021

or, D C = 14·5983 feet, the height of the object.

IV. *To find the distance between two inaccessible objects situated in the same plane with the observer.*

Let A B be a given line in the same plane with the objects H and M, and at A let the angles M A B, H A B, and at B the angles H B A, M B A, be observed.



Put angle $HAB = \alpha$, $MAB = \beta$, | $MBA = \gamma$, $HBA = \delta$, | $BHM + AMH = \beta + \delta = 2m$, $BHM - AMH = 2\theta$;
then $BHM = m + \theta$, $AMH = m - \theta$, $AHB = 180^\circ - (\alpha + \delta)$, and $AMB = 180^\circ - (\beta + \gamma)$. Hence by Art. 21,

$$\frac{HC}{MC} = \frac{\sin(m-\theta)}{\sin(m+\theta)}, \frac{AC}{HC} = \frac{\sin(\alpha+\delta)}{\sin(\alpha-\beta)}, \frac{BC}{AC} = \frac{\sin\beta}{\sin\delta}, \frac{MC}{BC} = \frac{\sin(\gamma-\delta)}{\sin(\beta+\gamma)}.$$

Whence by multiplication,

$$\frac{\sin(m-\theta)}{\sin(m+\theta)} = \frac{\sin(\alpha-\beta)\sin\delta\sin(\beta+\gamma)}{\sin(\alpha+\delta)\sin\beta\sin(\gamma-\delta)} \ldots\ldots (1).$$

But by (20) of Art. 16, $\dfrac{\sin(m-\theta)}{\sin(m+\theta)} = \dfrac{\tan m - \tan\theta}{\tan m + \tan\theta}$;

hence, putting $\dfrac{\sin(\alpha-\beta)\sin\delta\sin(\beta+\gamma)}{\sin(\alpha+\delta)\sin\beta\sin(\gamma-\delta)} = \tan h \ldots\ldots (2)$,

the equation (1) becomes $\dfrac{\tan m - \tan\theta}{\tan m + \tan\theta} = \tan h$;

from which we readily get by (37) of Art. 15, and (18) of Art. 16,

$$\tan\theta = \tan(45^\circ - h)\tan m \ldots\ldots (3).$$

If h be $> 45^\circ$ and $\tan m$ positive, θ will be negative. Hence, to find H M the required distance, we have by Art. 21 (putting $AB = a$),

$$HM = HA\frac{\sin(\alpha-\beta)}{\sin(m-\theta)}, \text{ and } HA = a\frac{\sin\delta}{\sin(\alpha+\delta)}, \text{ or}$$

$$HM = a\frac{\sin\delta\sin(\alpha-\beta)}{\sin(\alpha+\delta)\sin(m-\theta)} \ldots\ldots (4).$$

Cor. By (4),

$$a \text{ or } AB = HM\frac{\sin(\alpha+\delta)\sin(m-\theta)}{\sin\delta\sin(\alpha-\beta)},$$

which is a solution of the converse problem *to find the distance between the stations* A *and* B, *when* H M *and the angles at* A *and* B *are given.*

EXAMPLE.

Let $a = 600{\cdot}07$,

$\alpha = 95^\circ\ 20'\ 10''$, | $\gamma = 98^\circ\ 45'\ 15''$,
$\beta = 58^\circ\ 20'\ 12''$, | $\delta = 53^\circ\ 30'\ 6''$;

then

$\alpha - \beta = 36^\circ\ 59'\ 58''$, | $\alpha + \delta = 148^\circ\ 50'\ 16''$,
$\gamma - \delta = 45^\circ\ 15'\ 9''$, | $\phi + \delta = 111^\circ\ 50'\ 18''$,
$\beta + \gamma = 157^\circ\ 5'\ 27''$, | $m = \frac{1}{2}(\beta + \delta) = 55^\circ\ 55'\ 9''$.

To find h by (2):—

$\log\sin\ 36^\circ 59' 58'' = 9\cdot7794574$
$\log\sin\ 53^\circ 30'\ 6'' = 9\cdot9051881$
$\log\sin\ 157^\circ\ 5' 27'' = 9\cdot5902524$
$\log\text{cosec}\ 148^\circ 50' 16'' = 10\cdot2861208$
$\log\text{cosec}\ 58^\circ 20' 12'' = 10\cdot0699953$
$\log\text{cosec}\ 45^\circ 15'\ 9'' = 10\cdot1486095$

$\log\tan h = 9\cdot7796235$

Hence $h = 31^\circ\ 2'\ 58''\cdot25$, and
$45^\circ - h = 13^\circ 57' 1''\cdot74$.

To find θ by (3):—

$\log\tan 13^\circ 57' 1''\cdot74 = 9\cdot3951694$
$\tan 55^\circ 55' 9'' = 10\cdot1696916$

$\log\tan\theta = 9\cdot5648610$

Hence $\theta = 20^\circ 9' 41''$.

Finally, to find H M by (4):—

$\log 600\cdot07 = 2\cdot7782019$
$\sin 53^\circ 30'\ 6'' = 9\cdot9051881$
$\sin 36^\circ 59' 58'' = 9\cdot7794574$
$\text{cosec}\ 148^\circ 50' 16'' = 10\cdot2861208$
$\text{cosec}\ 35^\circ 45' 28'' = 10\cdot2333196$

$\log \text{H M} = 2\cdot9822878$

Hence H M $= 960\cdot036$, the distance required.

Another Method.

By Art. 21.

$$\text{H A} = a\frac{\sin\delta}{\sin(\alpha+\delta)}, \text{ and M A} = a\frac{\sin\gamma}{\sin(\beta+\gamma)},$$

or $\log \text{H A} = \log a + \log\sin\delta + \log\text{cosec}(\alpha+\delta) - 20 \ldots.$ (5),

$\log \text{M A} = \log a + \log\sin\gamma + \log\text{cosec}(\beta+\gamma) - 20 \ldots.$ (6).

From these, H A and M A can be determined, and as their included angle, H A M $= \alpha - \beta$, is given, the third side H M can be found by Art. 22 or Art. 24.

By Art. 24 we have

$\log \text{H M} = \log(\text{M A} - \text{H A}) + \log\sec\phi - 10 \ldots..$ (7),

$2\log\tan\phi = \log 4 + \log \text{M A} + \log \text{H A} + 2\log\sin\tfrac{1}{2}(\alpha-\beta)$
$- 2\log(\text{M A} - \text{H A}) \ldots..$ (8).

To determine H A by (5):—

$\log 600\cdot07 = 2\cdot7782019$
$\log\sin\ 53^\circ 30'\ 6'' = 9\cdot9051881$
$\log\text{cosec}\ 148^\circ 50' 16'' = 10\cdot2861208$

$\log \text{H A} = 2\cdot9695108$
or H A $= 932\cdot2036$

To find M A by (6):—

$\log 600\cdot07 = 2.7782019$
$\log\sin\ 98^\circ 45' 15'' = 9\cdot9949110$
$\log\text{cosec}\ 157^\circ 5' 27'' = 10\cdot4097476$

$\log \text{M A} = 3\cdot1828605$
or M A $= 1523\cdot563$

Hence M A $-$ H A $= 591\cdot3594$.

Next, to find ϕ by (8):—

$\log 4 = 0\cdot6020600$
$\log \text{M A} = 3\cdot1828605$
$\log \text{H A} = 2\cdot9695108$
$2\log\sin 18^\circ 29' 59'' = 19\cdot0029402$

$25\cdot7573715$
$2\log 591\cdot3594 = 5\cdot5437030$

$2\log\tan\phi = 20.2136685$
$\log\tan\phi = 10\cdot1068342$
or $\phi = 51^\circ 58' 37''$

To find H M by (7):—

$\log 591\cdot3594 = 2\cdot7718515$
$\log\sec 51^\circ 58' 37'' = 10\cdot2104328$

$\log \text{H M} = 2\cdot9822843$

or H M $= 960\cdot03$, the same as before nearly.

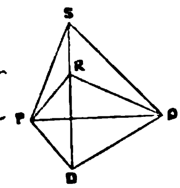
V. *Given the distances of three remote objects P, Q, R, from each other, and the angles they subtend at a station S, in the same horizontal plane, to find the distance of the station S from each of the objects.*

Denote the distances R Q, P Q, and P R, by a, b, c, and the angles P, R, Q, by A, B, C, respectively. Put also

angle $\mathrm{RSP} = \alpha$, $\mathrm{RPS} + \mathrm{RQS} = \mathrm{B} - (\alpha + \beta) = 2m$,

$\mathrm{RSQ} = \beta$, $\mathrm{RPS} - \mathrm{RQS} = 2\theta$;

then $\mathrm{RPS} = m + \theta$, and $\mathrm{RQS} = m - \theta$.



$$\text{Now } \frac{\mathrm{RP}}{\mathrm{RS}} = \frac{\sin\alpha}{\sin(m+\theta)}, \quad \frac{\mathrm{RS}}{\mathrm{RQ}} = \frac{\sin(m-\theta)}{\sin\beta}, \quad \frac{\mathrm{RQ}}{\mathrm{RP}} = \frac{\sin \mathrm{A}}{\sin \mathrm{C}}.$$

Proceeding with these as with the similar expressions in the first problem, we get

$$\frac{\sin\beta \ \sin \mathrm{C}}{\sin\alpha \ \sin \mathrm{A}} = \tan k \ \ldots . \ (1), \quad \tan\theta = \tan\,(45^\circ - k) \tan m \ \ldots . \ (2).$$

From (1), k can be found, and thence θ, by (2).

Whence finally

$$\mathrm{PS} = c\,\frac{\sin(m+\theta+\alpha)}{\sin\alpha}, \ \mathrm{RS} = c\,\frac{\sin(m+\theta)}{\sin\alpha}, \ \mathrm{QS} = a\,\frac{\sin(m-\theta+\beta)}{\sin\beta} \ldots (3).$$

If k be greater than 45° and $\tan m$ positive, θ will be negative.

Cor. 1. When S is anywhere in the angular space formed by R P, R Q (or these produced in the direction R P, R Q), then $2m = 360^\circ - (\alpha + \beta + \mathrm{B})$.

Cor. 2. Let D be the point in which the line S R meets the circle which circumscribes the triangle P S Q; join P D, Q D. Then the angles P Q D, P S D are equal, being in the same segment, as are also the angles Q P D, Q S D. Hence this geometrical construction:—

Describe the triangle P Q R so that its sides may be equal to the three given distances. At P and Q make the angles Q P D, P Q D, equal, respectively, to the angles Q S R, P S R, and describe a circle about the triangle P D Q. Produce D R to meet this circle in S; then will S P, S Q, S R, be the distances required.

Example.

Let $a = 262$, $b = 404$, $c = 213$, $\alpha = 13^\circ\,30'$, $\beta = 29^\circ\,50'$.

The angles A, B, C are found to be

$$\mathrm{A} = 35^\circ\,35'\,54'', \ \mathrm{B} = 116^\circ\,9'\,27'', \ \mathrm{C} = 28^\circ\,14'\,39''.$$

Hence $m = \frac{1}{2}\{\mathrm{B} - (\alpha + \beta\} = 36^\circ\,24'\,43''$.

First, then, to find k and θ by (1) and (2):

log sin 29° 50′ 0″ = 9·6967745	As k is greater than 45°, θ is negative.
log sin 28° 14′ 39″ = 9·6750723	log tan 15° 0′ 21″·34 = 9·4282321
log cosec 13° 30′ 0″ = 10·6318147	log tan 36° 24′ 45″ = 9·8678124
log cosec 35° 35′ 54″ = 10·2350029	
log tan k = 10·2386644	log tan θ = 9·2960445
Hence k = 60° 0′ 21″·34	Whence θ = − (11° 11′ 3″)
and $45^\circ - k$ = − (15° 0′ 21″·34)	and $m + \theta + \alpha$ = 38° 43′ 40″

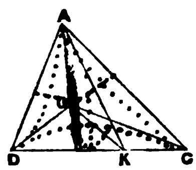
Next to find P S by (3):—

$$\begin{aligned} \log 213 &= 2\cdot3283796 \\ \log \sin\ 38^\circ\, 43'\, 40'' &= 9\cdot7963112 \\ \log \operatorname{cosec} 13^\circ\, 30' &= 10\cdot6318147 \\ \hline \log PS &= 2\cdot7565055 = \log 570\cdot828, \end{aligned}$$

or $\quad PS = 570\cdot828.$

The distances R S, Q S are found in a similar way.

VI. *To find the height of an object from angles of elevation taken on a horizontal plane, at three given stations in a straight line, not in the direction of the object.*



Let A O be the object perpendicular to the plane D O C in which are the given stations D, K, C, in the same straight line. Denote the angles of elevation at C, K, D, that is, the angles A C O, A K O, A D O, by α, β, γ; and put $CK = a$, $KD = b$, and $AO = x$. Then $AC = x \operatorname{cosec} \alpha$, $AK = x \operatorname{cosec} \beta$, $AD = x \operatorname{cosec} \gamma$.

Hence by (1) of Art. 23,

$$\cos ACD = \frac{a^2 + x^2 \operatorname{cosec}^2\alpha - x^2 \operatorname{cosec}^2\beta}{2\,a\,x \operatorname{cosec}\alpha} = \frac{(a+b)^2 + x^2\operatorname{cosec}^2\alpha - x^2 \operatorname{cosec}^2\gamma}{2\,(a+b)\,x \operatorname{cosec}\alpha},$$

or $\{(a+b)(\operatorname{cosec}^2\alpha - \operatorname{cosec}^2\beta) + a(\operatorname{cosec}^2\gamma - \operatorname{cosec}^2\alpha)\}\, x^2 = a\, b\,(a+b)$.

This equation, by means of (21), Art. 16, reduces to the following:—

$$\{(a+b)\sin(\beta+\alpha)\sin(\beta-\alpha)\sin^2\gamma + a\sin(\alpha+\gamma)\sin(\alpha-\gamma)\sin^2\beta\}\, x^2 = a\,b\,(a+b)\sin^2\alpha \sin^2\beta \sin^2\gamma,$$

or $\quad (1+\lambda)\, x^2 = \mu \quad . \quad . \quad . \quad . \quad (1),$

$$\text{where } \lambda = \frac{a\sin(\alpha+\gamma)\sin(\alpha-\gamma)\sin^2\beta}{(a+b)\sin(\beta+\alpha)\sin(\beta-\alpha)\sin^2\gamma},\ \mu = \frac{a\,b\sin^2\alpha\sin^2\beta}{\sin(\beta+\alpha)\sin(\beta-\alpha)}.$$

Now if λ and μ be both positive, as in (1), by assuming

$$\lambda = \tan^2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

we get by means of (1), the equation

$$x^2 \sec^2\phi = \mu, \text{ or } x^2 = \mu\cos^2\phi \quad . \quad . \quad . \quad . \quad (3),$$

which is in a form for logarithms. The subsidiary angle ϕ is determined by (2).

Again, if λ and μ be both negative; then by assuming

$$\lambda = \sec^2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

we get

$$x^2 \tan^2\phi = \mu, \text{ or } x^2 = \mu\cot^2\phi \quad . \quad . \quad . \quad . \quad (5).$$

Hence in this case x is determined from (4) and (5).

Lastly, if λ be negative and μ positive (these are the only cases that can arise*), by assuming

$$\lambda = \sin^2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

we get for x the equation

$$x^2\cos^2\phi = \mu, \text{ or } x^2 = \mu\sec^2\phi \quad . \quad . \quad . \quad . \quad (7).$$

---

* The signs of λ and μ depend on α, β, γ, and as these can be combined, two and two, in three ways only, the three cases that have been discussed are the only cases that can arise. It is possible to reduce two of these to the same, but the solution seems to be more easily effected by the formulæ given in the text.

Example.

Let $a=50$, $b=60$, $\alpha=30^\circ\ 40'$, $\beta=40^\circ\ 33'$, $\gamma=50^\circ\ 23'$.
Now

$$a+b=110, \quad \alpha+\gamma=81^\circ\ 3', \quad \beta+\alpha=71^\circ\ 13',$$
$$\alpha-\gamma=-(19^\circ\ 43'), \quad \beta-\alpha=9^\circ\ 53'.$$

Hence as λ is negative and μ positive, the formulæ of solution are (6) and (7).

To find ϕ by (6):—

$\log 50$	=	1·6989700
$\log\sin\ 81^\circ\ 3'$	=	9·9946798
$\log\sin\ 19^\circ\ 43'$	=	9·5281053
$2\log\sin\ 40^\circ\ 33'$	=	19·6259756
$\log\operatorname{cosec} 71^\circ\ 13'$	=	10·0237679
$\log\operatorname{cosec}\ 9^\circ\ 53'$	=	10·7653751
$2\log\operatorname{cosec} 50^\circ\ 23'$	=	20·2266488
		21·8635225
$\log 110$	=	2·0413927
$2\log\sin\phi$	=	19·8221298
$\log\sin\phi$	=	9·9110649
Hence ϕ	=	$54^\circ\ 34'\ 12''{\cdot}61$

To find x by (7):—

$\log 50$	=	1·6989700
$\log 60$	=	1·7781513
$2\log\sin\ 30^\circ\ 40'$	=	19·4152128
$2\log\sin\ 40^\circ\ 33'$	=	19·6259756
$2\log\sec\phi$	=	20·4735852
$\log\operatorname{cosec} 71^\circ\ 13'$	=	10·0237679
$\log\operatorname{cosec}\ 9^\circ\ 53'$	=	10·7653751
$2\log x$	=	3·7810379
$\log x$	=	1·8905189

or $x = 77{\cdot}7175$, the required height.

Exercises.

1. At what horizontal distance from a column 200 feet high will it subtend an angle of $31^\circ\ 17'\ 12''$? *Ans.* 329·114 feet.

2. A staff one foot in length stands on the top of a tower 200 feet high. Find the angle which it subtends at a point in the horizontal plane 100 feet from the base of the tower. *Ans.* $6'\ 51''$ nearly.

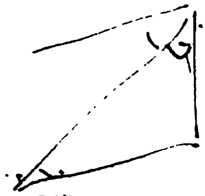
3. Two observers on the same side of a balloon, and in the same vertical plane with it, a mile apart, find its angles of elevation to be respectively $15^\circ\ 17'\ 18''$ and $62^\circ\ 30'\ 20''$. Find the height of the balloon. *Ans.* Height = 560·89 yards.

4. A tower A B stands on the summit A of a sloping hill A C, which is inclined to the horizon in an angle of 30°; the length of the hill A C is 225 yards, and the elevation of B above the horizon when observed from the middle D of the hill A C is $34^\circ\ 18'\ 30''$. Find the height of the tower. *Ans.* Height = 10·2315 yards.

5. A column stands on the opposite bank of a river, and from two stations in a line with its base, and distant 100 yards, the angles of elevation of its top were observed to be $58^\circ\ 5'\ 18''$ and $32^\circ\ 17'\ 30''$. Find the height of the column and the breadth of the river. *Ans.* Height = 104·2073 yards, breadth = 64·89 yards.

6. From the bottom of a building, 100 feet high, the angle of elevation of the top of a tower in the same horizontal plane was $45^\circ\ 17'\ 20''$, and at the top the angle of depression of the same was $8^\circ\ 10'\ 6''$. Find the height and distance of the tower. *Ans.* Height = 87·5582 feet; distance = 86·6796 feet.

7. A tower B C is in the same straight line with two objects A and D (A and D being on opposite sides of the tower), and the distance A B



is found to be 236·759 feet; also the angles of depression of the two objects A and D from the top of the tower C are $42^\circ\ 10'\ 38''$ [redacted] $58^\circ\ 12'\ 39''$. Find the height of the tower and the [redacted]

Ans. Height = 214·508 feet; [redacted] 365 feet.

8. From the top of a tower 50 feet high, [redacted] near the edge of a cliff, the angle of depression of a ship's hull was observed $14^\circ\ 20'\ 24''$, and from the base of the tower the angle of depression of the ship's hull was $12^\circ\ 30'\ 18''$: find the horizontal distance of the ship from the tower and the height of the tower's base above the level of the sea.

Ans. Distance = 1476·92 feet; height = 327·562 feet.

9. At noon a column in the direction E.S.E. from an observer cast upon the ground a shadow, the extremity of which was in the direction N.E. from the same: the angle of elevation of the column being 45°, and the distance of the extremity of the shadow from the base of the column 80 feet; determine the height of the column.

Ans. Height $= 80\sqrt{\{2 - \sqrt{2}\}} = 61{\cdot}229$ feet.

10. Two ships of war, a mile apart, find that the angles subtended by the other ship and a fort on the opposite cliff, are $56^\circ\ 19'\ 20''$ and $63^\circ\ 41'\ 18''$. Find the distance of each ship from the fort.

Ans. 1·0352 and ·96101 mile.

11. Prove that the following rule for finding the distance of the horizon at sea is very nearly correct:—Take the number of feet in the height of the station above the level of the sea, and increase it by half that number, the square root of this quantity will give the distance of the horizon in miles.

12. A person standing at the distances 130 and 218 yards, respectively, from two towers, and in the same straight line with them, observes their apparent altitudes to be the same; he then walks 120 yards towards the towers, and finds that the angle of elevation of one is double that of the other. Find the heights of the towers.

Ans. 47·3969 and 79·481 yards.

13. Show how the angle between two visual rays A B, A D, in the direction of two objects M and N, can be determined without any instrument for measuring angles.

14. A person wishing to determine the length of an inaccessible wall, places himself due south of one end, and then due west of the other, at such distances that the angles which the wall subtends at the two positions each $= 20^\circ\ 17'\ 18''$. Find the length of the wall, the distance between the two stations being 121 yards. *Ans.* 44·731 yards.

15. At each end of a horizontal base A B, 112 yards in length, the angle which the distance of the other end and an inaccessible object C subtends is observed, as also the angle of elevation of the object at one end of the base, viz., angle B A C $= 20^\circ\ 18'$, A B C $= 32^\circ\ 16'$, and C A O $= 10^\circ\ 29'$. Find the height and bearing of the object.

Ans. Height C C = 13·7 yards, bearing, or $<$ B A O $= 17^\circ\ 28'\ 56''$.

16. A person standing on the sea-shore can just see the top of a mountain, whose height he knows to be 1284·8 yards. After ascending vertically in a balloon to a certain height, he observes the angle of depression of the mountain's summit to be $2^\circ\ 15'$. Now, assuming the earth's diameter to be 7986·4 miles, it is required to find the height of the balloon at the time of the observation. *Ans.* 3 miles.

17. Having given the angle which the earth's radius (r) subtends at the sun $8'' \cdot 58$, to find the circular measure of this angle, and thence the distance of the sun from the earth.

Ans. Cir. measure = $\cdot 000041597$, distance = $24040\, r$.

18. Show how the diameter of an inaccessible circular basin is determined without any instrument for measuring angles.

19. Two objects M and L of a fortified town are seen from a station N. The distances M N, N L, are known to be 1020 and 1680 yards, respectively, and the angle M N L = $49^\circ\ 25'$. Find the distance between the two objects. *Ans.* The distance = $1277 \cdot 97$ yards.

20. A church-steeple C, and a tower D, are observed from the extremities of a line A B, $6265 \cdot 88$ feet in length, in the same horizontal plane, and the following angles were measured :—

$$C\,A\,B = 139^\circ\ 15'\ 45'',\ D\,A\,B = 53^\circ\ 30'\ 23'',\ A\,B\,C = 31^\circ\ 49',$$
$$A\,B\,D = 114^\circ\ 24'\ 55''.$$

Find the distance between the church and tower.

Ans. Distance = $33336 \cdot 4$ feet.

21. From a convenient station, P, there could be seen three objects, A. B, and C, whose distances from each other were A B = 8 miles, A C = 6 miles, B C = 4 miles; also the horizontal angles A P C and B P C were $33^\circ\ 45'$ and $22^\circ\ 30'$. Required the respective distances of the station P from each object. (The station P, and the object C, are on *opposite* sides of the line A B.)

Ans. A P = $7 \cdot 10199$, B P = $9 \cdot 34285$, C P = $10 \cdot 4252$ miles.

22. If in the last, A B = 275405, B C = 190826, A C = 184335 feet, and angle A P C = $136^\circ\ 48'\ 38''$, B P C = $84^\circ\ 54'\ 13''$, what is the distance of P from each object?

Ans. P A = 114476, B P = $179207 \cdot 4$, C P = $83392 \cdot 6$ feet.

23. From Plymouth the Lizard is distant $54 \cdot 44$ miles; from the Lizard the Start Point is distant $71 \cdot 15$ miles; and from the Start Point to Plymouth the distance is $23 \cdot 31$ miles.

From Eddystone Light, Plymouth bears N. $25^\circ\ 4'\ 5''$ E.
,, ,, Lizard ,, S. $70^\circ\ 13'\ 15''$ W.
,, ,, Start ,, N. $83^\circ\ 52'\ 20''$ E.

Find the distance of Eddystone Light from each of the other places.

Ans. The distances are $13 \cdot 10$, $44 \cdot 42$, and $27 \cdot 22$ miles.

24. Three stations A, B, C, are in a straight horizontal line, such that A B = 80 ,and B C = 75 yards. At A, B, and C, the angles of elevation of an inaccessible object D (not in the direction of the line A B C), are found to be $72^\circ\ 19'$, $78^\circ\ 16'$, and $70^\circ\ 10'$. From these data it is required to find the height of D above the horizon of the line of observation. *Ans.* Height = $286 \cdot 321$ yards.

EXPANSIONS AND DEVELOPMENTS OF TRIGONOMETRICAL EXPRESSIONS.

29. *The following formulæ, involving an arc α and multiples of ϖ, are used in Astronomy and some other branches of Mathematics* :—

Produce Q R (fig. to Art. 2) to meet the circle again in q, then the four arcs A B, Q E, q E, and A b, are evidently all equal. Hence the arcs A B, A C Q, A C q, and A b, may be denoted by α, $\varpi - \alpha$,

$\varpi + \alpha$, and $-\alpha$; consequently by (13, 14, 28, 29, 32, 33) of Art. 15,

$$\sin\alpha = \sin(\varpi-\alpha) = -\sin(\varpi+\alpha) = -\sin(-\alpha) \ldots (1),$$
$$\cos\alpha = -\cos(\varpi-\alpha) = -\cos(\varpi+\alpha) = \cos(-\alpha) \ldots (2).$$

These relations will also hold, if we add to each arc, or take from it, any number of circumferences; for the resulting arcs will have the same extremities as at present. Whence, adding $2n\varpi$ to each in (1), we get

$$\sin\alpha = \sin(2n\varpi+\alpha) = \sin\{(2n+1)\varpi-\alpha\}$$
$$= -\sin\{(2n+1)\varpi+\alpha\} = -\sin(2n\varpi-\alpha)$$
$$= \pm\sin(2n\varpi\pm\alpha) = \pm\sin\{(2n+1)\varpi\mp\alpha\} \ldots (3).$$

By taking away $2n\varpi$ from each arc, we get, in a similar way,

$$\sin\alpha = \pm\sin(-2n\varpi\pm\alpha) = \pm\sin\{-(2n-1)\varpi\mp\alpha\} \ldots (4).$$

Hence if p be a positive or negative integer of the series, 0, 2, 4, . . ., and q a positive or negative integer of the series 1, 3, 5 . . ., we have by (3) and (4),

$$\sin(p\varpi+\alpha) = \sin\alpha \ldots (5),\ \sin(p\varpi-\alpha) = -\sin\alpha \ldots (6),$$
$$\sin(q\varpi+\alpha) = -\sin\alpha \ldots (7),\ \sin(q\varpi-\alpha) = \sin\alpha \ldots (8).$$

Also, by (2), we get in a similar way,

$$\cos(p\varpi+\alpha) = \cos\alpha \ldots (9),\ \cos(p\varpi-\alpha) = \cos\alpha \ldots (10),$$
$$\cos(q\varpi+\alpha) = -\cos\alpha \ldots (11),\ \cos(q\varpi-\alpha) = -\cos\alpha \ldots (12).$$

30. *To develop the sine and cosine of an angle x in a series of ascending powers of x.*

Let
$$\sin x = e_0 + ex + e_2x^2 + e_3x^3 + \text{etc.} \ldots (1),$$
$$\cos x = c_0 + cx + c_2x^2 + c_3x^3 + \text{etc.} \ldots (2);$$

then by changing x into $-x$, and remembering (Art. 15) that $\sin(-x) = -\sin x$, and $\cos(-x) = \cos x$, we obtain

$$-\sin x = e_0 - ex + e_2x^2 - e_3x^3 + \text{etc.} \ldots (3),$$
$$\cos x = c_0 - cx + c_2x^2 - c_3x^3 + \text{etc.} \ldots (4).$$

By taking half the difference of (1) and (3), and half the sum of (2) and (4), we get

$$\sin x = ex + e_3x^3 + e_5x^5 + \text{etc.} \ldots (5),$$
$$\cos x = c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \text{etc.} \ldots (6).$$

Again, by (23, 25) of Art. 16, we have the relations

$$\sin x - \sin z = 2\cos\tfrac{1}{2}(x+z)\sin\tfrac{1}{2}(x-z),$$
$$\cos z - \cos x = 2\sin\tfrac{1}{2}(x+z)\sin\tfrac{1}{2}(x-z),$$

which by (5) and (6), become

$$e(x-z) + e_3(x^3-z^3) + \text{etc.} \ldots =$$
$$2\cos\tfrac{1}{2}(x+z)\left\{e\left(\frac{x-z}{2}\right) + e_3\left(\frac{x-z}{2}\right)^3 + \text{etc.} \ldots\right\},$$
$$c_2(x^2-z^2) + c_4(x^4-z^4) + \text{etc.} \ldots =$$
$$-2\sin\tfrac{1}{2}(x+z)\left\{e\left(\frac{x-z}{2}\right) + e_3\left(\frac{x-z}{2}\right)^3 + \text{etc.} \ldots\right\}.$$

Dividing both sides of each of these by $x - z$, and taking $z = x$ in the results, we get

$$e\cos x = e + 3e_3x^2 + 5e_5x^4 + \text{etc.} \ldots,$$
$$-e\sin x = 2c_2x + 4c_4x^3 + 6c_6x^5 + \text{etc.} \ldots;$$

and by substituting in these the values of $\sin x$ and $\cos x$ given in (5) and (6), we obtain

$$e\,(c_0+c_2x^2+c_4x^4+c_6x^6+\ldots\text{etc.}) = e+3e_3x^2+5e_5x^4+7e_7x^6+\ldots\text{etc.}$$

$$-e\,(ex+e_3x^3+e_5x^5+\text{etc.}\ldots) = 2c_2x+4c_4x^3+6c_6x^5+\text{etc.}\ldots$$

Since these two equations are *identical*, we have (Algebra, Art. 135),

$$e = ec_0,\ 3e_3 = ec_2,\ 5e_5 = ec_4,\ 7e_7 = ec_6,\ \text{etc.},$$

$$2c_2 = -e^2,\ 4c_4 = -ee_3,\ 6c_6 = -ee_5,\ \text{etc.};$$

hence, $c_0 = 1,\ c_2 = -\dfrac{e^2}{2},\ e_3 = -\dfrac{e^3}{1.2.3},\ c_4 = \dfrac{e^4}{1.2.3.4},$

$$e_5 = \frac{e^5}{1.2.3.4.5},\ c_6 = -\frac{e^6}{1.2.3.4.5.6},\ \text{etc.}$$

Substituting these values in (5) and (6), we get

$$\sin x = ex - \frac{e^3x^3}{1.2.3} + \frac{e^5x^5}{1.2.3.4.5} - \text{etc.}\ldots\ldots(7),$$

$$\cos x = 1 - \frac{e^2x^2}{1.2} + \frac{e^4x^4}{1.2.3.4} - \frac{e^6x^6}{1.2.3.4.5.6} + \text{etc.}\ .\ .\ (8).$$

To find the value of e, let $x = \dfrac{1}{e}$; then we have by (7),*

$$\sin\left(\frac{1}{e}\right) = 1 - \frac{1}{1.2.3} + \frac{1}{1.2.3.4.5} - \ldots\ldots = \cdot 84147098\ldots;$$

$$\text{or, } \frac{1}{e} = 57^\circ\cdot 2958\ldots = \frac{90^\circ}{\frac{1}{2}\varpi}.$$

Hence $\dfrac{1}{e}$ is the unit of circular measure, as in Art. 4. If, then, we take $\dfrac{1}{e} = 1$, or $e = 1$, the equations (7) and (8) become

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ .\ .\ .\ .\ .\ .\ (9),$$

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ .\ .\ .\ .\ .\ .\ .(10),$$

in which it is to be understood that x is an arc of the circle to radius unity. The sine and cosine of an angle A° will be obtained from (9) and (10), by writing for x in the right-hand members of these equations the expression (Art. 4) $\dfrac{\frac{1}{2}\varpi}{90^\circ}A^\circ$.

This elegant method of deducing $\sin x$ and $\cos x$ is due to Mr. W. Finlay, Professor of Mathematics at Manchester.—(*Mathematician*, page 299, vol. iii.)

Cor. Since $\tan x = \dfrac{\sin x}{\cos x}$, we see by the preceding expressions for $\sin x$ and $\cos x$, that the development of $\tan x$ begins with x. Hence assume

$$\tan x = \frac{\sin x}{\cos x} = x + a_2x^2 + a_3x^3 + \ .\ .\ .\ .\ .$$

* As x may be *any* angle in (7), whatever value is found for e for any value of x, the same value of e will belong to the series in general.

In this write the values of $\sin x$ and $\cos x$, already given, multiply out, and equate the coefficients of the like powers of x; we then get

$$\frac{\sin x}{\cos x} = \tan x = x + \frac{2\,x^3}{1.2.3} + \frac{2^4\,x^5}{1.2.3.4.5} + \ldots$$

Similarly

$$\frac{\cos x}{\sin x} = \cot x = \frac{1}{x} - \frac{2\,x}{1.2.3} - \frac{2^4\,x^3}{1.2.3.4.5.6} - \ldots.$$

31. *To prove Demoivre's Theorem, that for all values of n,*

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^n = \cos n\,\theta \pm \sqrt{-1}\sin n\,\theta.$$

By multiplication,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)\,(\cos\theta \pm \sqrt{-1}\sin\theta)$$
$$= \cos^2\theta - \sin^2\theta \pm \sqrt{-1}\;2\sin\theta\cos\theta,$$

or $(\cos\theta \pm \sqrt{-1}\sin\theta)^2 = \cos 2\theta \pm \sqrt{-1}\sin 2\theta$, by (5, 6,) of Art. 16.
Also, by multiplication and the preceding,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^2\,(\cos\theta \pm \sqrt{-1}\sin\theta)$$
$$= (\cos 2\,\theta \pm \sqrt{-1}\sin 2\,\theta)\,(\cos\theta \pm \sqrt{-1}\sin\theta)$$
$$= \cos 2\,\theta\cos\theta - \sin 2\,\theta\sin\theta \pm \sqrt{-1}\,(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta),$$

or $(\cos\theta \pm \sqrt{-1}\sin\theta)^3 = \cos 3\,\theta \pm \sqrt{-1}\sin 3\,\theta$, by (15, 16), Art. 16.
Hence when the index n is a *positive integer*,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^n = \cos n\,\theta \pm \sqrt{-1}.\sin n\,\theta.$$

Next let the index be a *negative integer*; then because $1 = \sin^2\theta + \cos^2\theta$,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^{-n} = \left(\frac{1}{\cos\theta \pm \sqrt{-1}\sin\theta}\right)^n = \left(\frac{\cos^2\theta + \sin^2\theta}{\cos\theta \pm \sqrt{-1}\sin\theta}\right)^n$$

$= (\cos\theta \mp \sqrt{-1}\sin\theta)^n$, by actual division.

Whence $(\cos\theta \pm \sqrt{-1}\sin\theta)^{-n} = (\cos\theta \mp \sqrt{-1}\sin\theta)^n$

$$= \cos n\,\theta \mp \sqrt{-1}\sin n\,\theta$$
$$= \cos(-n\,\theta) \pm \sqrt{-1}\sin(-n\,\theta), \text{ by (33), of Art. 15.}$$

This proves the theorem for *negative indices*.

Again, $\quad (\cos\theta \pm \sqrt{-1}\sin\theta)^n = \cos n\,\theta \pm \sqrt{-1}\sin n\,\theta,$

and $\quad \left(\cos\frac{n}{m}\,\theta \pm \sqrt{-1}\sin\frac{n}{m}\,\theta\right)^m = \cos n\,\theta \pm \sqrt{-1}\sin n\,\theta.$

Hence $\quad (\cos\theta \pm \sqrt{-1}\sin\theta)^n = \left(\cos\frac{n}{m}\,\theta \pm \sqrt{-1}\sin\frac{n}{m}\,\theta\right)^m,$

or $\quad (\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{n}{m}} = \cos\frac{n}{m}\,\theta \pm \sqrt{-1}\sin\frac{n}{m}\,\theta;$

which proves the theorem for *fractional indices*.

CONSTRUCTION OF TRIGONOMETRICAL TABLES.

32. In the construction of trigonometrical tables, the numerical expressions of some function (sine or cosine) for certain values of the arc or angle, at large intervals, are first determined, as in Art. 15, and tabulated; and then the intervals are filled up by the calculation of those

values of the function that belong to the intermediate values of the arc or angle. To secure accuracy, which is of the greatest importance in tables for use, *formulæ of verification*, as they are called, are employed, by which the value of any function already computed is again calculated by some independent method. The agreement of the value thus found, with that obtained by the other method, is the test of accuracy. As all the tables referred to in this Article have been computed already to the greatest accuracy, it will be sufficient to describe here, as briefly as possible, the mode of computation usually employed; and first to find sin 1'.

By (9) of Art. 16 (A being less than 45°),

$$\sin A = \tfrac{1}{2}\{\sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\}.$$

Let A = 15°; then we have by (36) of Art. 15,

$$\sin 15° = \tfrac{1}{2}\{\sqrt{(1 + \sin 30°)} - \sqrt{(1 - \sin 30°)}\} = \tfrac{1}{2}(\sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{1}{2}}).$$

Hence $\sin 15°$ or $\sin \frac{30°}{2}$ is known, and thence cos 15° by (1) of Art. 15.

In a similar way we find,

$$\sin \frac{30°}{2^2}, \sin \frac{30°}{2^3}, \sin \frac{30°}{2^4}, \ldots \sin \frac{30°}{2^{11}}, \text{ or } \sin 52''\cdot734375.$$

Now, by noticing the sines of the last of these and its double, it will be found that the sines of small arcs are nearly as the arcs themselves.

Hence if $\sin \frac{30°}{2^{11}}$ be denoted by s,

$$\text{arc } 52''\cdot734375 : \text{arc } 1' :: s : \sin 1'.$$

This gives sin 1', and thence cos 1' by (1) of Art. 15. The values of sin 1' and cos 1', found in this way, are

$$\sin 1' = \cdot0002908882, \cos 1' = \cdot9999999577, \text{ nearly.}$$

The tangent or any other function may now be obtained, as

$$\tan 1' = \frac{\sin 1'}{\cos 1'}, = \cdot0002908882, \text{ etc.}$$

It will be seen by this that the sine and tangent of 1' agree to ten decimal places.

The sine of 1' may also be calculated by the formula (Art. 29),

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{etc.}$$

Again by (15) of Art. 16,

$$\sin (n + 1) B = 2 \sin n B . \cos B - \sin (n - 1) B.$$

Hence, if in this we put $n = 1, 2, 3, \ldots$, and $B = 1'$, we shall be able to calculate the sines of all angles from 0° to 30°, for every minute of a degree, and consequently all the other trigonometrical functions.

After proceeding as far as 30°, the labour of computation is considerably reduced by the formula (11), Art. 16. By transposition that formula becomes

$$\sin (A + B) = 2 \sin A \cos B - \sin (A - B).$$

In this let A = 30°; then remembering (Art. 15) that sin 30° = cos 60° = ½, we have

$$\sin (30° + B) = \cos B - \sin (30° - B).$$

Hence writing 1′, 2′, 3′, etc., for B, we get

$$\sin 30^\circ\, 1' = \cos 1' - \sin 29^\circ\, 59',$$
$$\sin 30^\circ\, 2' = \cos 2' - \sin 29^\circ\, 58'$$
$$\text{etc.} \qquad \text{etc.}$$

Consequently the values of the sines of all angles from 30° to 45° may be found by simply taking the differences of previously found values. And similarly for the corresponding values of the cosines, by means of the formula (12), Art. 16.

It is unnecessary to carry the operation beyond 45°, as the sines of all angles less than 45° would give the cosines of their complements, etc. By this method, then, the functions of the entire quadrant are computed.

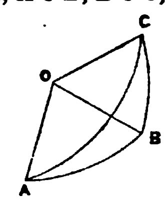
The labour of "filling up the intervals" is very much reduced by the *method of differences.* (See *Algebra*, Art. 169.)

SPHERICAL TRIGONOMETRY.

DEFINITIONS AND FIRST PRINCIPLES.

Art. 1. Spherical Trigonometry is that part of mathematical science which investigates the relations that exist between the sides and angles of triangles, formed by the intersections of three planes with the surface of a sphere. It may also be defined (from the mode of investigation employed) to be the science which investigates the relations between the several parts of a *solid angle*, formed by three planes.

2. Let O be a solid angle formed by the three planes, A O B, B O C, and A O C. With O as centre, describe a sphere, intersecting the three planes in A B, B C, A C; then the portion of the surface of the sphere intercepted by the planes A O B, B O C, A O C, is called a *spherical triangle*. The arcs A B, B C, A C, are the sides of this spherical triangle, and the three *dihedral* angles formed by the planes, taken two and two, are its angles. Thus, the dihedral angle formed by the planes B A O, C A O, is the angle A; and so on.



Cor. The sides of a spherical triangle are arcs of *great* circles (Geo. of the Sphere).

3. Since A B, B C, A C, are arcs of circles whose radii are equal, they are measures of the angles A O B, B O C, A O C, at the centre, and hence when the side of a spherical triangle is spoken of, the angle which that side subtends at the centre of the sphere is meant.

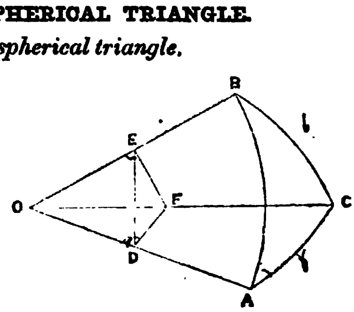
4. The sides of a spherical triangle are usually denoted by the letters a, b, c, and the opposite angles by A, B, C, as in plane trigonometry.

5. The inclination of two great circles is the angle made by their tangents at the point of intersection. Since each of these tangents is perpendicular to the radius in which the planes of the circles intersect; the same angle measures the inclination of the planes of the circles (Geo. of Planes).

Other definitions and principles referred to in the subsequent investigations are given in the *Geometry of the Sphere*.

THE RIGHT-ANGLED SPHERICAL TRIANGLE.

6. *Properties of the right-angled spherical triangle.*



Let A B C be a spherical triangle, right-angled at C, so that the planes (O being the centre of the sphere) A O C, B O C, are perpendicular to each other (Art. 2). Take any point E in O B, and draw E F perpendicular to O C; from F draw F D perpendicular to O A, and join E D. Then (Geo. of Planes) E F is perpendicular to the plane A O C, and D E

to O A. Hence the plane angle E D F, which measures the inclination (Geo. of Planes) of the two planes A O B, A O C, is equal to A (Art. 5), and the plane angles E O F, D O F, D O E, are respectively equal to a, b, and c, (Art. 3); the notation being as in Art. 4.

Now (Plane Trig. Art. 14),

$$\sin a = \frac{EF}{OE}, \ \cos a = \frac{OF}{OE}, \ \tan a = \frac{EF}{FO},$$
$$\sin b = \frac{FD}{FO}, \ \cos b = \frac{OD}{OF}, \ \tan b = \frac{DF}{DO},$$
$$\sin c = \frac{ED}{EO}, \ \cos c = \frac{OD}{OE}, \ \tan c = \frac{DE}{DO},$$
$$\sin A = \frac{EF}{DE}, \ \cos A = \frac{DF}{DE}, \ \tan A = \frac{EF}{DF}.$$

Hence the following relations:—

$$\cos c = \frac{DO}{EO} = \frac{DO}{FO} \cdot \frac{FO}{EO} = \cos b \cos a \ldots . (1).$$
$$\sin a = \frac{EF}{EO} = \frac{DE}{EO} \cdot \frac{EF}{DE} = \sin c \sin A \ldots . (2).$$
$$\tan b = \frac{DF}{DO} = \frac{DE}{DO} \cdot \frac{DF}{DE} = \tan c \cos A \ldots . (3).$$
$$\tan a = \frac{EF}{FO} = \frac{EF}{FD} \cdot \frac{FD}{FO} = \tan A \sin b \ldots . (4).$$

Each of the last three has a similar one deduced from the other side, viz. :—

$$\sin b = \sin c \sin B \ldots . (5),$$
$$\tan a = \tan c \cos B \ldots . (6),$$
$$\tan b = \tan B \sin a \ldots . (7).$$

Again, by (4) and (7),

$$\tan a \tan b = \tan A \tan B \sin a \sin b = \frac{\sin a \sin b}{\cot A \cot B},$$

or, $$\cot A \cot B = \frac{\sin a \sin b}{\tan a \tan b} = \cos a \cos b;$$

hence by (1),

$$\cos c = \cot A \cot B \ldots . (8).$$

Also, by (2) and (6),

$$\sin a \tan c \cos B = \sin c \sin A \tan a,$$

or, $$\cos B = \frac{\sin c \tan a}{\sin a \tan c} \sin A = \frac{\cos c}{\cos a} \sin A;$$

whence, by (1),

$$\cos B = \cos b \sin A \ldots . (9).$$

Similarly, $$\cos A = \cos a \sin B \ldots . (10).$$

These equations comprehend every case of right-angled spherical triangles. They are all expressed by the two following general rules, which are called *Napier's Rules for Circular Parts*, viz. :—

Sine middle part = product of tangents of adjacent parts;
Sine middle part = product of cosines of opposite parts.

But by (1),

$$1 - \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B = \frac{\sin s - \sin (s - c)}{\sin s} \quad \ldots\ldots\ldots (5).$$

Consequently by (4) and (5),

$$\frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{1 - \tan \frac{1}{2} A \tan \frac{1}{2} B} \tan \tfrac{1}{2} C = \frac{\sin (s - b) + \sin (s - a)}{\sin s - \sin (s - c)};$$

or, by (17, 22, 23) Art. 16, of Plane Trig.,

$$\tan \tfrac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \cot \tfrac{1}{2} C \quad \ldots\ldots\ldots (6).$$

Similarly,

$$\tan \tfrac{1}{2} (A - B) = \frac{\tan \frac{1}{2} A - \tan \frac{1}{2} B}{1 + \tan \frac{1}{2} A \tan \frac{1}{2} B}$$

$$= \frac{\sin (s - b) - \sin (s - a)}{\sin s + \sin (s - c)} \cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \cot \tfrac{1}{2} C \quad \ldots (7).$$

Cor. Dividing (6) by (7),

$$\frac{\tan \frac{1}{2} (A + B)}{\tan \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} (a + b)}{\tan \frac{1}{2} (a - b)}.$$

The formulæ (6) and (7) constitute what are called ***Napier's First Analogies.*** They are employed in the solution of a spherical triangle when two sides and the included angle are given.

When two sides and the included angle are given, it is sometimes desirable, as in Plane Trigonometry, to find the third side without finding the opposite angles. The formulæ of solution in this case are the following:—

Let a and b be the given sides, and C the included angle; then by Art. 7,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$
$$= \cos (a - b) - \sin a \sin b\, (1 - \cos C).$$

But

$$\cos c = 1 - \operatorname{vers} c,\ \cos (a - b) = 1 - \operatorname{vers} (a - b), \text{ etc.; hence}$$
$$\therefore \operatorname{vers} c = \operatorname{vers} (a - b) + \sin a \sin b \operatorname{vers} C;$$

or, since (Plane Trig., Art. 16) vers $c = 2 \sin^2 \frac{1}{2} c$, etc.,

$$\sin^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2} (a - b) \left\{ 1 + \frac{\sin a \sin b \sin^2 \frac{1}{2} C}{\sin^2 \frac{1}{2} (a - b)} \right\}.$$

Assume $$\tan^2 \phi = \frac{\sin a \sin b \sin^2 \frac{1}{2} C}{\sin^2 \frac{1}{2} (a - b)} \quad \ldots (8);$$

then

$$\sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a - b) \sec \phi \quad \ldots (9).$$

The third side c may therefore be found from (8) and (9).

10. *To find expressions for the sum and difference of two sides of a spherical triangle in terms of the other side and the sum and difference of the opposite angles.*

Applying the formulæ (6) and (7) of last Art. to the polar triangle, or, which is the same thing (Geo. of the Sphere), replacing A, B, C, a, b, c, by $\pi - a$, $\pi - b$, $\pi - c$, $\pi -$ A, $\pi -$ B, $\pi -$ C, we get

$$\tan \tfrac{1}{2} (a + b) = \frac{\cos \frac{1}{2} (B - A)}{\cos \frac{1}{2} (B + A)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

the required relations. These formulæ are *Napier's Second Analogies.*

The following are left as exercises:—

$$\sin \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b) \cos \frac{1}{2} C}{\cos \frac{1}{2} c}, \ \sin \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b) \cos \frac{1}{2} C}{\sin \frac{1}{2} c},$$

$$\cos \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a+b)\, \cancel{\cos \frac{1}{2} C}\ \sin \frac{1}{2} C}{\cos \frac{1}{2} c}, \ \cos \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a+b) \sin \frac{1}{2} C}{\sin \frac{1}{2} c},$$

$$\sin \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B) \sin \frac{1}{2} c}{\sin \frac{1}{2} C}, \ \sin \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B) \sin \frac{1}{2} c}{\cos \frac{1}{2} C},$$

$$\cos \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A+B) \cos \frac{1}{2} c}{\sin \frac{1}{2} C}, \ \cos \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A+B) \cos \frac{1}{2} c}{\cos \frac{1}{2} C}.$$

SOLUTION OF SPHERICAL TRIANGLES.

11. *The right-angled spherical triangle.*

The solutions of all the cases of right-angled spherical triangles are given in Art. 6. Before proceeding to the solution of numerical examples, it will be well to examine whether any of those solutions are *ambiguous.*

Now if the value of any quantity is to be determined by its cosine, tangent, or cotangent, there will be no ambiguity; for each quantity is supposed to be less than 180°, and therefore the *sign* will show whether it is greater or less than 90° (Plane Trig. Art. 15.) If, however, a quantity is to be found from its *sine*, there will in general be two values (Plane Trig. Art. 15,) which will satisfy the equation. It will be sufficient therefore to examine those solutions only in which a quantity is determined by its sine.

And first let A and a be given; then to find c, b, and B, we have by (2), (4), and (10) of Art. 6,

$$\sin c = \frac{\sin a}{\sin A}, \ \sin b = \frac{\tan a}{\tan A}, \ \sin B = \frac{\cos A}{\cos a}.$$

Hence in each of these the solution is *ambiguous*, as there is nothing to enable us to determine whether the smallest corresponding angles, or their supplements, are to be taken.

The same also follows when B and b are given.

Next, let A and c be given, then to find the other parts, we have by (2), (3), and (8), of Art. 6,

$$\sin a = \sin c \sin A, \ \tan b = \tan c \cos A, \ \cot B = \cos c \tan A.$$

Hence a only is given by its *sine*, and in this case there is only *one* solution; for by (4) of Art. 6,

$$\tan a = \sin b \tan A,$$

and *sin* b is always *positive*, since b is supposed to be less than 180° hence tan a and tan A must have the *same sign*, that is (Plane Trig. Art. 15), a must be greater or less than 90°, as A is greater or less than 90°. Consequently as A is known, there can be no ambiguity with respect to a. If a and c be given there will also be only one solution. Hence in right-angled spherical triangles, the only case which is really *ambiguous* is *when the data are a side and its opposite angle.*

EXAMPLES.

1. Given $b = 78° 20'$, A $= 37° 25'$, and B $= 90°$ of a spherical triangle A B C, to find c.

Take $90° -$ A for the middle part; then by the first of Napier's rules (Art. 6),

$$\sin (90° - A) = \tan c \tan (90° - b),$$

or

$$\tan c = \frac{\sin (90° - A)}{\tan (90° - b)} = \frac{\cos A}{\cot b} = \cos A \tan b :$$

$$\begin{aligned} \cos 37° 25' &= 9{\cdot}8999506 \\ \tan 78° 20' &= \underline{10{\cdot}6851149} \\ \tan c &= 10{\cdot}5850655 \\ \text{or } c &= 75° 25' 37'' \end{aligned}$$

This problem, by the preceding discussion, is not ambiguous, and c is acute (Geo. of the Sphere).

2. Given $a = 36° 31'$, A $= 37° 25'$, and B $= 90°$, to find c.

Let c be the middle part; then by the first of Napier's rules,

$$\sin c = \tan (90 - A) \tan a = \cot A \tan a :$$

$$\begin{aligned} \cot 37° 25' &= 10{\cdot}1163279 \\ \tan 36° 31' &= \underline{9{\cdot}8694731} \\ \sin c &= 9{\cdot}9858010 \\ \text{or } c &= 75° 25' 42''. \end{aligned}$$

As the side a and its opposite angle A are given, the problem is *ambiguous*. Whence $c = 75° 25' 42''$, or $104° 34' 18''$.

Scholium.—A *quadrantal* spherical triangle which has one of its sides equal to 90° may be solved in the same manner by means of the *polar* triangle.

EXERCISES.

1. Given $a = 48° 24' 16''$, $c = 70° 23' 42''$, and C $= 90°$, to find the other parts. *Ans.* $b = 59° 38' 27''$, A $= 52° 32' 55''$, B $= 66° 20' 40''$.

2. Given $c = 50° 30' 30''$, A $= 47° 54' 20''$, and B $= 90°$, to find b. *Ans.* $b = 61° 4' 56''$.

3. Given $a = 148° 27' 10''$, $c = 37° 10' 20''$, and C $= 90°$, to find the other parts. *Ans.* A $= 59° 59' 17''$, B $= 144° 3' 40''$, $b = 159° 13' 46''$.

4. Given $a = 40° 30' 20''$, A $= 47° 54' 20''$, and B $= 90°$, to find c. *Ans.* $c = 50° 30' 31''$, or $129° 29' 29''$.

5. Given $a = 98° 20' 20''$, B $= 57° 43' 12''$, and C $= 90°$, to find the other parts. *Ans.* $b = 57° 26' 40''$, $c = 94° 28' 33''$, A $= 97° 2' 35''$.

6. Given $a = 51° 30'$, A $= 58° 35'$, and B $= 90°$, to find c. *Ans.* $c = 50° 9' 51''$, or $129° 50' 9''$

7. Given $a = 104° 12' 40''$, $b = 97° 29' 15''$, and C $= 90°$, to find the other parts. *Ans.* $c = 88° 10'$, A $= 104° 5' 42''$, B $= 97° 15' 40''$.

12. OBLIQUE-ANGLED SPHERICAL TRIANGLES.

CASE I.

(1.) *Given two sides* (a, b) *of a spherical triangle, and one of the opposite angles* (A), *to find the other parts:*

(2.) *Or given two angles* (A, B) *and one of the opposite sides* (a), *to find the other parts.*

[This is usually divided into *two* cases, according as there are given two sides and an angle, or two angles and a side, but as both are solved by the same formulæ, it seems better to include both in one general case, as in the analogous one of Plane Trigonometry.]

When a, b, and A are given, then by Art. 8,

$$\sin B = \sin b \cdot \frac{\sin A}{\sin a} \quad . \quad . \quad . \quad . \quad (\alpha).$$

This will give B, and then by (6) of Art. 9,

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2} (A + B) \cdot \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)} \quad . \quad . \quad . \quad . \quad (\beta),$$

which will give the angle C. The remaining side c may then be found by Art. 8; or the side c may be found independently of C by Art. 10.

Since $\sin B = \sin (180 - B)$, the solution is sometimes *ambiguous*, as in the corresponding case of plane triangles (Geo. of the Sphere).

If two angles and a side be given, the mode of solution is exactly similar.

EXAMPLES.

1. Given $a = 115^\circ 20' 10''$, $b = 57^\circ 30' 6''$, and $A = 126^\circ 37' 30''$, to find the other parts.

To find the angle B by (α):—

$$\begin{array}{rll} b = & 57^\circ 30' \ 6'' \ \sin = & 9 \cdot 9260372^* \\ A = & 126^\circ 37' 30'' \ \sin = & 9 \cdot 9044761 \\ a = & 115^\circ 20' 10'' \operatorname{cosec} = & 10 \cdot 0439214 \\ \hline & \sin B = & 9 \cdot 8744347 \\ \text{or,} & B = & 48^\circ 29' 48'' \end{array}$$

As a is between b and $180^\circ - b$, the solution is not ambiguous, (Geo. of the Sphere); and since B and b are of the same kind, the value of B is acute, as given above.

To find C by (β):—

$$\begin{array}{rll} \frac{1}{2}(a + b) = & 86^\circ 25' \ 8'' \cos = & 8 \cdot 7956130 \\ \frac{1}{2}(a - b) = & 28^\circ 55' \ 2'' \sec = & 10 \cdot 0578335 \\ \frac{1}{2}(A + B) = & 87^\circ 33' 39'' \tan = & 11 \cdot 3706210 \\ \hline & \cot \frac{1}{2} C = & 10 \cdot 2240675 \end{array}$$

$$\text{Hence } \tfrac{1}{2} C = 30^\circ 50' \ 5''$$
$$\text{or } C = 61^\circ 40' 10''.$$

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* A somewhat different method of putting down the work to that employed in Plain Trigonometry is adopted in this and some of the subsequent examples. By *sin, cosec*, etc., are meant *log sin b, log cosec a*, etc.

To find c by Art. 8:—

$$\begin{aligned} b &= 57^\circ\,30'\ \ 6''\ \sin = 9\cdot9260372 \\ C &= 61^\circ\,40'\,10''\ \sin = 9\cdot9445934 \\ B &= 48^\circ\,29'\,48''\ \operatorname{cosec} = 10\cdot1255663 \\ \hline & \qquad\qquad \sin c = 9\cdot9961969 \\ & \qquad\qquad \text{or } c = 82^\circ\,25'\,42'' \end{aligned}$$

2. Given $a = 24^\circ\,4'\,12''$, $b = 30^\circ$, and $A = 36^\circ\,8'\,20''$, to find the other parts.

To find B from the relation:—

$$\sin B = \sin A\,\frac{\sin b}{\sin a} = \sin A \sin b \operatorname{cosec} a:$$

$$\begin{aligned} A &= 36^\circ\,8'\,20''\ \sin = 9\cdot7706640 \\ b &= 30^\circ\qquad\ \sin = 9\cdot6989700 \\ a &= 24^\circ\,4'\,12''\ \operatorname{cosec} = 10\cdot3894969 \\ \hline & \qquad \sin B = 9\cdot8591309 \\ & \qquad \text{or } B = 46^\circ\,18'\,6''. \end{aligned}$$

The value of a is not comprehended between b and $180^\circ - b$, and hence (Geo. of the Sphere), the solution is *ambiguous;* whence,

$B = 46^\circ\,18'\,6''$, or $B' = 180^\circ - 46^\circ\,18'\,6'' = 133^\circ\,41'\,54''$.

Consequently to find C and C′, we have (Art. 9),

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2}(A + B)\frac{\cos \frac{1}{2}(b + a)}{\cos \frac{1}{2}(b - a)},$$

$$\cot \tfrac{1}{2} C' = \tan \tfrac{1}{2}(A + B')\frac{\cos \frac{1}{2}(b + a)}{\cos \frac{1}{2}(b - a)}.$$

By means of these and the formula of Art. 8, each of the triangles A B C, A B′ C′ may be solved.

3. Given $A = 51^\circ\,30'\,6''$, $B = 59^\circ\,16'\,10''$, and $a = 63^\circ\,50'\,30''$, to find b.

By Art. 8,

$$\sin b = \sin a\,.\,\frac{\sin B}{\sin A} = \sin a \sin B \operatorname{cosec} A:$$

$$\begin{aligned} \sin 63^\circ\,50'\,30'' &= 9\cdot9530728 \\ \sin 59^\circ\,16'\,10'' &= 9\cdot9342862 \\ \operatorname{cosec} 51^\circ\,30'\ \ 6'' &= 10\cdot1064455 \\ \hline \sin b &= 9\cdot9938045 \\ \text{or } b &= 80^\circ\,20'\,42''. \end{aligned}$$

As the value of the angle A does not lie between B and $180^\circ - B$, the solution is *ambiguous* (Geo. of the Sphere), and therefore

$b = 80^\circ\,20'\,42''$, or $= 180^\circ - 80^\circ\,20'\,42'' = 99^\circ\,39'\,18''$.

If C and c were also required, the former could be obtained from the formula of Art. 9, and the latter from that of Art. 8. As b has two values, it will be obvious that C and c have also two values; that is, there are two triangles A B C, A B C′, that fulfil the conditions.

Exercises.

1. Given $a = 80^\circ\,5'\,4''$, $b = 109^\circ\,49'\,30''$, and $A = 33^\circ\,15'\,7''$, to find B. *Ans.* $B = 148^\circ\,25'\,22''\cdot3$.

2. Given A $= 47^\circ\ 19'\ 10''$, B $= 51^\circ\ 32'\ 15''$, and $a = 56^\circ\ 17'\ 12''$, to find b. *Ans.* $b = 62^\circ\ 22'\ 25''\cdot 6$, or $117^\circ\ 37'\ 34''\cdot 4$.

3. Given $b = 80^\circ\ 19'$, $a = 63^\circ\ 50'$, and A $= 51^\circ\ 30'$, to find B. *Ans.* B $= 59^\circ\ 15'\ 57''$, or $= 120^\circ\ 44'\ 3''$.

4. Given A $= 34^\circ\ 15'\ 2''$, B $= 42^\circ\ 15'\ 13''\cdot 25$, and $a = 40^\circ\ 0'\ 10''$, to find b. *Ans.* $b = 50^\circ\ 10'\ 30''$.

5. Given $a = 40^\circ\ 36'\ 37''$, C $= 50^\circ\ 17'\ 5''$, and A $= 35^\circ\ 57'\ 15''$, to find c. *Ans.* $c = 58^\circ\ 30'\ 56''$.

Case II.

(1.) *Given two sides* (a, b) *of a spherical triangle and the included angle* (C), *to find the other parts:*

(2.) *Or given two angles* (A, B) *and the included side* (c), *to find the other parts.*

When two sides and the included angle are given, A + B and A − B can be determined, and thence A and B, by (6) and (7) of Art. 9. The side c may then be found by Art. 8. The side c may also be obtained independently of the angles A and B, by means of (8) and (9) of Art. 9.

If two angles (A, B) and the included side (c) be given, by taking the polar triangle, there will then be two given sides and the included angle of a spherical triangle, which may be solved as the preceding. Or the formulæ of Art. 10 may be employed.

The sub-cases (1) and (2) constitute *one general case* for the reason stated in Case I.

Examples.

1. Given $b = 80^\circ\ 19'$, $c = 120^\circ\ 47'$, and A $= 51^\circ\ 30'$, to find B, C, and a.

By (6) of Art. 9,

$$\tan \tfrac{1}{2}(C + B) = \frac{\cos \tfrac{1}{2}(c - b)}{\cos \tfrac{1}{2}(c + b)} \cot \tfrac{1}{2} A,$$

$$\tan \tfrac{1}{2}(C - B) = \frac{\sin \tfrac{1}{2}(c - b)}{\sin \tfrac{1}{2}(c + b)} \cot \tfrac{1}{2} A.$$

Now because $\frac{1}{2}(c + b)$ is greater than 90°, $\cos \frac{1}{2}(c + b)$ is negative, and hence (Trig. Art. 15) $\frac{1}{2}$(C + B) is *obtuse*.

To find C + B and C − B by the preceding formulæ:—

$\frac{1}{2}(c - b) = 20^\circ\ 14'$	cos $= 9\cdot 9723380$	sin $=$	$9\cdot 5388804$
$\frac{1}{2}(c + b) = 100^\circ\ 33'$	sec $= 10\cdot 7373271$	cosec $=$	$10\cdot 0074043$
$\frac{1}{2}$ A $= 25^\circ\ 45'$	cot $= 10\cdot 3166443$	cot $=$	$10\cdot 3166443$
$\tan \frac{1}{2}$ (C + B) $=$	$11\cdot 0263094$	$\tan \frac{1}{2}$ (C − B) $=$	$9\cdot 8629290$

From these we get $\frac{1}{2}$ (C + B) $= 84^\circ\ 37'\ 23''$, and $\frac{1}{2}$ (C − B) $= 36^\circ\ 6'\ 17''$. But $\frac{1}{2}$ (C + B) is *obtuse*, and therefore the supplement of this value must be taken, that is, $\frac{1}{2}$ (C + B) $= 180^\circ - 84^\circ\ 37'\ 23'' = 95^\circ\ 22'\ 37''$. Whence,

$$C = 131^\circ\ 28'\ 54'' \text{ and } B = 59^\circ\ 16'\ 20''.$$

Lastly, to find a by the formula of Art. 10, viz,

$$\tan \tfrac{1}{2} a = \tan \tfrac{1}{2}(c - b) \cdot \frac{\sin \tfrac{1}{2}(C + B)}{\sin \tfrac{1}{2}(C - B)}:$$

$$\begin{array}{llr}
\tfrac{1}{2}(c - b) = 20^\circ 14' & \tan = & 9 \cdot 5665424 \\
\tfrac{1}{2}(C + B) = 95^\circ 22' 37'' & \sin = & 9 \cdot 9980848 \\
\tfrac{1}{2}(C - B) = 36^\circ \ 6' 20'' & \text{cosec} = & 10 \cdot 2296909 \\
\hline
 & \tan \tfrac{1}{2} a = & 9 \cdot 7943181
\end{array}$$

or $\tfrac{1}{2} a = 31^\circ 54' 46''$ and $a = 63^\circ 49' 32''$.

To verify this result, and to apply the formulæ (8) and (9) of Art. 9, the side a may now be found independently of the angles C and B.

To find ϕ by (8) :—

$$\begin{array}{rlr}
c = 120^\circ 47' & \sin = & 9 \cdot 9340482 \\
b = \ 80^\circ 19' & \sin = & 9 \cdot 9937679 \\
\tfrac{1}{2} A = \ 25^\circ 45' & \sin^2 = & 19 \cdot 2758702 \\
\tfrac{1}{2}(c-b) = 20^\circ 14' & \text{cosec}^2 = & 20 \cdot 9222392 \\
\hline
 & 2 \tan \phi = & 20 \cdot 1259255 \\
 & \tan \phi = & 10 \cdot 0629627 \\
 & \text{or } \phi = & 49^\circ 8' 19'' \cdot 7
\end{array}$$

To find a by (9) :—

$$\begin{array}{lr}
\tfrac{1}{2}(c-b) = 20^\circ 14' \ \sin = & 9 \cdot 5388804 \\
\phi = 49^\circ 8' 19'' \cdot 7 \ \sec = & 10 \cdot 1842703 \\
\hline
\sin \tfrac{1}{2} a = & 9 \cdot 7231507 \\
\text{or } \tfrac{1}{2} a = & 31^\circ 54' 46''
\end{array}$$

Hence $a = 63^\circ 49' 32''$, the same as before.

2. Given $A = 51^\circ 30'$, $C = 131^\circ 30'$, and $b = 80^\circ 19'$, to find the other parts.

Let a', b', c', be the sides, and A', B', C', the angles, of the polar triangle, then

$$a' = 180^\circ - A = 128^\circ 30', c' = 180^\circ - C = 48^\circ 30',$$
$$B' = 180^\circ - b = 99^\circ 41'.$$

Whence there are given of the polar triangle $A' B' C'$, the two sides a' and c', and the included angle B', from which by the preceding method, its remaining parts may be found, and thence the remaining parts of the primitive triangle.

To find the angles A' and C' by the formulæ,

$$\tan \tfrac{1}{2}(A' + C') = \frac{\cos \tfrac{1}{2}(a' - c')}{\cos \tfrac{1}{2}(a' + c')} \cot \tfrac{1}{2} B',$$

$$\tan \tfrac{1}{2}(A' - C') = \frac{\sin \tfrac{1}{2}(a' - c')}{\sin \tfrac{1}{2}(a' + c')} \cot \tfrac{1}{2} B'.$$

Now $\tfrac{1}{2}(a' - c') = 40^\circ, \tfrac{1}{2}(a' + c') = 88^\circ 30', \tfrac{1}{2} B' = 49^\circ 50' 30''$;

$$\begin{array}{rr}
\cos 40^\circ = & 9 \cdot 8842540 \\
\sec 88^\circ 30' = & 11 \cdot 5820810 \\
\cot 49^\circ 50' 30'' = & 9 \cdot 9262497 \\
\hline
\tan \tfrac{1}{2}(A' + C') = & 11 \cdot 3925847
\end{array}
\qquad
\begin{array}{rr}
\sin 40^\circ = & 9 \cdot 8080675 \\
\text{cosec } 88^\circ 30' = & 10 \cdot 0001488 \\
\cot 49^\circ 50' 30'' = & 9 \cdot 9262497 \\
\hline
\tan \tfrac{1}{2}(A' - C') = & 9 \cdot 7344660
\end{array}$$

Hence $\tfrac{1}{2}(A' + C') = 87^\circ 40' 51''$, $\tfrac{1}{2}(A' - C') = 28^\circ 29'$,
and $A' = 116^\circ 9' 51''$, $C' = 59^\circ 11' 51''$.

Whence $a = 180^\circ - A' = 63^\circ 50' 9''$, $c = 180^\circ - C' = 120^\circ 48' 9''$.

The third side b' is now found in the same way as a in the last example, and thence B by the relation $B = 180^\circ - b'$.

This example may also be solved by means of *Napier's Second Analogies*, given in Art 10.

Exercises.

1. Given $a = 84^\circ 14' 29''$, $b = 44^\circ 13' 45''$, and $C = 36^\circ 45' 28''$, to find the angles A and B. *Ans.* $A = 130^\circ 5' 22''$, $B = 32^\circ 26' 6''$.

2. Let $A = 39^\circ 23'$, $B = 33^\circ 45' 3''$, and $c = 68^\circ 46' 2''$, to find the other parts. *Ans.* $a = 43^\circ 37' 36''$, $b = 37^\circ 10'$, $C = 120^\circ 59' 46''$.

3. Given $A = 31^\circ 34' 26''$, $B = 30^\circ 28' 12''$, and $c = 70^\circ 2' 3''$, to find C. *Ans.* $C = 130^\circ 3' 50''$.

Case III.

(1.) *Given the three sides* (a, b, c) *of a spherical triangle, to find the angles:*

(2.) *Or given the three angles* (A, B, C) *to find the sides.*

When the three sides are given, the angles are found at once from (4), (5), or (6), of Art. 7.

When the three angles are given, the same formulæ may be applied to the polar triangle. This, however, never occurs in any applications of Spherical Trigonometry. Professor De Morgan remarks (in his Elements of Spherical Trigonometry), "Delambre, who probably calculated more spherical triangles than any man of his day, says, he never met with this case (the three angles given to find the sides) but once, and then he could have done without it."

Example.

1. Given $a = 63^\circ 50'$, $b = 80^\circ 19'$, $c = 120^\circ 47'$, to find the angles A, B, C.

The formula (6) of Art. 7, is employed in preference to the others, for the reason given in the analogous case of plane triangles (Art. 23).

To find A:—		To find B:—
$s = 132^\circ 28'$ cosec	$= 10\cdot1321377$	cosec $= 10\cdot1321377$
$s - a = 68^\circ 38'$ cosec	$= 10\cdot0309254$	sin $= 9\cdot9690746$
$s - b = 52^\circ 9'$ sin	$= 9\cdot8974181$	cosec $= 10\cdot1025819$
$s - c = 11^\circ 41'$ sin	$= 9\cdot3064303$	sin $= 9\cdot3064303$
$2 \tan \frac{1}{2} A$	$= 19\cdot3669115$	$2 \tan \frac{1}{2} B = 19\cdot5102245$
$\tan \frac{1}{2} A$	$= 9\cdot6834557$	$\tan \frac{1}{2} B = 9\cdot7551122$
or $\frac{1}{2} A$	$= 25^\circ 45' 19''$	or $\frac{1}{2} B = 29^\circ 38' 24''$
and A	$= 51^\circ 30' 38''$	and $B = 59^\circ 16' 48''$

The angle C is found in a similar way.

Exercises.

1. In a spherical triangle A B C, $a = 120^\circ 28' 10''$, $b = 83^\circ 10'$, and $c = 96^\circ 50'$, to find the angles.
Ans. $A = 120^\circ$, $B = 86^\circ 4' 13''$, $C = 93^\circ 55' 47''$.

2. Given $a = 68^\circ 46' 2''$, $b = 43^\circ 37' 38''$, and $c = 37^\circ 10'$, to find the angles.
Ans. $A = 120^\circ 59' 46''$, $B = 39^\circ 23'$, $C = 33^\circ 45' 2''$.

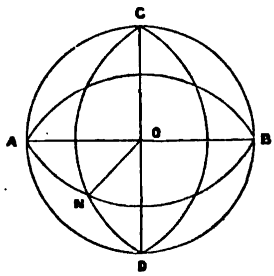
3. The three sides of a spherical triangle are $48^\circ 24' 16''$, $59^\circ 38' 27''$, and $70^\circ 23' 42''$, what are the angles?
Ans. $52^\circ 32' 54''$, $66^\circ 20' 40''$, and 90°.

4. Given $a = 40^\circ\ 0'\ 10''$, $b = 50^\circ\ 10'\ 30''$, $c = 76^\circ\ 35'\ 36''$, to find A, B, C.

Ans. $A = 31^\circ\ 15'\ 2''$, $B = 42^\circ\ 15'\ 13''\cdot 26$, $C = 121^\circ\ 36'\ 20''$.

THE AREA OF A SPHERICAL TRIANGLE.

13. *To find the area of a spherical triangle.*



Let the planes of the great circles C A D, C N D, intersect in the diameter C D, and let A N be the arc of another great circle whose poles are C and D; then O being the centre of the sphere whose radius O A or O D is r, we have by the Mensuration, Art. 13, and Euc. vi. 33,

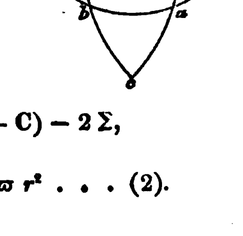
Lune A C N D : surface of sphere :: arc A N : $2\,\varpi\, r$
:: angle A O N : 360°.

Denote the angle at C, or the angle AON(Geo. of the Sphere) by A°, then because the surface of sphere $= 4\,\varpi\, r^2$ (Mensuration, Art. 13), we have

$$\text{Lune A C N D} = \frac{A^\circ}{360^\circ} \cdot 4\,\varpi\, r^2 = \frac{A^\circ}{180^\circ} \cdot 2\,\varpi\, r^2 \quad . \; . \; . \quad (1).$$

By means of this expression for the lune, the area of a spherical triangle may be found in terms of its angles and the radius of the sphere.

Let A C B be a spherical triangle on the surface of the sphere whose radius is r. Produce the sides A C, B C, till they meet again in c in the opposite hemisphere; then (Geo. of the Sphere), the triangles A B C, $a\,b\,c$, are equal in all respects. The whole hemisphere A B $a\,b$ is therefore equal to the sum of the lunes A B b, B A a, C c, minus twice the triangle A B C, or $a\,b\,c$. Hence, if Σ be the area of the triangle A B C, and A, B, C, be its angles, we have by the preceding expression for the lune (since $2\,\varpi\, r^2$ is the measure of the hemisphere),



$$2\,\varpi\, r^2 = \frac{2\,\varpi\, r^2}{180^\circ}(A + B + C) - 2\,\Sigma,$$

$$\text{or,} \quad \Sigma = \frac{A + B + C - 180^\circ}{180^\circ} \cdot \varpi\, r^2 \quad . \; . \; . \quad (2).$$

Hence, since the surface of the hemisphere is $2\varpi r^2$, *the area of a spherical triangle is to the surface of the hemisphere as the excess of its three angles above two right angles is to four right angles.*

The expression $A + B + C - 180°$ is called the *spherical excess*, being the excess of the angles above two right angles.

Cor. If Δ be a spherical triangle, and E the spherical excess, then

$$\Delta = E \cdot \frac{\varpi r^2}{180°}, \text{ or } E = \Delta \cdot \frac{180°}{\varpi r^2}.$$

Exercises.

1. The angles of a spherical triangle, measured on the surface of the earth, are $83° 10' 57''$, $66° 15' 16''$, and $30° 33' 48\frac{1}{2}''$: find the spherical excess, and the area of the triangle, the earth's diameter being 7957·5 miles, and its form being taken as spherical.

Ans. $E = 1\frac{1}{2}''$; area = 115·1223 miles.

2. The area of a spherical equilateral triangle is one-fourth of the surface of the sphere; what are its angles?

3. Prove that if E be the spherical excess,

$$\sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \sin C.$$

4. Prove that

$$\cos \tfrac{1}{2} E = \frac{\cos \frac{1}{2} a \cos \frac{1}{2} b + \sin \frac{1}{2} a \sin \frac{1}{2} b \cos C}{\cos \frac{1}{2} c}.$$

5. Deduce from Exercises 3 and 4 the area of a spherical triangle in terms of two sides and the included angle.

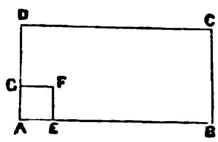
MENSURATION OF PLANES.

The term *mensuration* is applied to those methods or formulæ by which the lengths of curve lines, and the areas and volumes of superficial and solid figures, are determined.

PARALLELOGRAM AND TRIANGLE.

1. *To find the area of a square, a rectangle, or any parallelogram.*

First let A B, A D, be two adjacent sides of a rectangle, and A E, A G, each equal to the unit with which A B and A D are compared; then the square A E F G, described on A E or A G, will be the square unit with which the area A B C D is compared. And since the parallelograms A B C D, A E F G, are equiangular, they are to one another (Euc. vi. 23) in a ratio compounded of the ratios of their sides; that is, they are to one another as A B.A D to A E.A G. But A E F G = A E.A G = 1; hence A B C D = A B.A D. Consequently, if the adjacent sides of the rectangle be denoted by a and b,

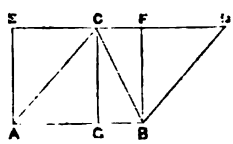


$$\text{area} = a\,b.*$$

Cor. The area of a square is found by squaring one of its sides.

For in this case, $a = b$, and $a\,b = a^2$.

Next, let A B D C be any parallelogram. Upon A B describe the rectangle A B F E, so that this rectangle and the parallelogram A B D C may be on the same base, and between the same parallels. Then, by Euc. i. 35, and the preceding expression for the area of a rectangle,



A B D C = A B F E = A B.A E = A B.C G.

But (Plane Trig., Art. 20), G C = A C sin C A B. Hence denoting the sides A C, A B, by b and c, and the included angle by A, we have for the area of the parallelogram A B D C,

$$\text{area} = c\,p = b\,c \sin A,$$

p being the altitude of the parallelogram.

2. *To find the area of a triangle.*

Since the triangle A B C (fig. to last Art.), is half the parallelogram A B D C (Euc. i. 41), we have,

$$A\,B\,C = \tfrac{1}{2}\,A\,B.C\,G = \tfrac{1}{2}\,A\,B.A\,C \sin A,$$

or, $$\Delta = \tfrac{1}{2}\,c\,p = \tfrac{1}{2}\,b\,c \sin A,$$

in which b, c, are two sides of the triangle, A their included angle, p the perpendicular on c from the opposite angle C, and Δ the area of the triangle.

* This proof, it will be noticed, applies only to cases in which the two sides of the rectangle can be exactly measured by a common linear unit. It is easily extended, however, to those cases in which the sides are incommensurable with the linear unit.

Cor. Since $\Delta = \frac{1}{2} b c \sin A$, and (Plane Trig., Art. 21) $b : c :: \sin B$, $\sin C$, or $b = c \frac{\sin B}{\sin C} = c \sin B \operatorname{cosec} C$; the expression for the area may be thus written,

$$\Delta = \tfrac{1}{2} c^2 \sin A \sin B \operatorname{cosec} C.$$

This is a convenient formula for the area of a triangle when two angles and their included side are given.

When the three sides of a triangle are given, the formula for the area is,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)};$$

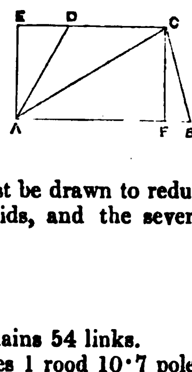
in which a, b, c are the sides opposite, respectively, to the angles A, B, C, and $s = \frac{1}{2}(a + b + c)$. The investigation of this expression is given in the *Application of Algebra to Geometry, Problem VII.*

TRAPEZIUM AND TRAPEZOID.

3. *To find the areas of a trapezium and a trapezoid.**

(1.) Divide the *trapezium* or quadrilateral into two triangles by a diagonal, find the areas of these triangles by Art. 2, and then add them together for the whole area.

(2.) Let A B C D be a *trapezoid*, of which A B, C D, are the parallel sides, and E A or C F the perpendicular distance between them. Put A B = a, D C = b, and A E = C F = d. Then by the preceding Article,



$$ABCD = ABC + ADC = \tfrac{1}{2} FC.AB + \tfrac{1}{2} AE.DC,$$

$$\text{or area} = \frac{d}{2}(a + b).$$

Scholium.—For all *irregular polygons* lines must be drawn to reduce the figures to quadrilaterals, triangles, or trapezoids, and the several areas found by the preceding formulæ.

Exercises.

1. Find the area of a square whose side is 11 chains 54 links.
Ans. 13 acres 1 rood 10·7 poles.

2. Find the area of a square whose diagonal is 5 feet.
Ans. 12·5 square feet.

3. The two adjacent sides of a rectangular board are 5 feet 7 inches and 7 feet 4 inches; what is its area in yards?
Ans. 4·549382 yards.

4. One side of a rhombus is 2 feet 4 inches, and its perpendicular height 3 yards; find its area in square feet. *Ans.* 21 square feet.

5. Two adjacent sides of a parallelogram are 4 chains 82 links and 5 chains, and their included angle 15° 10′; find its area.
Ans. 2 roods 20·88 poles.

6. How much paper, $\frac{3}{4}$ yard wide, will be required for a room that is 22 feet long, 14 feet wide, and 9 feet high, if there be 3 windows and 2 doors, each 6 feet by 3 feet? *Ans.* $82\frac{2}{3}$ yards.

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* A trapezoid is a quadrilateral figure, having two of its sides parallel.

7. The length of a certain railway is $47\frac{1}{2}$ miles, and the average breadth of land required for its formation 57 yards; what will be the amount of purchase of the land at 50*l.* per acre?

Ans. 49227*l.* 5*s.* $5\frac{1}{4}$*d.*

8. What is the area of a triangular field, the base and perpendicular of which are 3568 and 1589 links?

Ans. 28 acres 1 rood 15·6 perches.

9. What is the area of a triangle, two adjacent sides of which are 24 and 17·6 feet, and included angle 30°? *Ans.* 105·6 feet.

10. Find the area of a triangle whose three sides are 1350, 672, and and 1460 links. *Ans.* 4 acres 2 roods 3·5 poles.

11. The three sides of a triangular garden are 400, 348, and 312 yards; find its area. *Ans.* 10 acres 3 roods 8 perches $12\frac{1}{4}$ yards.

12. The area of an equilateral triangle is 4 acres 1 rood 10 poles; what is each side? *Ans.* 9·98 chains or 998 links.

13. Two sides of a triangle are $3 - \sqrt{2}$ and $3 + \sqrt{2}$, and the area is $\sqrt{10}$; find the third side. *Ans.* $2\sqrt{7}$ or 4.

14. How many square feet are contained in a plank whose length is 10 feet 10 inches, and breadths at the two ends $3\frac{1}{4}$ feet and $2\frac{1}{2}$ feet?

Ans. 31·1458 feet.

15. Of a field, the following measures were taken:—

The diagonal A C measured 11 chains 82 links, and perpendiculars from this to the corners B and D (the former to the right, and the latter to the left, of A C), 4 chains 18 links, and 5 chains 32 links, respectively. The straight line from A to D falling *within* the fence, perpendiculars 57, 61, 78, and 184 links, respectively, were measured to bendings in the fence at the distances 2 chains 4 links, 3 chains 40 links, 4 chains 5 links, and 4 chains 90 links, from A; and the line A D measured 6 chains 81 links. Find the area of the field.

Ans. 6 acres 0 roods 13·6 poles.

THE REGULAR POLYGON AND CIRCLE.

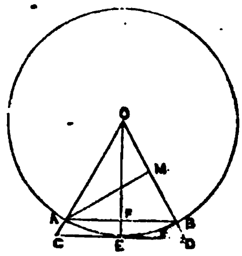
4

4. *To find the perimeter and area of a regular polygon of n sides:—*

(1.) *When it is inscribed in a given circle;*

(2.) *When it is circumscribed about the same.*

Let A B be a side of the inscribed polygon, C D a side of the circumscribed one, and $OA = r$ the radius of the given circle. Now there are as many equal angles at the centre O as sides (n) of the polygon, and as all these equal angles are together equal to 360°, the angle $AOB = \frac{360}{n}$, and, consequently, if the radius OE be drawn meeting AB in F (E being the point of contact), there evidently results



$$AB = 2AF = 2AO.\sin AOF = 2r.\sin\frac{180^\circ}{n};$$

hence, $$n.AB = 2nr\sin\frac{180^\circ}{n} \dots (1),$$

is the perimeter of the inscribed polygon of n sides.

Also, $\quad CD = 2CE = 2EO \tan COE = 2r \tan \dfrac{180^\circ}{n}$;

whence $$nCD = 2nr \tan \frac{180^\circ}{n} \ \ldots\ldots (2),$$

is the perimeter of the circumscribed one.

Again, for the *areas*. As there are n inscribed triangles each equal to the triangle AOB, therefore by Art. 2, and (5) of Art. 16, Trig.,

$$nAOB = nAF.FO = nr \sin \frac{180^\circ}{n}.r \cos \frac{180^\circ}{n} = \tfrac{1}{2} n r^2 \sin \frac{360^\circ}{n}. \ \ldots (3),$$

is the expression for the area of the inscribed polygon.

Similarly,

$$nCOD = nEC.EO = nr^2 \tan \frac{180^\circ}{n} \ \ldots\ldots (4),$$

is the area of the circumscribed one.

5. *To find the area of a regular polygon of n sides in terms of its side (a).*

Since (fig. of last Art.) $AF = \frac{1}{2}AB = \frac{1}{2}a$, and $FO = FA \cot AOF = \frac{1}{2} a \cot \dfrac{180^\circ}{n}$; hence, if the triangle AOB be denoted by Δ,

$$n\,\Delta = nAF.FO = \frac{n}{4} a^2 \cot \frac{180^\circ}{n},$$

is the required expression for the area of the polygon.

RATIO OF THE CIRCUMFERENCE OF A CIRCLE TO ITS DIAMETER AND LENGTH OF ARC.

6. *To find the circumference of a circle of given radius.*

Let AB (see fig. to Art. 4) be a side of the inscribed polygon of n sides, CD a side of the circumscribed one of the same number of sides, and $AO = r$ the radius of the given circle. Then the circumference of the circle is evidently greater than the perimeter (p) of the former polygon, but less than the perimeter (P) of the latter, whatever be the number (n) of the sides. Moreover, the number of sides of the polygons may be taken so large, that P and p may differ by less than any given quantity. For put $OF = h$, then by similar triangles,

$$P : p :: CD : AB :: OE\,(r) : OF\,(h);$$

or by division, $P - p : P :: r - h : r$; or, $P - p = \dfrac{P(r-h)}{r}$.

Hence, by increasing the number of sides of the polygons, h approximates to r, and consequently P to p. Now because the circumference of the circle is always comprehended between P and p, to the same number of decimal places as the perimeters of the polygons agree, for any value of n, to the same extent will the circumference of the circle be obtained.

By Art. 4,

$$p = 2nr \sin \frac{180^\circ}{n}, \text{ and } P = 2nr \tan \frac{180^\circ}{n}.$$

Let $n = 2^4 . 3^3 . 5^2$; then (Art. 32, Trig.), since in this case $\frac{180^\circ}{n} = 1'$,

$$p = 2r . 2^4 . 3^3 . 5^2 \sin \frac{180^\circ}{2^4 . 3^3 . 5^2} = 2r \times 3 \cdot 141592653616 \ldots$$

$$P = 2r . 2^4 . 3^3 . 5^2 \tan \frac{180^\circ}{2^4 . 3^3 . 5^2} = 2r \times 3 \cdot 141592653576 \ldots$$

Hence the circumference of the circle to radius r, or diameter d, is $2r \times 3.141592653 \ldots$; or, $d \times 3.141592653 \ldots$

Whence the diameter of a circle is to its circumference as 1 to 3·14159 . . .

This approximate value of the ratio which the circumference of a circle bears to its diameter is denoted by ϖ (Trig., Art. 5), and the value of it commonly used is 3·1416.*

Cor. If a be the length of an arc of a circle to radius r, and A° the degrees which the arc subtends at the centre, then by the preceding expression for the circumference ($2r\varpi$), and Euc. vi. 33, we have

$$360^\circ : A^\circ :: 2r\varpi : a = 2r\varpi . \frac{A^\circ}{360^\circ}.$$

Hence if any two of the quantities a, A°, r, be given, the third can be found by this formula.

AREAS OF CIRCLE, SECTOR, SEGMENT, AND ANNULUS.

7. *To find the area of a circle of given radius.*

The area of the circle is obviously contained between the areas of its inscribed and circumscribed polygons, whatever be the number of sides of the polygons. Hence it will be measured by the expression to which each polygon approximates, as the number of sides of the polygons is continually increased. Let the notation for the circle and circumscribed polygon be as in Art. 4; then we have for the area of the circumscribed polygon (Art. 4),

$$\text{area} = n r^2 \tan \frac{180^\circ}{n} = 2nr \tan \frac{180^\circ}{n} \times \tfrac{1}{2} r.$$

Now the expression to which the factor

$$2nr \tan \frac{180^\circ}{n}$$

approximates, as n is continually increased, is $2r\varpi$, the circumference of the circle, by Art. 6. Hence the area of the circle is,

$$\text{area} = 2r\varpi . \tfrac{1}{2} r = r^2 \varpi = \frac{d^2}{4} \varpi = \frac{c^2}{4\varpi};$$

in which r is the radius, d the diameter, c the circumference, and $\varpi = 3 \cdot 14159 \ldots$, as in Art. 6.

The form $2r\varpi . \frac{1}{2}r$, or *half the product of the radius and circumference* (from which all the others are derived), should be committed to memory.

* Other methods for the computation of ϖ by means of the series to which P and p approximate as we increase the number of sides of the polygons have been employed by different mathematicians. Dr. Rutherford has calculated its value to 208 places of decimals by means of the formula,

$$\frac{\varpi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99}.$$

Cor. 1. Since by Euc. vi. 33, the *sectors* of a circle are as the angles corresponding to them, we have

$$360^\circ : A^\circ :: r^2\varpi : \text{area of sector} = \frac{A^\circ}{360^\circ}r^2\varpi;$$

in which A° denotes the degrees which the arc of the sector subtends at the centre.

Cor. 2. In the expression just deduced for the area of the sector, write the value of A° in *Cor.* to Art. 6, then

area of sector $= \frac{1}{2}\alpha r$; another convenient form.

Cor. 3. The area of an *annulus*, or the surface enclosed by the circumferences of two concentric circles whose radii are r_1 and r_2, is obviously

$$\text{area} = (r_1^2 - r_2^2)\varpi = (r_1 + r_2)(r_1 - r_2)\varpi.$$

Cor. 4. The area of the *segment* A E B A (fig. to Art. 4) is the difference of the areas of the sector A B O and the triangle A B O. Denote the arc A E B by α and perpendicular A M by p; then because the sector A O B $= \frac{1}{2}\alpha r$, by *Cor.* 2, and the triangle A O B $= \frac{1}{2}rp$, the area of the segment is

$$\text{area} = \frac{r}{2}(\alpha - p),^*$$

or *the area of the segment of a circle has for its measure the half of the radius multiplied by the excess of the arc of the segment above the perpendicular from one extremity of the arc on the radius passing through the other extremity.*

If the segment be greater than a semicircle, the formula for the area is

$$\text{area} = \frac{r}{2}(\alpha + p).$$

Exercises.

1. Find the areas of a regular pentagon, hexagon, heptagon, and octagon, each of whose sides is 15 feet.

Ans. 387·11, 584·568, 817·635, and 1086·4 square feet.

2. The area of a regular hexagon is 400; find the length of its side, and the radii of its inscribed and circumscribed circles.

Ans. 12·40806, 10·7457, and 12·40806.

3. The diameter of a circle is 20 inches; find the area of a regular pentagon inscribed in it, and a regular hexagon circumscribed about the same. *Ans.* 1·6511397 and 2·40562 square feet.

4. If the diameter of a circle be 4 feet 2 inches, what is its circumference? *Ans.* 13·08996 feet.

5. If the circumference of a circle be 8 chains, what is its radius in feet? *Ans.* 84·03 feet.

6. What is the length of an arc of a circle which contains 26° 17′ 4″, the radius of the circle being 2 feet? *Ans.* 11 inches.

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* Let α_1 and p_1 be the length of the arc and sine to radius unity, corresponding to α and p to radius r; then (Plane Trig., Art. 14),

$$\alpha = r\alpha_1, \text{ and } p = rp_1.$$

Substituting these in the expression for the area of the segment, we have

$$\text{area} = \frac{r^2}{2}(\alpha_1 - p_1);$$

a convenient form when the arc is given in degrees, more especially with a table of circular arcs for degrees, minutes, etc., to radius unity.

7. How many square feet are contained in a circle whose circumference is 18·17 yards? *Ans.* 236 459 square feet.

8. How much will the paving of a circular piece of ground come to at 2*s*. 8*d*. per square foot, if it be 126 feet round? *Ans.* 168*l*. 9*s*. 1*d*.

9. The paving of a semicircular alcove with marble, at 2*s*. 6*d*. a-foot, came to 10*l*.; what was the length of the semicircular arc?
Ans. 22·42 feet.

10. What is the area contained between two concentric circles whose radii are 2 feet 2 inches and 4 feet? *Ans.* 35·5174 square feet.

11. What is the area of the sector of a circle of radius 8 feet, if the arc of the sector contain 159°? *Ans.* 88·802 square feet.

12. The length of an arc of a sector is 18 inches and radius of the circle 2 feet; find the area of the sector. *Ans.* $1\frac{1}{2}$ square feet.

13. Find the area of a sector whose arc of 40°·2 is 8 feet long.
Ans. 45·6 square feet.

14. If the radius of a circle be 3 feet, what is the area of that segment of it whose height is 13 inches? *Ans.* 3·476 square feet.

15. The chord of a segment is 2 feet and height 8 inches; find the area of the segment. *Ans.* 138·74 square inches.

MENSURATION OF SOLIDS.

THE PRISM AND PYRAMID.

8. *To find the content of a cube, rectangular parallelopiped, or any prism.*

LET a, b, c be respectively the number of linear units contained in the three adjacent edges of a rectangular parallelopiped; then the cube described upon the unit with which these are compared will be the *unit of volume or content.* Now (Geo. of Solids) this unit of volume and the rectangular parallelopiped are to one another in a ratio compounded of their bases and altitudes. Hence proceeding, as in the analogous case for the rectangle (Art. 1), the content of the parallelopiped is found to be equal to $a\,b\,c$, or *the area of the base by the perpendicular height.* And since (Geo. of Solids) prisms and parallelopipeds which have equal bases and altitudes are equal to one another, *the volume of any prism, parallelopiped, or cylinder, is expressed by the area of its base into its perpendicular height;* that is, in a prism or cylinder, if a^2 be the area of its base, aud h its perpendicular height,

$$\text{volume} = a^2 h.$$

Cor. The content of a cube is found by cubing the length of one of its edges; for in this case $a = b = c$, and hence $a\,b\,c = a^3$.

9. *To find the content of a pyramid or cone.*

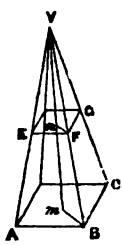
If a^2 be the area of the base of the pyramid or cone, and h its perpendicular height, then

$$\text{volume} = \tfrac{1}{3}\,a^2 h.$$

This expression is deduced at once from the preceding Article and the Geometry of Solids.

10. *To find the content of the frustum of a pyramid or cone.*

Let A B G be the frustum of a square pyramid whose vertex is V and altitude V m. Let V m meet the other end of the frustum in the point n, and join n F, m B. Then (Geo. of Planes) B m and F n are parallel, as are also B A, E F.



Put B A $= a$, E F $= b$, V $n = c$, and the height $m\,n$ of the frustum $= h$. Then by similar triangles

$$c + h : c :: \text{V B} : \text{V F} :: a : b, \text{ or } a\,c = b\,(c + h).$$

From this we get

$$c = \frac{b\,h}{a - b}, \text{ and by addition, } c + h = \frac{a\,h}{a - b}.$$

Again, because the volume (V) of the frustum is equal to the difference of the volumes of the pyramids A B V, E F V, we have, by Art. 9 and the preceding values of c and $c + h$,

$$\text{V} = \tfrac{1}{3}\,a^2 (c + h) - \tfrac{1}{3}\,b^2 c = \tfrac{1}{3}\,a^2 \frac{a\,h}{a - b} - \tfrac{1}{3}\,b^2 \frac{b\,h}{a - b}$$
$$= \tfrac{1}{3}\,h\,(a^2 + a\,b + b^2).$$

It will be seen by this formula that a^2 and b^2 are the areas of the two ends of the frustum, and $a\,b$ the mean proportional between them.

A similar demonstration is applicable whatever be the figures of the ends.

Cor. 1. If the base of the frustum of a pyramid be any regular

polygon of n sides, and a denote one side at the greater end and b one side at the less; then the altitude of the frustum being h,

$$V = \tfrac{1}{12} n\, h\, (a^2 + a\, b + b^2) \cot \frac{\pi}{n}.$$

This is deduced in a similar way by means of Art. 5.

Cor. 2. If the frustum be that of a cone, the diameters of the two ends being d and d' then

$$V = \tfrac{1}{3} h\, (d^2 + d\, d' + d'^2) \frac{\varpi}{4};$$

or if the circumferences c and c' be given,

$$V = \tfrac{1}{3} h\, (c^2 + c\, c' + c'^2) \frac{1}{4\varpi}.$$

11. The surfaces of solids bounded by plane figures (and the surfaces of cylinders and cones), are found by the appropriate methods given in the Mensuration of Planes. The following formulæ will be obvious:—

(A). In a parallelopiped, if a, b, c be the edges,

$$\text{surface} = 2\,(a\, b + a\, c + b\, c):$$

(B). In a prism or cylinder, if p be the perimeter of either end, and h the height,

$$\text{curve surface} = p\, h:$$

(C). In a pyramid or cone, if p be the perimeter of the base, and l the slant height,

$$\text{curve surface} = \tfrac{1}{2} p\, l:$$

(D). In a frustum of a pyramid or cone, if p and q be the perimeters of the two ends, and l the slant height,

$$\text{curve surface} = \tfrac{1}{2}\,(p + q)\, l.$$

THE SPHERE.

12. Before proceeding to the mensuration of the sphere, it is necessary to establish the two following lemmas:—

Lemma 1.—Let A Z, B C, be two lines in the same plane not parallel to each other; B b, C c, perpendicular to the former, and M O perpendicular to the latter, M being the middle of B C: then if the plane in which are the lines A Z, B C, be supposed to revolve about A Z as an axis, the line B C will generate a conical surface which has for measure

$$S = 2\, b\, c\,.\, MO\,.\, \varpi.$$

For produce C B to meet A Z in L, and join B O, C O; then by similar triangles,

$$Cc : Bb :: LC : LB; \text{ hence (Euc. v. 18)},$$

$$Cc + Bb : Bb :: LC + LB : LB :: 2\,LM : LB,$$

$$\text{or } (Cc + Bb)\, LB = 2\,LM\,.\,Bb \;.\;.\;.\;.\; (1).$$

Also, $LM : MO :: Lb : Bb$, or $LM\,.\,Bb = MO\,.\,Lb$, and $Lb : LB :: bc : BC$, or $Lb\,.\,BC = LB\,.\,bc$.

Whence by multiplication

$$LM\,.\,Bb\,.\,BC = MO\,.\,LB\,.\,bc \;.\;.\;.\;.\;.\; (2).$$

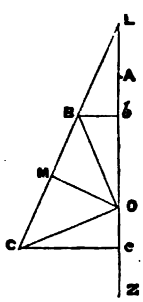
Comparing (1) and (2),

$$(Cc + Bb)\; BC = 2\,MO\,.\,bc \;.\;.\;.\;.\; (3).$$

Hence (Art. 11),

$$S = (Cc + Bb)\, BC\,.\,\varpi^{*} = 2\,bc\,.\,MO\,.\,\varpi,$$

is the measure of the surface generated by B C in a complete revolution round A Z.



* It will be obvious that the surface generated by B C is that of a conical frustum, the radii of whose ends are C c and B b.

Lemma 2.—The volume (V) generated by the triangle B O C, in revolving round A Z, has for measure the expression

$$V = \tfrac{2}{3}\, O M^2 \,.\, bc \,.\, \varpi.$$

Since the volume V is evidently equal to the difference between the cone generated by the triangle L C c and the cones generated by C O c, B b O, B b L, we have, by Art. 9,

$$V = \tfrac{1}{3} Lc.Cc^2.\varpi - \tfrac{1}{3} Oc.Cc^2.\varpi - (\tfrac{1}{3} Ob.Bb^2.\varpi + \tfrac{1}{3} Lb.Bb^2.\varpi)$$
$$= \tfrac{1}{3} LO\,(Cc^2 - Bb^2)\,\varpi = \tfrac{1}{3} LO\,(Cc + Bb)\,(Cc - Bb)\,\varpi \ldots\ldots (4).$$

But by similar triangles, and Euc. v. 17,

$Cc - Bb : Bb :: BC : LB$, or $Cc - Bb : BC :: Bb : LB :: MO : LO$;

hence $$(Cc - Bb)\, LO = BC \,.\, MO.$$

This gives by means of (3),

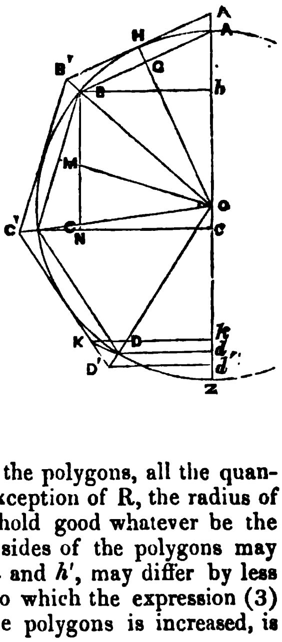
$$(Cc - Bb)\,(Cc + Bb)\, LO = 2\, MO^2 \,.\, bc \ldots\ldots (5).$$

Whence by (4) and (5),

$$V = \tfrac{2}{3}\, MO^2 \,.\, bc \,.\, \varpi.$$

13. *To find the surfaces of a spherical segment, zone, and sphere.*

The segment of the sphere may be conceived to be generated by the revolution of the arc A D about the diameter A Z. Inscribe in this arc and circumscribe about it two regular polygons, A B C D, A′ B′ C′ D′, similar to each other. Represent by A and A′ the areas of the surfaces of revolution generated by the inscribed and circumscribed polygons in turning round A Z; by r and R, the perpendiculars O G, O H, on the sides A B, A′ B′, from the centre O; and by h and h', the distances A d, A′ d' (D d, D′ d', etc., being perpendiculars on A Z). Then the whole surface generated by the inscribed polygon, by Lemma 1, is (the perpendiculars O G, O M, etc., being equal),



$$A = 2\,(Ab + bc + cd)\, GO.\varpi = 2h.r.\varpi \ldots (1).$$

Similarly, $$A' = 2\,h' \,.\, R \,.\, \varpi \ldots\ldots (2).$$

Consequently, $$\frac{A'}{A} = \frac{R}{r} \cdot \frac{h'}{h} \quad \ldots\, (3).$$

Now by increasing the number of sides of the polygons, all the quantities in these equations will vary with the exception of R, the radius of the circle; and these equations will still hold good whatever be the number of sides. Moreover, the number of sides of the polygons may be taken so large that r and R, as well as h and h', may differ by less than any given quantity. Hence the value to which the expression (3) approximates as the number of sides of the polygons is increased, is

$$\frac{A'}{A} = 1, \text{ or } A = A'.$$

If it can now be shown that the surface of the segment is always comprehended between A and A′, whatever be the number of sides of the polygons, it will obviously follow that the expression for the area of the spherical segment will be the same as that of the inscribed polygon (1),

when R is written for r; for the resulting expression for A in (1) is what the inscribed and circumscribed polygons approximate to, as we increase the number of sides.

Now the surface of the segment is evidently greater than A, since every convex surface is less than any other surface which wholly envelopes it. It is moreover less than A'; for if we draw to the point D the tangent D K, it is clear that the surface generated by A' B' C' K will be greater than the surface of the segment. But the surface generated by A' B' C' K is less than A'; for these surfaces have a common part, produced by the revolution of A' B' C' K; and the surfaces of the frustums of the cones generated by D K, D' K, are (Art. 11), respectively, $(\mathrm{K}k + \mathrm{D}d)\,\varpi\,.\,\mathrm{KD}$ and $(\mathrm{K}k + \mathrm{D}'d')\,\varpi\,.\,\mathrm{KD}'$, of which $\mathrm{D}d < \mathrm{D}'d'$, and $\mathrm{KD} < \mathrm{KD}'$. Hence the surface of the segment lies between A and A'. Consequently if S be the surface of the segment, h its height, and R the radius of the sphere,

$$\mathrm{S} = 2\,\varpi\,\mathrm{R}\,h,$$

or the area of the spherical segment equals the circumference of a great circle multiplied by the height of the segment (Art. 6).

Cor. 1. The surface of a zone b B C D d is equal to the circumference of a great circle multiplied by its height. For the zone is the difference of the two segments A B b, A D d, and hence it has for measure the circumference of a great circle multiplied by $\mathrm{A}d - \mathrm{A}b$, or by $b\,d$.

Cor. 2. The surface of a sphere is equal to the circumference of a great circle multiplied by its diameter; for it may be regarded as a segment whose height is equal to the diameter. Hence, in a sphere whose radius is r,

$$\text{surface} = 2\,\varpi\,r \times 2\,r = 4\,\varpi\,r^2.$$

14. *To find the volumes of a sphere and spherical sector.*

The volume of the sector may be conceived to be generated by the revolution of the sector A C D O (fig. to Art. 13) round the radius A O. Then if v denote the volume generated by the polygonal sector A B C D O, we have by lemma 2, Art. 12,

$$v = \tfrac{2}{3}\,r^2\,(\mathrm{A}b + b\,c + c\,d)\,\varpi = \tfrac{2}{3}\,r^2\,h\,\varpi\;.\;.\;.\;.\;(1),$$

in which r is the perpendicular on each of the equal sides A B, B C, etc., from the centre O, and h, as in last Article.

Similarly $\quad v' = \tfrac{2}{3}\,\mathrm{R}^2\,h'\,\varpi\;.\;.\;.\;(2),$

is the volume generated by A' B' C' D' O.

Hence $\quad \dfrac{v'}{v} = \dfrac{\mathrm{R}^2}{r^2}\cdot\dfrac{h'}{h}\;.\;.\;.\;.\;(3).$

Reasoning with these equations as in the preceding Article, it is easily shown that by increasing the number of sides of the polygons, v and v' may be made to differ by less than any assignable quantity, and that the volume of the sector is always comprehended between v and v'. Consequently if V be the volume of the sector of a sphere whose height is h and radius of sphere R,

$$\mathrm{V} = \tfrac{2}{3}\,\mathrm{R}^2\,h\,\varpi.$$

Cor. In a sphere whose radius is R,

$$\text{volume} = \tfrac{4}{3}\,\varpi\,\mathrm{R}^3.$$

In this case the height $h = 2\,\mathrm{R}$.

15. *To find the volumes of a spherical zone and spherical segment.*

The zone or slice may be supposed to be generated by the revolution of the segment on B C, and the trapezoid B c, round the diameter A Z (see fig. to Art. 13).

Put B $b = p$, C $c = q$, and $b\,c = m$; then because the volume generated by the segment on B C is equal to the difference of the volumes generated by the sector B O C and triangle B O C, and that the trapezoid B c generates a conical frustum, the radii of whose ends are p, q, and height m, we have by Arts. 10, 13, 14,

$$V = \tfrac{2}{3}\,OB^2 \,.\, m\,\varpi - \tfrac{2}{3}\,MO^2 \,.\, m\,\varpi + \tfrac{1}{3}\,m\,(p^2 + p\,q + q^2)\,\varpi$$
$$= \tfrac{1}{6}\,BC^2\,m\,\varpi + \tfrac{1}{3}\,m\,(p^2 + p\,q + q^2)\,\varpi.$$

But $BC^2 = BN^2 + NC^2 = m^2 + (p-q)^2$.

Substituting this value of A B^2 in the preceding, we get for the volume of the zone,

$$V = \tfrac{1}{6}\,\varpi\,m^3 + \frac{m\,\varpi}{2}\,(p^2 + q^2)$$

of which p, q, are the radii of the two ends, and m the height.

Cor. 1. Let $p = 0$; then

$$V = \tfrac{1}{6}\,\varpi\,m^3 + \tfrac{1}{2}\,m\,\varpi\,q^2 = \tfrac{1}{6}\,\varpi\,m\,(m^2 + 3\,q^2),$$

which is the expression for the volume of a spherical segment whose height is m, and the radius of whose base is q.

Cor. 2. Since $q^2 = Cc^2 = Ac\,.\,cZ = m\,(2\,R - m)$; hence, by substitution,

$$V = \tfrac{1}{3}\,\varpi\,m^2(3\,R - m).$$

This is a convenient expression for the volume of a spherical segment when the height (m) and radius (R) of the sphere are given.

Exercises.

1. A ship's hold is 100 feet long, 50 feet broad, and 4 feet deep; how many bales of goods, each 2 feet 4 inches long, 2 feet 1 inch broad, and 2 feet 1 inch deep, can be stowed into it, leaving a gangway of 3 feet broad and of the same length and depth as the hold? *Ans.* 1856½.

2. A stone 18 inches long, 17 broad, and 7 deep, weighs 278 lbs.; how many cubic feet of this kind of stone will freight a vessel of 230 tons burthen? *Ans.* 2297¼ feet, nearly.

3. The diagonal of a cube is 300 feet; what is its solidity? *Ans.* 5,196,152 cubic feet.

4. How many gallons will a cistern hold whose length, breadth, and depth are respectively 5 feet 2 inches, 3 feet 4 inches, and 1 foot 10 inches? *Ans.* 196·77 gallons.

5. The length of a sack is 3 feet 10 inches; what must be its breadth that it may contain 3 bushels of flour? *Ans.* 21·318 inches.

6. Find the content of a prism whose length and perimeter of its base are, respectively, 12 feet and 42 inches, the base being a regular hexagon. *Ans.* 10·6088 feet.

7. Find the whole superficies and solid content of a square pyramid each side of the base being 2⅔ yards, and the perpendicular height 2 feet. *Ans.* Surface = 15·96 yards, volume = 45⅗ feet.

8. Find the superficies and volume of a square pyramid, each side of the base being 4 feet and the slant height 24 feet. *Ans.* Surface = 192 feet, volume = 127·5 feet.

9. Find the content of a conic frustum, the diameters of the two ends being 4 and 9 inches, respectively, and its altitude 12 feet 7 inches.

Ans. 3·042 cubic feet.

10. How many cubic feet of water can be contained in a ditch of the form of an inverted frustum of a pyramid, if it measure 300 feet by 30 at the top, and 200 by 15 at the bottom, the depth being 5 feet?

Ans. 28660 cubic feet.

11. What quantity of canvas is necessary for a conical tent, the altitude of which is 7 feet, and radius of base 5 feet? *Ans.* 135·124 feet.

12. Find the surface and volume of a sphere whose radius is 6 feet 9 inches. *Ans.* surface = 572·55 feet; volume = 1288 feet.

13. If the diameter of the earth be 8000 miles, and the interior to the depth of 5 miles were known, how much of the earth's interior would still be unknown? *Ans.* $\frac{797}{800}$ part, nearly.

14. Admitting the height of the atmosphere to be 45 miles, what would be its solid content? *Ans.* 9,149,976,000 cubic miles.

15. If the pressure of the atmosphere at the earth's surface be 15 lbs. to a square inch, how much weight of atmosphere does the earth support?

Miscellaneous Exercises.

1. The diagonal of a rectangle is 5 yards, and one of its sides is 4 yards; what is its area? *Ans.* 12 square yards.

2. A room, having a surface of 150 square yards, is to be papered. The paper is in rolls of $1\frac{1}{4}$ yard wide and 12 yards long; how many of these rolls will be required, and what will be the expense at 3*s.* 9*d.* per roll? *Ans.* $8\frac{1}{2}$ rolls, expense = 1*l.* 11*s.* 3*d.*

3. A drawing is to be copied on a sheet of paper of half the size of that on which it is represented; in what ratio must we diminish the dimensions of the objects?

4. When we copy a drawing by doubling its dimensions, in what ratio is its surface augmented?

5. What must be the side of an equilateral triangle, so that its area may be equal to that of a square of which the diagonal is 180 feet?

Ans. 193·442 feet.

6. The three sides of a triangle are 6, $6 + \sqrt{2}$, and $6 - \sqrt{2}$; find its area. *Ans.* 13·7477271.

7. What is the side of that equilateral triangle whose area and perimeter are expressed by the same number? *Ans.* $4\sqrt{3}$.

8. Find the area of a right-angled triangle whose sides are in arithmetical progression, the sum of the sides being 48 inches.

Ans. 96 inches.

9. Having given the lengths (a, b, c) of three lines drawn from a point within a square to three of its angular points; to find the area of the square.

Ans. $\frac{1}{2}[a^2 + c^2 \pm \sqrt{\{4(a^2b^2 + a^2c^2 + b^2c^2 - b^4) - (a^2 + c^2)^2\}}]$.

10. In a right-angled triangle A B C, C D, C E are drawn from the right angle C, making angles C D A $= \alpha$, C E A $= \beta$, with the hypothenuse; prove that the area of the triangle C D E $= \dfrac{a^2 b^2}{2c^2}(\cot\alpha - \cot\beta)$.

11. Having given one side (c) of a triangle, the opposite angle $= 120°$, and the line (d) joining the given angle with the point of bisection of the opposite side; to find the area of the triangle.

Ans. $\dfrac{c^2 - 4d^2}{8}\sqrt{3}$.

12. A triangular field is bequeathed to three persons, whose respective ages are 20, 30, and 50 years, and it is to be portioned out to them in proportion to their ages by means of two fences drawn parallel to one of the sides; find the positions of the fences, the sides of the triangle being $a = 60$, $b = 70$, and $c = 80$ chains.

Ans. If the fences be parallel to the side c, they will meet C B or a in D and E, so that $CD = 12\sqrt{5}$, $CE = 30\sqrt{2}$ chains.

13. Compare the areas of an equilateral triangle, a square, and a regular hexagon, of equal perimeter.

14. Prove that the area of the pentagon (Euc. iv. 11) is

Area $= \frac{5}{4} r . CE$, r being the radius of the circle.

15. The extremity of the minute-hand of a clock moves 5 inches in 3 minutes; what is its length? *Ans.* 15·91 inches.

16. Prove that the area of the ring contained between two concentric circumferences is equal to the area of the circle which has for its diameter that chord of the larger circumference which is tangent to the smaller one.

17. From the arc A B of a circle whose centre is O an arc A C is cut off equal to the sine of A B; prove that the area of the sector C O B is equal to the area of the segment A C B.

18. If in a circle whose diameter is 180 feet, the chord of an arc 21 yards long be 15 inches shorter than the arc; find the area of the segment. *Ans.* 26 square yards.

19. If a cubic foot of iron weigh 4 cwt., what will be the weight of a water-pipe of that material the length of which is 10 feet 4 inches, the interior diameter 8 inches, and thickness of metal half an inch: also what will be the cost of two miles of such pipe, at 10*l.* per ton?

Ans. 3·8 cwt., and 1958·26*l.*

20. Given the surface (not including the base) of a square prism 580, and the square of its diagonal 344, to find the volume. Explain also the meaning of the double result.

Ans. Volume = 1200, or 1206·677.

21. Compare the contents of a triangular and hexagonal pyramid, one side of the base in each being 5 feet, and their altitude $3\frac{1}{2}$ feet.

Ans. Ratio 1 : 6.

22. A regular triangular pyramid is contained by four equilateral triangles, the side of each triangle being 24 inches; find the content of the solid in feet. *Ans.* ·9427 feet.

23. If from a right cone whose slant height is 21 feet and circumference of base 8 feet there be cut off by a plane parallel to the base a cone of 5 feet in slant height; find the surface of the remaining frustum.

Ans. 79·238 feet.

24. What is the radius of a sphere of gold which weighs 1000 oz., a cubic foot of gold weighing 19,000 oz. nearly? *Ans.* $r = \left(\frac{3}{76\varpi}\right)^{\frac{1}{3}}$.

25. The height of a conical frustum is 31, and the radius of one end 10; determine the radius of the other end, so that the frustum may be equal in volume to a right cylinder whose height is $\frac{1}{4}$ and radius of its base 62. *Ans.* Radius = 2.

26. Of two cylinders, one contains 154 lbs. of water, and its dimensions are half those of the other; how many cubic feet of water will the latter cylinder hold, a cubic foot of water weighing 1000 ozs. nearly?

Ans. 19·712 cubic feet.

COORDINATE GEOMETRY OF TWO DIMENSIONS.

EXPOSITION OF PRINCIPLES AND FUNDAMENTAL THEOREMS.

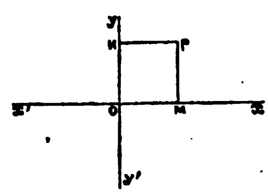
1. The ordinary process, in the application of algebra to geometry, of putting letters for lines, etc., and then solving the questions algebraically, though of occasional use, is not calculated to extend the boundaries of geometry or improve our modes of investigation, inasmuch as it is not founded upon any general principle. Each question requires its own peculiar method of solution, and therefore such solution can furnish but little aid towards the investigation of other questions. It is hence desirable that investigations of this class should be based on one general principle. It is, moreover, necessary that not only the absolute magnitudes of geometrical quantities, such as lines, angles, areas, etc., but also *the position of points* should be represented by symbols, in order that geometrical properties which involve the ideas of figure and position, as well as magnitude, may be investigated by the language and notation of algebra. These omissions in the old system are completely supplied by coordinate geometry, which may with propriety be denominated *general* geometry.

As an instrument of research, too, the coordinate theory is unequalled for power and facility, and therefore it is now extensively used in every branch of mathematical science. The limits of this treatise, however, will admit of little more than the development of the method; some of its applications will be found in subsequent parts of the course.

Coordinate geometry is said to be of two or three dimensions, according as the points and lines are in the same or different planes.

It will be necessary first to consider the position of a point in a plane, and then proceed to the consideration of a series of points which constitute a straight line or curve.

2. The position of a point in a plane is determined by finding its situation relatively to two fixed lines in the plane called the *coordinate axes*. Thus if $x\,O\,x'$ and $y\,O\,y'$ be two lines which intersect in O, and P any point in the same plane with these lines; then if P M be drawn parallel to O y and P N to O x, the position of P will be known, when O M and O N, or O M and M P are known. The line O M is named the *abscissa* of the point P; O N or its equal M P, the *ordinate;* O M and M P together, the *coordinates* of P; and the intersection O, the *origin of coordinates*. Moreover, the lines $x\,O\,x'$ and $y\,O\,y'$ are the *coordinate axes*, and they are *rectangular* or *oblique*, according as the angle at O is a right angle or an oblique one.



3. Any distance described from O along Ox is generally denoted by the letter x, and any distance along Oy by y. Hence Ox is sometimes named the axis of x, and Oy the axis of y.

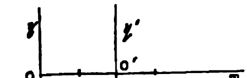
4. If the abscissas measured to the right of the axis of y be *positive*, those measured in the opposite direction are *negative*: also if the ordinates above the axis of x be *positive*, those below it are *negative*; and *vice versâ*. Throughout this treatise, Ox, Oy will be considered the positive, and Ox', Oy' the negative, directions.*

5. The point whose coordinates are x and y is simply denoted by (x, y). When a point is given, and consequently its coordinates are known, these are usually represented by a, b, etc.; or by x', y', etc.; and such point is designated (a, b), or (x', y'), etc. Hence, $(0, 0)$ denotes the origin of coordinates; $(0, b)$ a point in the axis of y; $(a, 0)$ a point in the axis of x; and $\left(\frac{1}{0}, \frac{1}{0}\right)$ a point at an infinite distance.

6. Hence if $x = a$, and $y = b$, the point (x, y) or (a, b) is determined by making $OM = a$, $ON = b$, and then drawing MP parallel to Oy and NP to Ox. The intersection P in this case is in xOy. If $x = -a$, $y = b$, be the coordinates of P, O M must be taken to the left of the origin; the point P in such case is in yOx'. If $x = -a$, $y = -b$, P is in $x'Oy'$; and if $x = a$, $y = -b$, P is in xOy'. If, moreover, $x = 0$, $y = 0$, the point P is at the origin.

7. Another method of determining the position of a point in a plane, that of *polar coordinates*, remains to be explained.

Let O be a given point, O A a fixed line, and P any point in a plane passing through O A. Join O and P; assume $OP = r$, and angle $POA = \theta$: then r, θ, are the polar coordinates of the point P, and P is known when r and θ are given. The point O is named the *pole*; O P the *radius vector*; O A the *prime radius*; and the angle P O A the *vectorial angle*.



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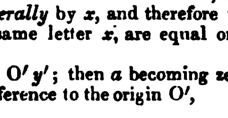
* This law of the algebraical signs of the coordinates, which is also the same as that of the sine and cosine in Trigonometry, is shown thus: Let Ox, Oy be rectangular axes which originate at O, and O′ any point in Ox. Draw $O'y'$ perpendicular to Ox, and cut off from $O'x$ and $O'O$ equal distances $O'A$ and $O'B$. Put $OO' = a$, $O'A = O'B = b$, and denote the distances O A, O B referred to the origin O, by x. Hence, then, in reference to the origin O and the points A and B,



$$x = a + b, \; x = a - b.$$

It must be kept in mind that though the abscissas O A, O B are each denoted by x, it does not follow that they are equal to one another. As has been stated in Art. 3, any distance taken along Ox when unknown is denoted *generally* by x, and therefore the abscissas of two points, though they are denoted by the same letter x, are equal only when the two points coincide.

Let the axis Oy be now supposed to coincide with $O'y'$; then a becoming zero, there remain for the abscissas of the points A and B in reference to the origin O′,

$$x = +b, \text{ and } x = -b.$$

In the one case x is measured from O′ to the right, and in the other it is measured in the contrary direction; and similarly for the ordinates.

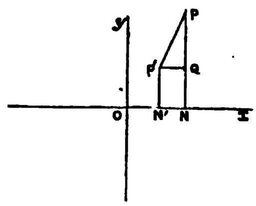
8. When one equation only is given between *two* variable quantities x and y, to one of them, as x, innumerable values can be given, and for each value of x there will be a corresponding value of y (real or imaginary), determined by the given equation (*Algebra*, Art. 133). Now the values given to x may differ as little as possible, and therefore it will be obvious that in general some straight line or curve will be given by the values of y, or by the *extremities* of the *ordinates*, in reference to a system of coordinates as in the preceding articles. *This straight line or curve is called the locus of the equation.* And conversely, *the equation which expresses the relation of the abscissa and ordinate for every point of a curve, is called the equation of the curve.*

9. Hence if h and k be quantities which being substituted for x and y in an equation between x and y, satisfy that equation; then (h, k) is a point in the locus of that equation. And conversely, if (h, k) be a point in a line or curve represented by an indeterminate equation between x and y; then if h and k be substituted for x and y, the equation will be satisfied. Thus, since the values $x = 3, y = 4$, satisfy the equation $5x - 6y + 9 = 0$, the point $(3, 4)$ is a point in the locus of the equation $5x - 6y + 9 = 0$; and so on.

10. Equations are said to be of different orders according to the highest degree of either of the coordinates (x or y), or the product of these. Thus $ax + bx + c = 0$, is an equation of the *first* order, $ay^2 + bxy + cx^2 + dy + ex + f = 0$, an equation of the *second* order, etc.

11. *To find the distance between two points, and the angle formed by the axis of x and the line which joins these points, the axes being rectangular.*

Let (x, y) and (x', y') be the two points P and P$'$, referred to rectangular axes Ox and Oy. Draw PN, P$'$N$'$ perpendicular, and P$'$Q parallel to the axis of x; and put PP$'$ = d, angle PP$'$Q $= \theta$. Then PQ = PN − QN = PN − P$'$N$'$ $= y - y'$, P$'$Q = NN$'$ = ON − ON$'$ $= x - x'$; hence (Euc. i. 47, and Art. 14, Plane Trigonometry)



$$d = \pm \sqrt{\{(x - x')^2 + (y - y')^2\}},$$

$$\text{and } \tan\theta = \frac{y - y'}{x - x'},$$

which express the distance and angle of inclination required.

Exercises.

1. Prove that the distance (d) between two points (h, k) and (h', k'), in reference to axes which make an angle $= \theta$ with each other is

$$d = \pm \sqrt{\{(h - h')^2 + (k - k')^2 + 2(h - h')(k - k')\cos\theta\}}.$$

2. Prove that the distance (d) between two points (r, θ) and (r', θ'), in reference to polar coordinates is

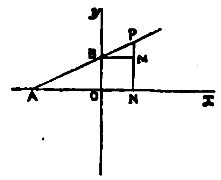
$$d = \sqrt{\{r^2 + r'^2 - 2rr'\cos(\theta - \theta')\}}.$$

3. Find the distance (d) between the points $(2, 3)$ and $(-5, 7)$, and the angle which this line makes with the line of abscissas.

THE STRAIGHT LINE REFERRED TO RECTANGULAR COORDINATES.

12. *To find the equation of a straight line, in reference to rectangular axes.*

Let A B be the line referred to the rectangular axes O x and O y; and P any point in it, the coordinates O N, N P, of which are x and y, respectively. Draw B M parallel to the axis of x; then whatever be the position of P, the ratio P M : B M = tan P B M (Art. 14, Plane Trigonometry) = tan B A O = a, suppose. Hence (putting O B = b), P M = P N − M N = P N − B O = $y - b$, and B M = O N = x, therefore



$$\frac{y-b}{x} = a, \text{ or } y = ax + b \ldots . (1),$$

which is a relation between the variable coordinates (x and y) of any point in the line, and therefore it is the equation required (Art. 8).

In general, therefore, the equation of a straight line is of the first degree (Art. 10), and contains two constants a and b; the former is the tangent of the angle which the line makes with the axis of x, that is, the angle formed by the part of the line *above* the axis of x, and the axis of x itself taken in the *positive* direction; the latter (b) is the part of the axis of y, intercepted between the line and the origin.*

The student must render himself familiar by frequent and varied exercise with the geometrical signification of all the algebraic circumstances proper to this fundamental equation, which will represent every straight line in the plane by attributing to the arbitrary constants a and b suitable values. The following are modifications of the general form (1), for particular positions of the line:—

If the line *pass through the origin*, then $b = 0$, and its equation is

$$y = ax \ldots\ldots (2).$$

If it be *parallel to the axis of* x, and meet the axis of y at a distance $= b$ from the origin; then $a = 0$, and the equation of the line is

$$y = b \ldots\ldots (3).$$

Similarly $$x = c \ldots\ldots (4),$$

denotes a *line parallel to the axis of* y, at a distance $= c$, from the origin.

Again, if the line *coincide with the axis of* x, then $a = 0$, and $b = 0$; hence the axis of x is denoted by the equation

$$y = 0 \ldots\ldots (5).$$

Similarly $$x = 0 \ldots\ldots (6),$$

represents the axis of y.

The following additional form of the equation of the line is frequently used:—

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* The number of constants (a, b), or parameters as they are sometimes called, corresponds, geometrically, to the number of points through which a line must pass, in order that its position may be completely determined. As a is *angular*, and b *linear*, a is frequently called the angular, and b the linear coefficient or parameter.

Put $OA = c$; then $\tan BAO = a = \frac{BO}{AO} = \frac{b}{c}$. The equation (1) consequently becomes

$$\frac{x}{-c} + \frac{y}{b} = 1 \ldots\ldots (7).$$

This is the *symmetrical* form; c and b, it will be seen, are the portions of the axes of x and y, between the line and the origin. The *negative* value of c merely implies that the intersection of the line with the axis of x is *to the left of the origin.*

Scholium.—In Art. 6 it is shown that the equations $x = a$ and $y = b$ give the point P. These are sometimes said to be *the equations of the point* P. This point may also be represented by the single equation

$$(x-a)^2 + (y-b)^2 = 0, \text{ or } m(x-a)^2 + n(y-b)^2 = 0,$$

as no other values of x and y, except $x = a$, $y = b$, will satisfy these equations. The symbols m and n denote any finite quantities.

13. *To prove that the locus of the general equation of the first degree between two variables x and y, is a straight line.*

The equation of the first degree between two variables x and y is, in its most general form (Art. 10),

$$Ax + By + C = 0 \ldots\ldots (1),$$

in which A, B, C, are independent of x and y.

Let (h, k), (h', k') be any two points in (1); then h, k and h', k', being put for x and y respectively, must (Art. 9) satisfy this equation.

$$\text{Hence } Ah + Bk + C = 0, \; Ah' + Bk' + C = 0.$$

Eliminating C between these, we get

$$A(h-h') + B(k-k') = 0, \text{ or } \frac{A}{B} = -\frac{k-k'}{h-h'}.$$

But if θ be the angle which the line joining the points (h, k) and (h', k'), makes with the axis of x, then (Art. 11),

$$\tan\theta = \frac{k-k'}{h-h'}, \text{ hence } \tan\theta = -\frac{A}{B}.$$

The same result will be obtained by taking any other two points in (1). It follows, therefore, that whatever two points be taken in (1), the line which joins these points makes the *same angle* with the axis of x, which could not be the case unless (1) were a straight line.

14. *To construct a line from its equation in reference to given coordinate axes.*

If in the equation of the line we put $x = 0$, and find the value of y in the resulting equation, and again in the original equation put $y = 0$, and find x; the values of y and x thus found will be, by (7) of Art. 12, the portions of the axes of y and x between the line and the origin. Hence, as a straight line is determined when any two points in it are known, we can, by means of this property, construct any line whose equation is of the form $y = ax + b$, or $Ax + By + C = 0$. The points in the axes found in this way must be taken along the positive or negative directions of the axes, according as the value of x in the one case and that of y in the other are positive or negative.

There is one form of the equation of the line to which this method

does not apply, that is, the form (2), Art. 12, or $y = ax$; for when $x = 0$, $y = 0$, and therefore this merely shows that the line passes through the origin. A second point, however, may be found, by giving to x some determinate value as 1, and then finding the value of y from the resulting equation. The construction of the line whose equation is of the form (3) or (4), Art. 12, will be obvious.

15. *The equations of two lines referred to the same coordinate axes being given, to find their point of intersection.*

The variable coordinates of the lines (x and y) are *identical* at their point of intersection,* and hence it will be sufficient and necessary to resolve the two equations for x and y, for the coördinates of the required point.

EXERCISES.

1. Construct the following equations to the same scale:—

(1.) $3x + 5y + 12 = 0$,	(4.) $2x + 6y - 9 = 0$,
(2.) $5x - 7y + 10 = 0$,	(5.) $5y + 8 = 0$,
(3.) $7x - 8y - 10 = 0$,	(6.) $x = 10$.

2 Find the angles which the lines (1) and (2) of the preceding make, respectively, with the axis of x.

3. Find the points of intersection, respectively, of the lines (1) and (2), (3) and (4), and (5) and (6), of *Ex.* 1.

16. *To find the equation of a line subject to the condition of passing through one given point, or two given points.*

$$\text{Let } y = ax + b \quad . \quad . \quad . \quad . \quad . \quad (1),$$

be the equation of the line to be determined, and (h, k) one of the given points; then, because (h, k) is a point in (1), we have, by Art. 9,

$$k = ah + b \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This is the condition that (h, k) is a point in (1). If this condition be combined with (1), the resulting equation will be that of a line passing through the point (h, k). Now this can be done by eliminating either a or b between (1) and (2), and as the elimination of b is effected by a simple subtraction, it is the one which is most easily performed. Hence

$$y - k = a(x - h) \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This is the equation of a line subject to the condition of passing through a given point (h, k).

Again, let (h', k') be another point in the line; then (3) must be satisfied when h' and k' are written for x and y,

$$\text{or } k' - k = a(h' - h), \text{ that is, } a = \frac{k' - k}{h' - h}.$$

This condition being combined with the equation (3), gives

$$y - k = \frac{k' - k}{h' - h}(x - h) \quad . \quad . \quad . \quad . \quad . \quad (4),$$

which is the equation of a line passing through the two points (h, k) and (h', k').

* It must be kept in mind that, in reference to the equations of two or more lines, a point (x, y) in one is not the same as a point (x, y) in another, *except at the point of intersection of the lines*, though the x and y seem to be the same in both.

Cor. If m be any arbitrary quantity (*Algebra*, Art. 97), the equation

$$y - ax - b = m(y - a'x - b') \;.\;.\;.\;.\;.\; (5),$$

denotes any line passing through the intersection of the lines

$$y = ax + b,\; y = a'x + b' \;.\;.\;.\;.\;.\; (6).$$

For (5) is a straight line by Art. 13, and as it is satisfied by the simultaneous equations

$y - ax - b = 0$, $y - a'x - b' = 0$, or $y = ax + b$, $y = a'x + b'$, which give (Art. 15) the point of intersection of the equations (6), it obviously represents an indefinite number of lines, all passing through the intersection of the equations (6), as m admits of all possible values.*

17. *To find the conditions that the two lines*

$$y = ax + b, \text{ and } y = a'x + b',$$

may be parallel or perpendicular to each other.

When the lines are parallel, they make equal angles with the axis of x, and hence, by Art. 12,

$$a = a' \;.\;.\;.\;.\;.\; (1).$$

Again, if the lines be perpendicular to each other, and θ, θ' be the angles they make with the axis of x, then $\theta' = \theta + \frac{1}{2}\pi$: hence, Art. 15, (Plane Trig.),

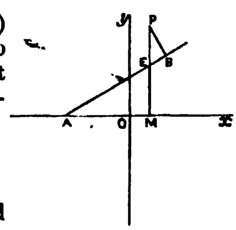
$$\tan\theta' = \tan(\theta + \tfrac{1}{2}\pi) = -\cot\theta = -\frac{1}{\tan\theta};$$

$$\text{or (Art. 12), } a' = -\frac{1}{a} \;.\;.\;.\;.\;.\; (2).$$

The required conditions are contained in (1) and (2).

18. *To find the perpendicular distance of a given point from a given line.*

Let $y = ax + b \;.\;.\;.\;.\;.\; (1)$,
be the equation of the given line A B, and (h, k) the given point P. Draw P B perpendicular to A B; then because P B passes through the point (h, k), and is perpendicular to A B, whose equation is (1); its equation by Arts. 16, 17, is



$$y - k = -\frac{1}{a}(x - h) \;.\;.\;.\;.\;.\; (2).$$

Also, if the point B be denoted by (x, y), and the distance P B by d, we have, by Art. 11,

$$d = \pm\sqrt{\{(x - h)^2 + (y - k)^2\}} \;.\;.\;.\;.\;.\; (3).$$

If now x and y be eliminated from these equations (the point B whose coordinates are x and y being common to all), the resulting equation will contain d and known quantities. Equating, then, the values of y in (1) and (2), and then subtracting ah from both sides of the resulting equation, we have

$$x - h = \frac{a(k - ah - b)}{1 + a^2} \;.\;.\;.\;.\;.\; (4).$$

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* It will be obvious that *every* locus whose equation is formed by the combination of the equations (6), *in any way whatever*, passes through the intersection of these lines; for it is tacitly assumed that a point (x, y) is common to the two lines, and the equation which results from their combination.

Hence, by (2), $y - k = -\dfrac{k - ah - b}{1 + a^2}$ . . . . . (5).

Substituting these values in (3), there results for the required distance

$$d = \pm \frac{k - ah - b}{\sqrt{(1 + a^2)}};$$

the upper or lower sign to be used according as the numerator is positive or negative. For as d represents the distance of two points, and is not susceptible of opposition of direction, being an absolute magnitude, it is necessary to reject that of the two signs which will give a negative value for d.

Exercises.

1. Find the equations of the lines which pass through the following points, taken two and two:—

$$(3, 4),\ (2, -5),\ (-6, 7),\ \text{and}\ (-1, -2).$$

2. Find the equation of the line which passes through the point $(5, -3)$, and makes an angle of $60°$ with the line whose equation is $7y + 6x - 8 = 0$.

3. Determine the equation of the line which meets the axis of x at a distance $= 4$ from the origin, and makes an angle of $45°$ with the line whose equation is $3y - 6x + 7 = 0$.

4. Find the equation of the line which meets the axis of y at a distance $= 3$ from the origin, and is perpendicular to the line which passes through the points $(3, 4)$ and $(-5, 2)$.

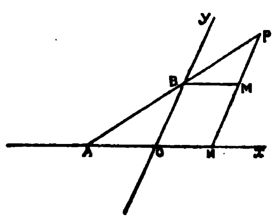
5. Find the angle included by the lines whose equations are $2y - 5x - 7 = 0$, and $3y + 6x - 10 = 0$.

6. Determine the distance of the point $(2, -3)$, from the line which passes through the points $(1, 2)$ and $(6, 7)$.

OBLIQUE COORDINATES.

Modifications necessary to adapt the preceding results to oblique axes.

19. *To find the equation of a straight line in reference to oblique axes.*



Let AB be the line referred to the oblique axes Ox, Oy, which make an angle $= \omega$ with each other; and P any point in the line, the coordinates ON, NP, of which are x and y. Then putting OB $= b$, and the angle BAO $= \theta$, we have, by Art. 21, (Plane Trig.,)

$$\frac{\mathrm{MP}}{\mathrm{MB}} = \frac{\sin \mathrm{PBM}}{\sin \mathrm{BPM}} = \frac{\sin \mathrm{BAO}}{\sin \mathrm{ABO}},$$

$$\text{or } \frac{y - b}{x} = \frac{\sin \theta}{\sin(\omega - \theta)}.$$

Hence $$y = \frac{\sin \theta}{\sin(\omega - \theta)}\, x + b \quad . \; . \; . \; . \; . \; (1),$$

is the equation required.

It appears, then, that if $y = ax + b$ be the equation of a straight

line referred to oblique axes, a, the angular coefficient, expresses the ratio of the sines of the angles which the line makes with the axes of x and y; and b, as in rectangular coordinates, is the part of the axis of y between the line and the origin. Moreover, if θ be the angle which the line makes with the axis of x, and ω the angle of ordination, or the angle contained by the axes, then

$$a = \frac{\sin\theta}{\sin(\omega - \theta)} = \frac{\sin\theta}{\sin\omega\cos\theta - \sin\theta\cos\omega} = \frac{\tan\theta}{\sin\omega - \cos\omega\tan\theta},$$

$$\text{or } \tan\theta = \frac{a\sin\omega}{1 + a\cos\omega} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

The forms (2), (3), (4), (5), (6), and (7), of Art. 12 are the same for oblique axes, except the form (2), in which a is the same as in this Article.

Scholium. This Article contains all the modifications necessary to adapt the results in Arts 12... 16, to oblique axes.

20. *To find the conditions that the two lines*

$$y = ax + b, \text{ and } y = a'x + b',$$

which are referred to oblique axes inclined at an angle $= \omega$, *may be parallel or perpendicular to each other.*

The condition of parallelism is obviously the same as for rectangular axes, or $a = a'$. The condition of perpendicularity must be modified thus:—

Let θ and θ' be the angles which the lines make with the axis of x; then by (2), Art. 19,

$$\tan\theta = \frac{a\sin\omega}{1 + a\cos\omega}, \tan\theta' = \frac{a'\sin\omega}{1 + a'\cos\omega}.$$

But because the lines are perpendicular to each other,

$$\theta = \theta' + \frac{1}{2}\pi, \text{ and hence (Plane Trig., Art. 15),}$$

$$\tan\theta = \tan\left(\theta' + \frac{1}{2}\pi\right) = -\cot\theta' = -\frac{1}{\tan\theta'};$$

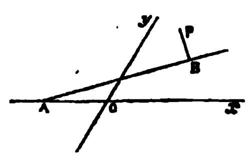
$$\text{whence } \frac{a\sin\omega}{1 + a\cos\omega} = -\frac{1 + a'\cos\omega}{a'\sin\omega}, \text{ or } a' = -\frac{1 + a\cos\omega}{a + \cos\omega},$$

which is the condition required.

21. *To find the perpendicular distance of a given point from a given line in reference to oblique axes.*

$$\text{Let } y = ax + b \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

be the equation of the given line A B, and (h, k) the given point P, referred to axes O x, O y, which make an angle $= \omega$ with each other. Draw P B perpendicular to the given line; then, because P B passes through the point (h, k), and is perpendicular to (1), its equation (Arts. 16, 20) is



$$y - k = -\frac{1 + a\cos\omega}{a + \cos\omega}(x - h) \ldots\ldots (2).$$

Moreover, if the point B be denoted by (x, y), and the distance P B by p, we have, by Art. 11,

$p = \pm \surd\{(x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega\} \ldots\ldots (3).$

If now we eliminate x and y from these equations (the point B whose coordinates are x and y being common to all), the resulting equation will contain p and known quantities.

Equating the values of y in (1) and (2), and taking ah from each side of the resulting equation, we get

$$x - h = \frac{(a + \cos\omega)(k - ah - b)}{a^2 + 2a\cos\omega + 1}.$$

Consequently by (2),

$$y - k = -\frac{(1 + a\cos\omega)(k - ah - b)}{a^2 + 2a\cos\omega + 1}.$$

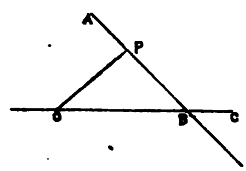
Hence by (3) the distance P B is

$$p = \mp \frac{(k - ah - b)\sin\omega}{\surd(a^2 + 2a\cos\omega + 1)};$$

the upper or lower sign to be used as in the analogous expression for rectangular coordinates (Art. 18).

22. *To find the polar equation of a straight line.*

Let O be the pole (Art. 7), O C the prime radius, A B the line whose equation is required, and O P $= r$, angle P O C $= \theta$, the polar coordinates of any point P in A B. Then if we put O B $= a$, and angle A B C $= \beta$, we have



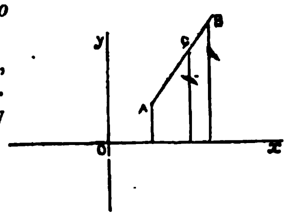
$$\frac{\mathrm{OP}}{\mathrm{OB}} = \frac{\sin\beta}{\sin(\beta - \theta)}, \text{ or } \frac{r}{a} = \frac{\sin\beta}{\sin(\beta - \theta)},$$

which is a relation between the polar coordinates of any point in the line, and hence it is the equation required.

INVESTIGATION OF PROPERTIES FOR ILLUSTRATION RELATIVE TO THE POINT AND STRAIGHT LINE.

Rectangular Axes.

23. *The coordinates of two points are given to find the coordinates of that point which divides in a given ratio the line joining the two points.*



Denote the given points A and B by (a, b), (c, d), and the required point C by (x, y). Then if $m : n$ be the given ratio, we have by Art. 11 and the conditions of the question,

$$m\surd\{(x-a)^2 + (y-b)^2\}$$
$$= \pm n\surd\{(x-c)^2 + (y-d)^2\},$$

or $m(x-a)\surd\left\{1 + \dfrac{(y-b)^2}{(x-a)^2}\right\} = \pm n(x-c)\surd\left\{1 + \dfrac{(y-d)^2}{(x-c)^2}\right\} \ldots (1).$

If, moreover, θ be the angle which A B makes with the axis of x, then, Art. 11,

$$\tan\theta = \frac{y-b}{x-a} = \frac{y-d}{x-c} \ldots\ldots\ldots\ldots\ldots (2).$$

Hence by (2) the equation (1) becomes

$$m(x-a) = \pm n(x-c) \ldots\ldots\ldots\ldots\ldots (3).$$

By (2) and (3) we readily get

$$x = \frac{ma \pm nc}{m \pm n},\ y = \frac{mb \pm nd}{m \pm n},$$

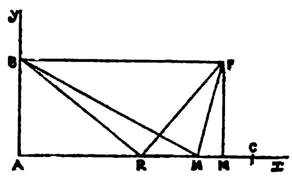
the required coordinates. The upper or lower sign must be used according as the point (x, y) or C is in the given line A B, or its extension. This property is much used in mechanics (*Centre of gravity*).

Cor. If A B be bisected in C, then $m = n$, and

$$x = \tfrac{1}{2}(a+c),\ y = \tfrac{1}{2}(b+d),$$

are the coordinates of the middle point of the line which joins the points (a, b) and (c, d).

24. *Two lines* B A *and* A C *are perpendicular to each other;* B *is a given point in* B A; R *a variable point in* A D; B R P *is a right angle; and* B R *is to* R P *in a constant ratio* 1 *to* m. *Find the locus of* P.



Since B R P is a right angle, the sum of the angles B R A and P R C is equal to the sum of the angles P R C and R P M. Hence the triangles B R A and P R M are similar, consequently

$$\mathrm{RM : AB :: RP : RB} :: m : 1, \text{ or } \mathrm{RM} = m \,.\, \mathrm{AB} \ldots (1).$$

Also $\mathrm{BP^2 = BR^2 + RP^2 = BA^2 + (AM - RM)^2 + PM^2 + RM^2} \ldots (2)$.

Whence putting B A $= a$, A M $= x$, M P $= y$ (C A and A B being the axes), we have by Art. 11, and (1) and (2) of the preceding,

$$(y-a)^2 + x^2 = a^2 + (x - ma)^2 + y^2 + m^2a^2, \text{ or } y = m(x - ma) \ldots (3),$$

for the equation of the locus, which (Art. 13) *is a straight line.*

Let $y = 0$ in (3); then $x = ma =$ A N, N being the point in which the line (3) meets the axis of x. Now the equation of the line which passes through the points B and N, or $(0, a)$ and $(ma, 0)$, is (Art. 16)

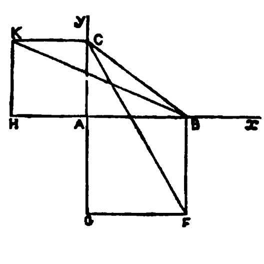
$$y = -\frac{1}{m}(x - ma) \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Hence (Art. 17) the line (4) is perpendicular to (3). The locus in question may hence be constructed thus:—

Make A N $= m\,.$ B A $= ma$, and draw N P perpendicular to N B, then is N P the locus required.

25. The following theorem may now be proved:—

Upon B A, A C, *the base and perpendicular of a right-angled triangle* A B C, *describe squares* A F, A K, *and join* F C, B K; *then will the lines* F C, B K, *intersect on the perpendicular from* A *on* B C.



Take B A and A C for axes, and put A B $= c$, A C $= b$, then the points B, K, F, C will be denoted thus

$$(c, 0),\ (-b, b),\ (c, -c),\ (0, b).$$

Hence (Art. 16) the equations of F C, B K, are, respectively,

$$y = -\frac{b+c}{c}x + b, \text{ or } \frac{y}{b} + \frac{b+c}{bc}x = 1 \ldots\ldots\ldots (1),$$

$$\text{and } y = -\frac{b}{b+c}x + \frac{bc}{b+c}, \text{ or } \frac{(b+c)}{bc}y + \frac{x}{c} = 1 \ldots (2).$$

Combine these by subtraction, and we have

$$by - cx = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which (Art. 12) is the equation of a straight line passing through the origin A; and as it is formed by the combination of (1) and (2), it also passes through the intersection of the lines F C and B K.

Again, the equation of B C is (Art. 16)

$$y = -\frac{b}{c}x + b \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Consequently the equation of a perpendicular to this line from the origin A is (Art. 17)

$$y = \frac{c}{b}x, \text{ or } by - cx = 0 \ldots\ldots\ldots\ldots (5).$$

Hence as (3) and (5) are *identical*, the theorem is established, viz., F C, B K, and the perpendicular from A on B C, intersect in the same point.

Scholium.—When the axes are oblique, the mode of investigation is exactly similar. In this case, however, the angular coefficients of the lines and the conditions of perpendicularity must be interpreted by Arts. 19, 20, 21.

Exercises on the Straight Line.

1. Prove that the sum of the perpendiculars on B C (Art. 25), from the points F and K, is equal to B C, and that if B G and C H be drawn, these lines will be parallel.

2. Prove that the perpendiculars from the angles of a triangle on the opposite sides pass through the same point.

3. In a given triangle A B C a variable straight line M N moves parallel to the side B C, so that the points M and N are always on the sides A B, A C. From B and C straight lines B N and C M are drawn to N and M; it is required to prove that the locus of the intersection P of these lines is a straight line passing through A and the middle of B C.

4. In the figure to Euclid, i. 43, let the diagonals A F, H C, and B K be drawn, then will these lines meet in the same point on B K produced. Prove this by coordinates.

5. Express the area of a triangle in terms of the coordinates of its three angles.

6. Three lines A B, A C, A D, which emanate from the same point A, are given in position. In A D any point P is taken, and lines P C F, P E B, are drawn to A B, A C (meeting A B in B, F, and A C in E, C); then if we draw the lines E F, C B, the locus of their intersection O will be a straight line passing through A, however the point P be taken in A D.

7. Prove that the lines joining the angular points and the middle points of the opposite sides of a triangle divide each other in the ratio of 1 : 2.

8. If L be the point of intersection of the lines drawn from the angular points of a triangle to the middle points of the opposite sides; M that of the perpendiculars upon the sides from the opposite angles; and N that of the perpendiculars from the middle points of the sides: then L, M, and N are in a straight line. Prove this theorem by coordinates.

CURVES OF THE SECOND ORDER.

26. In discussing the curves of the second order algebraically, two methods are employed. By the one it is shown that the general equation of the second order,

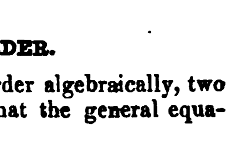
$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

comprehends all those curves, viz., the circle, ellipse, hyperbola, and parabola, when certain relations exist amongst the coefficients a, b, c, etc., and then, by a change of origin and direction of the coordinate axes, this general equation is simplified for the discussion of each particular case. By the other method (that which is adopted in this treatise) those curves are defined by some of their geometrical properties, and from these definitions their equations are deduced. It will be sufficient to give in this place, for subsequent use, the algebraic elements of those curves only, viz., their equations, and the equations of tangents and normals.

THE CIRCLE.

27. *To find the equation of the circle to rectangular coordinates.*

Definition.—The circle is the locus of a point P which is always at the same distance (r) from a given point C, which is called the centre.



Denote the given point C by (α, β), and the variable point P by (x, y), in reference to the rectangular axes O x, O y. Then, by Art. 11, or Euc. i. 47,

$$\left.\begin{array}{l}(y-\beta)^2 + (x-\alpha)^2 = r^2 \\ \text{or } y^2 + x^2 - 2\beta y - 2\alpha x + \beta^2 + \alpha^2 - r^2 = 0\end{array}\right\} \ldots (1).$$

As this is a relation between the coordinates x and y of any point P in the circumference, it is the equation in general of the circle to rectangular coordinates, the origin being any point in the plane of the circle, either within or without the circumference.

28. Hence, in order that the equation in general of the second order may represent a circle, in reference to rectangular coordinates, it must assume the form

$$y^2 + x^2 + dy + ex + f = 0 \quad . \quad . \quad . \quad . \quad (\alpha),$$

which does not contain xy, and in which the coefficients of x^2 and y^2 are equal. When these conditions are fulfilled, such equation cannot represent any other locus (with two exceptions, which will be noticed presently) than a circle. For by completing the squares the equation (α) becomes

$$(y + \tfrac{1}{2}d)^2 + (x + \tfrac{1}{2}e)^2 = \tfrac{1}{4}d^2 + \tfrac{1}{4}e^2 - f,$$

which (Art. 11) shows that the distance of a given point $(-\tfrac{1}{2}d, -\tfrac{1}{2}e)$,

from a variable one (x, y), is *constant*, and hence this equation denotes a circle, the centre and radius of which are, respectively,

$$(-\tfrac{1}{2}d, -\tfrac{1}{2}e), \text{ and } \sqrt{(\tfrac{1}{4}d^2 + \tfrac{1}{4}e^2 - f)}.$$

If, however, there be also the condition

$$\tfrac{1}{4}d^2 + \tfrac{1}{4}e^2 - f = 0,$$

the equation (α) becomes

$$(y + \tfrac{1}{2}d)^2 + (x + \tfrac{1}{2}e)^2 = 0,$$

which (Art. 12, *Scholium*) is a *point*.

If, moreover, the expression $\tfrac{1}{4}d^2 + \tfrac{1}{4}e^2 - f$ be *negative*, the equation in question is not susceptible of geometrical interpretation, for no values of x and y can make the left-hand member of the equation (α), in this case, negative. These are the exceptions to which reference has been made.

29. The form of the equation in Art. 27 suggests a simple method *of constructing a circle from its equation.* For let the equation of a circle be

$$y^2 + x^2 + 6y - 4x + 9 = 0,$$

which takes the form

$$(y + 3)^2 + (x - 2)^2 = 9 + 4 - 9 = 4;$$

then comparing this with (1) of Art. 27, we have

$$\alpha = 2, \beta = -3, \text{ and } r = 2,$$

which give the position of the centre and radius of the circle.

30. The following additional forms of the equation of the circle may be taken for exercises, as their investigation presents no difficulty:—

1. The equation of a circle to rectangular coordinates, the origin being at the centre, is

$$y^2 + x^2 = r^2.$$

2. If the origin be in the circumference of a circle, and a diameter be the axis of x, its equation to rectangular axes is

$$y^2 + x^2 - 2rx = 0.$$

3. The equation of a circle to rectangular coordinates, when the origin is in the circumference and any chord the axis of x, is

$$y^2 + x^2 - 2ky - 2hx = 0;$$

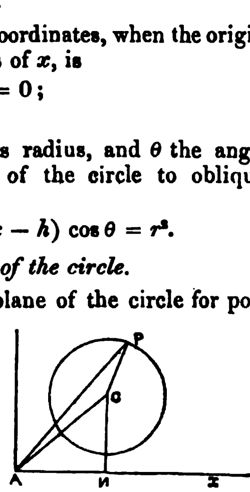
h and k being the coordinates of the centre.

4. Let (h, k) be the centre of a circle, r its radius, and θ the angle contained by the axes; then the equation of the circle to oblique coordinates is

$$(y - k)^2 + (x - h)^2 + 2(y - k)(x - h)\cos\theta = r^2.$$

31. *To find the polar equation of the circle.*

Taking any point A, and line Ax in the plane of the circle for pole and prime radius (Art. 7), let $\text{A P} = \rho$, $< \text{P A } x = \theta$, be the polar coordinates of any point P in the circumference, and A C $= \beta$, $< \text{C A } x = \alpha$, those of the centre; then if the radius P C be denoted by r, we have, by Euc. ii. 13,



$$\rho^2 + \beta^2 - 2\beta\cos(\theta - \alpha)\rho = r^2,$$

which being a relation between ρ, θ, the polar coordinates of any point P in the circumference is the required equation.

Cor. 1. If $\beta = r$, that is, if the pole be in the circumference, the equation becomes

$$\rho - 2\cos(\theta - \alpha)\, r = 0.$$

Cor. 2. If $\beta = 0$, that is, if the centre be the pole, the equation is

$$\rho = \pm r.$$

Cor. 3. If $\beta = r$, and $\alpha = 0$, the equation becomes

$$\rho - 2\beta \cos\theta = 0,$$

which is the polar equation of the circle when a diameter is the polar axis or prime radius, and one extremity of the diameter the pole.

Cor. 4. If p be the length of the perpendicular from the pole A on the tangent at the point P;

then $$p = \rho \cos \text{APC} = \rho \cdot \frac{\rho^2 + r^2 - \beta^2}{2\rho r} = \frac{\rho^2 + r^2 - \beta^2}{2r},$$

or $$2pr = \rho^2 + r^2 - \beta^2,$$

which gives a relation between ρ and p.

32. *To find the equation of a tangent to a given circle drawn from a given point in the circumference.*

Definition.—The equation of a tangent to a curve is the limit towards which the equation of a secant approximates, as one of its points of intersection, supposed to be variable, approaches indefinitely near to the other, supposed to be fixed.

Let (h, k) and (h', k') be two points in the circle,

$$y^2 + x^2 = r^2 \quad . \quad . \quad . \quad . \quad . (1),$$

referred to rectangular axes which originate at the centre, the point (h, k) being given. Then as the equation of the line which passes through the points (h, k) and (h', k') is (Art. 16),

$$y - k = \frac{k' - k}{h' - h}(x - h) \quad . \quad . \quad . \quad . \quad (2);$$

the equation, therefore, of the tangent to (1) at the point (h, k) is that to which (2) approximates, as k' approaches to k and h' to h.

Since (h, k) and (h', k') are points in (1),

$$k^2 + h^2 = r^2, \text{ and } k'^2 + h'^2 = r^2.$$

Taking the latter equation from the former, we get

$$k^2 - k'^2 + h^2 - h'^2 = 0, \text{ or } (k - k')(k + k') + (h - h')(h + h') = 0;$$

hence $$\frac{k' - k}{h' - h} = -\frac{h + h'}{k + k'}.$$

Whence (2) becomes

$$y - k = -\frac{h + h'}{k + k'}(x - h) \quad . \quad . \quad . \quad . \quad (3).$$

Now the limiting equation to which this approximates is

$$y - k = -\frac{h}{k}(x - h) \quad . \quad . \quad . \quad . \quad (4);$$

for as the point (h, k) approaches to (h', k'), or h to h' and k to k', the

expression $\frac{h+h'}{k+k'}$ approximates to $\frac{2h}{2k}$ or $\frac{h}{k}$. Hence by the definition, the equation (4) is that of a tangent to the circle (1), drawn from a point (h, k) in the circumference.* It will be obvious that (h, k) in (4) is any point in the circumference, and (x, y) any point in the tangent.

Cor. Since $k^2 + h^2 = r^2$, the equation (4) reduces to

$$k y + h x = r^2.$$

33. *To find the equation of the normal at any point (h, k) of a given circle $y^2 + x^2 = r^2$.*

Definition.—The straight line drawn through the point of contact perpendicular to the tangent is called the *normal.*

The equation of any line passing through the point (h, k) is (Art. 16), $y - k = A(x - h)$. In the case of the normal, as this line is perpendicular to the tangent at the same point whose equation (Art. 32) is $k y + h x = r^2$, hence (Art. 17) $A = \frac{k}{h}$.

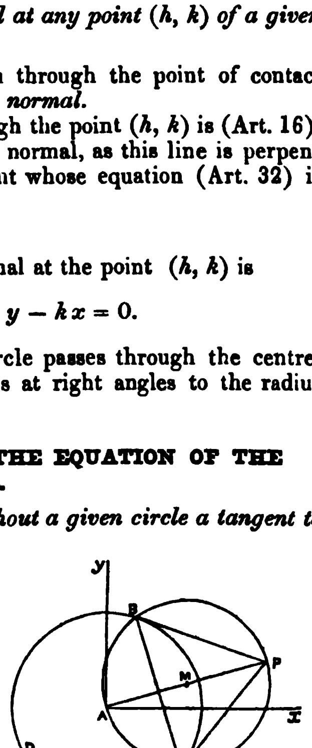
Consequently the equation of the normal at the point (h, k) is

$$y - k = \frac{k}{h}(x - h), \text{ or } h y - k x = 0.$$

This shows that the normal in the circle passes through the centre, and, therefore, the tangent at any point is at right angles to the radius drawn to that point.

SOLUTIONS ILLUSTRATIVE OF THE EQUATION OF THE CIRCLE.

34. *To draw from a given point without a given circle a tangent to the circle.*



Let (α, β) be the given point P, and r the radius of the given circle B C D; then the centre A being the origin of rectangular coordinates, the equations of the circle B C D and the tangent to it from P (x, y are the coordinates of the point of contact) are by Arts. 30 and 32,

$$y^2 + x^2 = r^2 \quad . \quad . \quad . \quad . \quad (1),$$

and $\beta y + \alpha x = r^2 \quad . \quad . \quad . \quad . \quad (2).$

Hence the values of x and y given by these equations will determine the points (there are evidently two points by the form of the equations) in which the tangent meets the circle, and thence the construction of the tangents will readily follow. The construction, however, is more elegantly effected in the following manner:—

As the equation

$$(y^2 + x^2 - r^2) - (\beta y + \alpha x - r^2) = 0,$$

or $$y^2 + x^2 - \beta y - \alpha x = 0 \quad . \quad . \quad . \quad . \quad (3),$$

* This method of finding the equation of a tangent to a circle is preferred, because it is applicable to any curve of the second order.

is satisfied by the simultaneous equations

$$y^2 + x^2 - r^2 = 0, \text{ and } \beta y + \alpha x - r^2 = 0,$$

or $$y^2 + x^2 = r^2, \quad \text{and } \beta y + \alpha x = r^2,$$

which give the points of intersection of (1) and (2); hence the locus of the equation (3) passes through the points common to (1) and (2). But (3) is the equation of a circle, the centre and radius of which are, respectively (Art. 27),

$$(\tfrac{1}{2}\alpha, \tfrac{1}{2}\beta) \text{ and } \tfrac{1}{2}\sqrt{(\alpha^2 + \beta^2)};$$

consequently, *the centre of this circle is in the middle* (M) *of the line* A P *which joins the given point and the centre of the given circle.* The geometrical construction is therefore obvious.

Cor. 1. The equation

$$(y^2 + x^2 - r^2) - (y^2 + x^2 - \beta y - \alpha x) = 0,$$

or $$\beta y + \alpha x = r^2. \quad . \quad . \quad . \quad (4),$$

is satisfied by the equations

$$y^2 + x^2 = r^2, \text{ and } y^2 + x^2 - \beta y - \alpha x = 0,$$

which give the points of intersection of (1) and (3), or of the circles B C D and P B A; hence (4) is the equation of the chord of contact B C.

Cor. 2. The equation of A P (Art. 12) is

$$y = \frac{\beta}{\alpha} x,$$

which (Art. 17) is perpendicular to (4), or the line B C; whence *if two circles intersect one another, the line joining the points of section is perpendicular to the line joining the centres.*

Scholium.—The points of intersection of a straight line and circle or other curve whose equations are given, are found by solving the equations for x and y as in the analogous case for two straight lines (Art. 15). If the resulting values of x or y be *imaginary*, the line and curve do not meet, and if they be *equal*, the line touches the curve, or is a tangent to it. Thus eliminating y between the line $y = a x + b$, and the circle $y^2 + x^2 = r^2$,

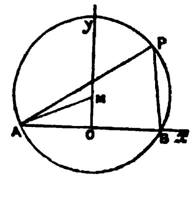
we have $$(1 + a^2) x^2 + 2 a b x + b^2 - r^2 = 0,$$

or $$x^2 + \frac{2ab}{1 + a^2} x + \frac{a^2 b^2}{(1 + a^2)^2} = \frac{(1 + a^2) r^2 - b^2}{(1 + a^2)^2}.$$

Hence if $(1 + a^2) r^2 - b^2$, be *negative*, the values of x are imaginary, and the line and circle do not meet; and if $(1 + a^2) r^2 - b^2 = 0$, the values of x are equal, and the line touches the circle.

35. *To find the locus of the points whence the line joining two given points is constantly seen under the same angle.*

Join the given points A and B, bisect A B in O, and take O B and a perpendicular O y for rectangular axes. Put A O = O B = a, and tangent of the given angle A P B = t. Then because A P and B P pass respectively through the points $(-a, 0)$ and $(a, 0)$, their equations (Art. 16) are



$$y = \text{A}(x + a), \text{and } y = \text{A}'(x - a);$$

of which A = tan P A B, A′ = tan P B x. And since t = tan A P B, we have, by Art. 16, Plane Trig.,

$$t = \frac{A' - A}{1 + A'A} = \frac{y(x+a) - y(x-a)}{x^2 - a^2 + y^2} = \frac{2ay}{x^2 - a^2 + y^2};$$

hence $t(x^2 - a^2 + y^2) - 2ay = 0$, or $\left(y - \frac{a}{t}\right)^2 + x^2 = \frac{a^2}{t^2}(1 + t^2)$,

is the equation of the required locus, which is a circle.

The coordinates (α, β) of the centre, and radius (r) of this circle, are

$$\alpha = 0,\ \beta = \frac{a}{t} = \frac{AO}{\tan APB},\ \text{and}\ r = \frac{a}{t}\sqrt{(1 + t^2)} = \frac{a \sec APB}{\tan APB}$$
$$= \frac{AO}{\sin APB}.$$

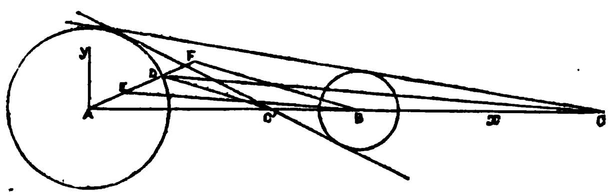
When $y = 0$, $x = \pm a$, hence the locus passes through the points A and B, and since $\alpha = 0$, the centre is in the axis of y. Also if the angle OAM be made equal to the complement of the angle APB, the point M, in which AM meets the axis of y, will be the centre. For

$$OM = AO \tan MAO = a \tan(90^\circ - P) = a \cot P = \frac{a}{\tan P} = \beta.$$

4

36. *To draw a common tangent to two given circles.*

Let the origin of rectangular coordinates be at the centre A of the greater circle, and the line which passes through the centres the axis of x.



Then if r' and r be the radii of the two circles, and d the distance of their centres, the equations of the circles (Art. 27) are

$$y^2 + x^2 = r'^2 \ \ldots\ldots (1),\quad y^2 + (d - x)^2 = r^2 \ \ldots\ldots (2).$$

Let $$y = ax + b \ \ldots\ldots (3),$$

be the common tangent, of which a and b are unknown.

Eliminating y between (1) and (3), there results for x the equation

$$(1 + a^2)x^2 + 2abx + b^2 - r'^2 = 0 \ \ldots\ldots (4).$$

Now (3) will be a tangent to (1), when the values of x in (4) are equal, and this will be the case (Art. 69, *Algebra*), when

$$(1 + a^2)(b^2 - r'^2) = a^2 b^2,\ \text{or}\ b^2 = r'^2(1 + a^2) \ \ldots\ldots (5).$$

Similarly, (3) will be a tangent to (2), when

$$(ad + b)^2 = r^2(1 + a^2) \ \ldots\ldots (6).$$

Dividing (6) by (5),

$$\frac{ad + b}{b} = \pm \frac{r}{r'} \ \ldots\ldots (7).$$

This, therefore, is the condition that (3) may be a common tangent to the circles (1) and (2).

Again, to find the intersection of the common tangent with the axis of x, let $y = 0$ in (3), then $x = -\frac{b}{a}$. Whence by (7),

$$x = -\frac{b}{a} = \frac{dr'}{r' \mp r}.$$

This gives the following analogies:—

$$r' - r : r' :: d : x,$$

and $$r' + r : r' :: d : x.$$

Whence this geometrical construction:—

Take any radius A D *of the greater circle and produce it, and make* D E = D F = r; *draw the lines* E B *and* F B, *and from* D *draw lines parallel respectively to these to meet* A B *in* C *and* C′: *these are the points in which the common tangents meet the line which joins the centres.* The problem is consequently reduced to that of "drawing from a given point a tangent to a given circle," which has been done in Art. 34.

Exercises on the Circle.

1. Find the radii and coordinates of the centres of the following circles, and thence construct them to the same scale:—

$2y^2 + 2x^2 - 4(x + y) - 1 = 0$, $(x + 2)x + (y - 4)y = 0$,
$y^2 + x^2 - 6x + 4y = 3$, $y^2 + x^2 = 4y$,
$y^2 - 8y + x^2 - 12x + 48 = 0$, $(y - 4)y + (x + 2)x - 4 = 0$.

2. Find the radii of the circles which pass respectively through the two following triads of points:—

$(-6, -1), (0, 0), (0, -1)$, and $(-2, 5), (4, -6), (-2, -6)$.

3. Determine whether the points $(0, 0), (0, 4), (1, 1)$, and $(1, -1)$, lie in the same circle.

4. A ladder of given length being placed vertically against a wall, is moved to a horizontal position, so that its extremities are always kept in a vertical plane perpendicular to the wall; it is required to prove that the locus of its middle point is a circle.

5. Prove that the locus of the vertex of a triangle, when the base and ratio of the other two sides are given, is a given circle. Find also the radius of this circle.

6. Find the locus of a point such that if straight lines be drawn from it to the four corners of a given square the sum of their squares shall be constant.

7. The equation of a circle is $y^2 + x^2 = a(x + y)$; what is the equation of that diameter which passes through the origin?

8. Find the locus of the different points of the vertex of a right-angled isosceles triangle, when its other two angles are constrained to move upon two fixed lines perpendicular to each other.

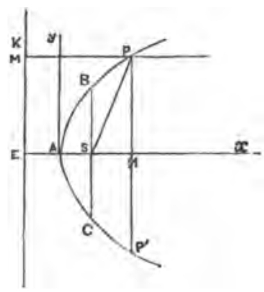
9. Two given circles are seen from points at which they subtend equal angles; prove that the locus of these points is a third circle, having its centre in the line joining the other two.

10. Find the locus of all the points from which equal tangents may be drawn to two given circles.

THE CONIC SECTIONS.

37. *To find the equation in general of a conic section to rectangular coordinates.*

Definition.—A conic section is the locus of a point P whose distance S P from a given point S is always proportional to its perpendicular distance P M from a line E K, given in position.



Let the given ratio PS : PM be $e : 1$, so that $PS = e \,.\, PM$. Draw S E perpendicular to E K, and divide S E in A in the given ratio $e : 1$; then by the definition A is a point in the curve. Take A S x and a perpendicular A y for axes, and denote A N, N P the coordinates of P by x and y; also put A S $= m$.

Then we have

$$PM^2 = EN^2 = \left(\frac{m}{e} + x\right)^2, \; SP^2 = SN^2 + NP^2 = (x - m)^2 + y^2;$$

hence since $PS^2 = e^2 \,.\, PM^2$, there results for the equation of the locus,

$$y^2 + (x - m)^2 = e^2\left(\frac{m}{e} + x\right)^2,$$

or $$y^2 + (1 - e^2)\,x^2 - 2m\,(1 + e)\,x = 0 \quad . \; . \; . \; . \quad (1).$$

Now this equation will represent three different curves, according as e is equal to, less than, or greater than unity. It will hence be necessary to consider each of these cases.

THE PARABOLA.

38. *To trace the locus when e is equal to unity.*

The equation (1) in this case becomes

$$y^2 = 4\,m\,x, \text{ or } y = \pm\, 2\sqrt{m\,x}.$$

It appears from this equation that for each positive value of x there are two equal values of y with contrary signs, and, consequently, that the curve is divided by the axis of x into two parts that are exactly similar, that is, the curve is symmetrical with respect to the axis of x. And, moreover, as x increases from zero to infinity, y also increases from zero to plus and minus infinity, but that no part of the curve is situated to the left of the origin, as a negative value of x makes y imaginary. Hence the form of the curve is evidently that represented in the figure (Art. 37).

The curve in this case is called the *parabola*.

The given point S is called the *focus;* A the *vertex;* the line E K the *directrix;* A x the *axis*, or *principal diameter;** and e the *eccentricity*.

Cor. 1. Draw the double ordinate B S C through the focus. Then at B we have by the preceding equation

$$y^2 \text{ or } BS^2 = 4m \,.\, x = 4m \,.\, AS = 4m^2, \text{ or } y = \pm\, 2m.$$

Hence $BS = CS = 2m$, and therefore $BC = 4m$.

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* The terms *diameter, vertex,* etc., of a curve, will be fully explained in a subsequent Article.

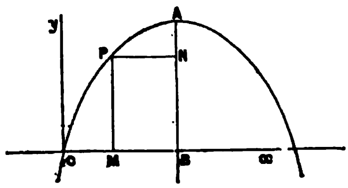
The double ordinate drawn through the focus is called the *latus rectum* or *principal parameter* of a conic section, hence the latus rectum of the parabola is $4m$.

Cor. 2. The construction of a parabola by points whose latus rectum ($4m$) is given is effected in the following manner:—

Make (fig. of Art. 37) $AS = AE = m$, and in EAS produced take any point N; with S as centre and radius $EN = MP = SP$, describe a circle, cutting a perpendicular PP' to AS through N in P and P'; then P and P' are obviously points in the curve; and similarly for other points.

39. *To find the equation of the parabola, the origin being any point in the curve.*

Let A be the vertex of a parabola and C any point in the curve. Take CB, which is perpendicular to the axis AB of the parabola, for the axis of x; and let $CM = x$, $MP = y$, be the rectangular coordinates of any point P; also $CB = \alpha$, $BA = \beta$, those of the vertex A.



Then, by Art. 38, we have the relation

$$PN^2 = 4m . AN,$$

or $$(\alpha - x)^2 = 4m . (\beta - y),$$

which, since $BC^2 = 4m . AB$, or $\alpha^2 = 4m\beta$, reduces to

$$y = \frac{\alpha}{2m} x - \frac{x^2}{4m};$$

the required equation.

This form of the equation of the parabola is frequently used.

Exercises on the Parabola.

1. Upon any line AB describe a semicircle ACB; from AB cut off a line AD of given length, and draw the ordinate DC; then if a parallel CP to AB be drawn to meet a perpendicular BP to AB in P, the point P will be in a parabola whose vertex is D and latus rectum AD. Prove this, and thence show how a parabola of given parameter may be constructed graphically by means of this property.

2. Let AB be the diameter of a circle ACB, and DC any ordinate. Join AC, and produce DC till DP is equal to AC; then is the locus of P a parabola whose parameter is equal to the diameter of the circle.

3. Find the equation of the parabola to rectangular coordinates; (1) when the origin is in the directrix; (2) when it is in the focus; the axis of x in each case coinciding with the principal axis.

THE ELLIPSE.

40. *To trace the curve represented by* (1), Art. 37, *when e is less than unity.*

The equation (1) takes the form

$$y^2 + (1 - e^2)\left(x^2 - \frac{2m}{1-e} x\right) = 0;$$

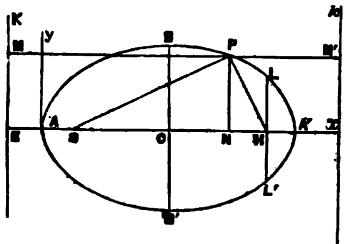
and because e is less than unity, $1 - e$ is positive, and therefore when

$x = \dfrac{2m}{1-e}$, the value of y is zero, that is, the curve meets the axis of x again in a point A′ such that $x = \mathrm{A\,A'} = \dfrac{2m}{1-e}$. Put, for brevity, A A, $= \dfrac{2m}{1-e} = 2a$; then we have for the equation of the locus

$$y^2 + (1 - e^2)(x^2 - 2ax) = 0.$$

Bisect A A′ in C, and in the last equation let $x = \mathrm{AC} = a$; then $\quad y^2 = (1 - e^2)\, a^2$,

or $\quad y = \pm a \sqrt{(1 - e^2)}$.



Hence if we draw B C B′ perpendicular to A A′, and make C B = C B′ $= a\sqrt{(1 - e^2)}$, B and B′ will be two points in the curve. Assume again, for brevity,

$$a\sqrt{(1 - e^2)} = b;$$

then $1 - e^2 = \dfrac{b^2}{a^2}$, and the equation (1) of Art. 37, becomes in terms of a and b,

$$y^2 = \frac{b^2}{a^2}(2ax - x^2) \quad . \quad . \quad . \quad . \quad (1),$$

in which $a = \mathrm{AC} = \dfrac{m}{1-e}$; $b = \mathrm{BC} = a\sqrt{(1 - e^2)}$; $\dfrac{b^2}{a^2} = 1 - e^2$.

It must also be kept in mind that

$$m = \mathrm{AS}, \text{ and } e = \frac{\mathrm{AS}}{\mathrm{AE}}.$$

The equation of the curve in this case may be simplified by removing the origin to the point C; this will be effected by writing $x + a$ for x in equation (1).*

Hence $\quad y^2 = \dfrac{b^2}{a^2}(a^2 - x^2)$, or $\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2} = 1 \quad . \quad . \quad . \quad . \quad (2).$

From this

$$y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)};$$

hence as x increases positively from zero to a, the two values of y are real, and diminish from b to zero, giving the portion of the curve B A′ B′; but when x exceeds a, the values of y are imaginary, and therefore no part of the curve is to the right of A′. Also since the portion of the curve to the left of B B′ (given by negative values of x) is evidently similar and equal to B A′ B′, the form of the curve is that represented in the figure.

The curve in this case is called the *ellipse*. It will be obvious from

* This simple transformation will be readily understood from the Note of Art. 4. Thus it will be seen that when $x + a$ is written for x, the origin is transformed from the point O to the point O′. An article on *Transformation* both in reference to the origin and the directions of the axes, was prepared for this work, as was also another on *Coordinate Geometry of Three Dimensions*, but want of room prevents their insertion.

its form that there are two foci, S and H, and two directrices, M K, M′ k, equally distant from the point C.

The point C is called the *centre*;* A A′, B B′, the *principal axes*, or *major* and *minor axes*; and the points A, A′, B, B′, the *vertices* of the curve.

Cor 1. If y' be any ordinate of the circle described on A A′ as a diameter (radius a), corresponding to any ordinate y of the ellipse; then by (2) of the preceding, and Art. 30,

$$y^2 : y'^2 :: \frac{b^2}{a^2}(a^2 - x^2) : a^2 - x^2 :: b^2 : a^2,$$

or $$y : y' :: b : a.$$

Whence it follows that the ellipse may be derived from the circle by diminishing proportionally, in the ratio $a : b$, all the ordinates relative to the same diameter. If the ellipse be compared with the inscribed circle, having for diameter the minor axis, it will be found in a similar way that the ordinate of the inscribed circle is to the corresponding ordinate of the ellipse as $b : a$. From this double comparison the construction of the ellipse by points, when the two axes are given, is easily effected by the aid of the two corresponding circles.

Cor. 2. Since $\text{A C} = a = \dfrac{m}{1-e}$, hence $a(1-e) = m = \text{A S} = \text{A}'\text{H}$, and therefore $\text{C H} = \text{C S} = a - a(1-e) = ae$. Whence to find the *latus rectum* L L′, let $x = ae$, in (2);

then $$y^2 = \frac{b^2}{a^2}(a^2 - a^2e^2) = b^2(1-e^2) = \frac{b^4}{a^2},$$

or $$y = \frac{b^2}{a} = a(1-e^2), \text{ since } b^2 = a^2(1-e^2).$$

Hence the latus rectum $\text{L L}' = 2a(1-e^2)$.

Cor. 3. When $a = b$, the equation (2) becomes

$$y^2 + x^2 = a^2,$$

which represents a circle; hence, *when its axes are equal, the ellipse becomes a circle.*

Cor. 4. By *Cor.* 2, $\text{A S} = a(1-e)$, and by the def., $\text{A S} = e \,.\, \text{A E}$, hence $\text{A E} = \dfrac{a(1-e)}{e}$, and $\text{E C} = a + \dfrac{a(1-e)}{e} = \dfrac{a}{e}$. Wherefore

$$\text{S P} = e \,.\, \text{P M} = e(\text{E C} + \text{C N}) = e\left(\frac{a}{e} + x\right) = a + ex.$$

Similarly, $$\text{H P} = a - ex.$$

Consequently $$\text{S P} + \text{H P} = 2a,$$

that is, *the sum of the focal distances of any point in the ellipse is constant, and equal to the major axis.*

This property furnishes a simple method of determining any number of points in an ellipse of which the foci and principal axis are given. For if in A A′ any point G be taken, and with centre S and radius A G

* The terms *centre* and *axes* will be explained in a subsequent Article.

a circle be described, also with centre H and radius A′ G another circle be described, then the points of intersection of these circles will evidently determine two points in the curve.

EXERCISES ON THE ELLIPSE.

1. Prove that the sum of two lines drawn from the foci to a point in the plane of the ellipse is greater or less than $2a$, according as the point is without or within the curve.

2. Prove that (fig. to Art. 40) A S . A′ S $= b^2$.

3. Show that the square of any ordinate of a given ellipse varies as the rectangle of the corresponding segments of the major axis.

4. Deduce the equation of the ellipse from the property established in *Cor.* 4, and show that it is identical with that deduced in Art. 40.

5. Find the equation of the ellipse when the origin is placed at one of the foci.

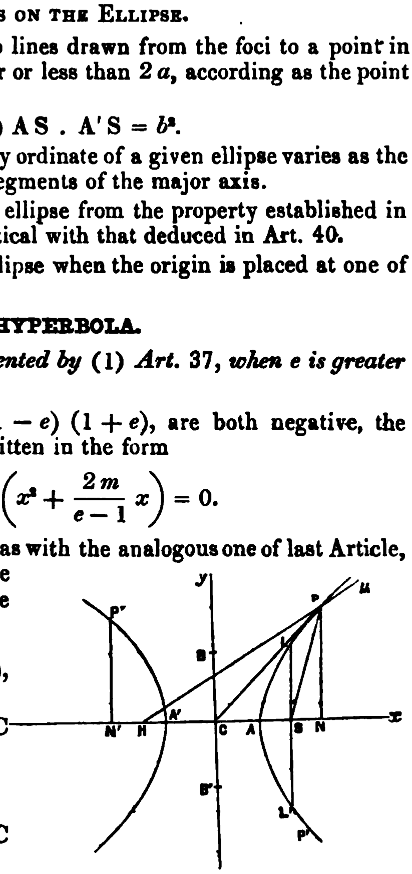
THE HYPERBOLA.

41. *To trace the curve represented by* (1) *Art.* 37, *when e is greater than unity.*

Since $1 - e$ and $1 - e^2 = (1 - e)(1 + e)$, are both negative, the equation of the curve may be written in the form

$$y^2 - (e^2 - 1)\left(x^2 + \frac{2m}{e-1}x\right) = 0.$$

Proceeding with this equation as with the analogous one of last Article, there results for the equation of the locus in this case, A being the origin,



$$y^2 = \frac{b^2}{a^2}(x^2 + 2ax) \quad . \quad . \quad . \quad . \quad (1),$$

in which $a = \text{AC} = \dfrac{m}{e-1}$; $b = \text{BC}$ $= a\sqrt{(e^2 - 1)}$; $\dfrac{b^2}{a^2} = e^2 - 1$.

Remove the origin to the point C by writing $x - a$ for x in (1);

then $$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (2),$$

is the equation of the curve when the middle point C of the line A A′ is the origin of coordinates.

It may be shown, as in the preceding Articles, that the curve in this case is symmetrical with respect to the axes of x and y, and that it has two foci, S and H, and two directrices, as in the case of the ellipse. It differs, however, from the ellipse in this respect, that the coefficients of x^2 and y^2 have opposite signs, which correspond geometrically to this,—that of the two right lines round which the curve is symmetrical, the one continues to meet the curve, but the other does not cut it; so that there exists only a single couple of vertices instead of two. And, moreover, the curve extends indefinitely to the right and left of the limits

$x = a$, and $x = -a$, that is, it is composed of two parts with infinite branches, as in the figure.

This curve is called the *hyperbola.**

The point C is the *centre;* A A′, B B′, the *axes*, or *transverse* and *conjugate* diameters; and A, A′ the *vertices.*

The curve does not meet the axis of y, as has been stated, for when $x = 0$ in (2), y is *imaginary*. Hence B B′ is sometimes called the *impossible* axis. Also because $\frac{b^2}{a^2} = e^2 - 1$, and as e may be of any magnitude greater than unity, it is obvious that b may be either greater or less than a.

Cor. When $b = a$, the equation (2) becomes

$$x^2 - y^2 = a^2.$$

The curve in this case is called the *rectangular hyperbola*, and it is to the ordinary hyperbola what the circle is to the ellipse.

Exercises on the Hyperbola.

1. Prove that the latus rectum of the hyperbola is $= 2a(e^2 - 1)$.

2. Prove that the difference of the focal distances of any point in the hyperbola is constant, and equal to $2a$.

3. Show that the square of any ordinate of the hyperbola varies as the rectangle of its distances from the extremities of the transverse axis.

42. *To find the polar equation of the parabola, ellipse, and hyperbola.*

Let S be the pole and Sx the prime radius (fig. to Art. 37). Put S P $= r$, angle P S $x = \theta$, and let the other lines be denoted as in Art. 37; then because S P $= e$. P M, and P M = E N = E A + A S + S N $= \frac{m}{e} + m + r \cos\theta$, we have

$$r = m(1+e) + e\,r\cos\theta, \text{ or } r = \frac{m(1+e)}{1 - e\cos\theta} \quad . \quad . \quad . \quad . \quad (1),$$

which is the polar equation in general of the three curves.

In the *parabola*, since $e = 1$, $\qquad r = \dfrac{2m}{1 - \cos\theta}$.

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* Developing the equation (2) by the binomial theorem,

we get $\quad y = \pm\frac{b}{a}x\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}} = \pm\frac{b}{a}x\left(1 - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} - \ldots\right).$

Now the larger x becomes, the more y tends to become equal to $\pm\frac{b}{a}x$, and when x = infinity, $y = \pm\frac{b}{a}x$.

Hence if two lines be drawn through the origin, making angles whose tangents are respectively $\frac{b}{a}$ and $-\frac{b}{a}$, with the axis of x, the curve will continually approximate to these lines, but will meet them only at an infinite distance. These lines are called the *asymptotes to the hyperbola.*

In the *ellipse*, $m = a(1 - e)$; hence $r = \dfrac{a(1-e^2)}{1 - e\cos\theta}$.

In the *hyperbola*, $m = a(e - 1)$; $\quad r = \dfrac{a(e^2-1)}{1 - e\cos\theta}$.

TANGENTS AND NORMALS.

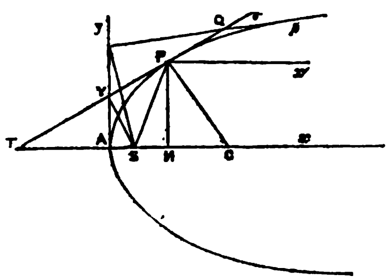
43. *To find the equation of the tangent to the parabola, ellipse, and hyperbola.*

[The definitions of tangent and normal are the same as for the circle, Arts. 32, 33.]

Let (h, k) be a point P in the parabola

$$y^2 = 4mx \quad . \quad . \quad . \quad . \quad (1),$$

referred to rectangular axes which originate at the vertex (Art. 38). Then if (h', k') be a point in the same parabola near to the point (h, k), the equation of the tangent to (1) at the point P or (h, k) will be (Art. 32) that to which the equation



$$y - k = \frac{k' - k}{h' - k}(x - h) \quad . \quad . \quad . \quad (2),$$

approximates, as k' approaches to k and h' to h.

Proceeding with these equations as with (1) and (2) of Art. 32, there results for the tangent to (1) at the point (h, k), the equation

$$k\,y = 2m(x + h) \quad . \quad . \quad . \quad . \quad (3);$$

in which (h, k) is the point of contact, as has been stated, and (x, y) any point in the tangent PT.

Similarly, $\quad a^2 k y + b^2 h x = a^2 b^2 \quad . \quad . \quad . \quad . \quad (4),$

is the equation of the taugent to the ellipse $a^2 y^2 + b^2 x^2 = a^2 b^2$ (Art. 40);

and $\quad a^2 k y - b^2 h x = - a^2 b^2 \quad . \quad . \quad . \quad . \quad (5),$

is the analogous equation to the hyperbola $a^2 y^2 - b^2 x^2 = - a^2 b^2$ (Art. 41).

Definition.—The straight line intercepted between the ordinate PN and the point T, in which the tangent meets the axis of x, is called the *subtangent.*

Scholium.—As the normal PG by the definition is perpendicular to the tangent, and passes through the point of contact, its equation follows at once from that of the tangent and Arts. 16, 17.

The equations, therefore, of the normals of the preceding curves at the point (h, k) are, respectively,

$$y - k = -\frac{k}{2m}(x - h) \quad . \quad . \quad . \quad . \quad (6),$$

$$y - k = \frac{a^2 k}{b^2 h}(x - h) \quad . \quad . \quad . \quad . \quad (7)$$

$$y - k = -\frac{a^2 k}{b^2 h}(x - h) \quad . \quad . \quad . \quad . \quad (8).$$

Definition.—The line G N intercepted between the ordinate and normal is called the *subnormal.*

Cor. 1. In (3), let $y = 0$, then $x = -h = -$ A N; hence *the subtangent in the parabola is bisected in the vertex.* The negative sign merely implies that the intersection T is to the left of the origin A. Whence this simple method of drawing a tangent to a parabola from a given point P:—

Take in N A *produced,* A T = A N, *the abscissa of the given point, and join* T P, *then is* T P *the required tangent.*

Cor. 2. The equation of the perpendicular S Y from the focus S, or $(m, 0)$, on the tangent P T whose equation is (3), is, by Arts. 16, 17,

$$y = -\frac{k}{2m}(x - m) \quad . \quad . \quad . \quad . \quad (9).$$

Eliminating y between (3) and (9), we get, since $k^2 = 4mh$ by the equation (1) of the curve, $x = 0$. Hence *a perpendicular from the focus of a parabola on any tangent intersects that tangent in the tangent from the vertex.*

Cor. 3. In (6), let $y = 0$; then $x - h = 2m$, or A G − A N = $2m$. Hence, *in the parabola, the subnormal* N G *is equal to half the latus rectum.*

Cor. 4. Since S T = A S + A T = A S + A N = $m + h$, and S G = S N + N G = $h - m + 2m = h + m$; hence S T = S G.

But by Art. 38 (fig. to Art. 37), S P = P M = E N = $m + h$.

Whence if P *be a point in the parabola,* S *the focus, and* G, T *the points of intersection of the normal and tangent at the point* P *with the axis of* x, *then* S P = S G = S T.

Cor. 5. Let a line P x' be drawn parallel to the axis; then because of the equals S P, S T by *Cor.* 4, and the parallels P x', A x, the angle t P x' = S T P = S P T; and consequently since G P is perpendicular to the tangent P T, G P x' = G P S. Hence *the tangent and normal at any point of a parabola make equal angles with the focal distance of that point, and with a line drawn through it parallel to the axis.*

In a similar way are all the properteis of the conic sections established.

Exercises on Tangents and Normals.

1. Prove that the tangents at the points B and B′ in the ellipse (Art. 40) are perpendicular to the minor axis.

2. Let the ordinate N P (Art. 40) meet the circle described on A A′ in Q, and let T be the intersection of the tangent to the ellipse at the point P, with A A′ produced; then will the tangent to the circle at Q also pass through the point T.

3. Prove that the normal at any point of an ellipse bisects the angle contained by the focal distances of that point, and that the focal distances make equal angles with the tangent.

4. Show that in the hyperbola the normal at any point bisects the exterior angle between the focal distances of that point.

CENTRES OF CURVES.

44. *To find the centre of any curve of the second order.*

Definition.—The centre of a curve is a point such that every secant passing through it meets the curve in two points which are equidistant from the point in question.

It follows immediately, from this definition, that if the origin of coordinates be placed at the centre of a curve of the second order, the equation of the curve will be such as to give equal values of x when $y = 0$, or equal values of y when $x = 0$; that is, the equation will not contain the first powers of x and y. This property furnishes a simple method of finding the centres of such curves from their equations.

Thus taking the equation (1) of the ellipse (Art. 40), viz.:—

$$a^2 y^2 = b^2 (2 a x - x^2):$$

remove the origin to a point (α, β), by writing $x + \alpha$ for x and $y + \beta$ for y; then the equation becomes

$$a^2 (y + \beta)^2 = b^2 (2 a x + 2 a \alpha - x^2 - 2 \alpha x - \alpha^2),$$

or $a^2 y^2 + b^2 x^2 + 2 a^2 \beta y + 2 b^2 (\alpha - a) x = b^2 (2 a \alpha - \alpha^2) - a^2 \beta^2.$

Now in order that (α, β) may be the centre, the coefficients of x and y in this equation must vanish;

hence $\quad 2 a^2 \beta = 0,\ \alpha - a = 0$, or $\beta = 0$, and $a = \alpha$.

The equation of the curve consequently becomes

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which is the same as the equation (2) of the ellipse (Art. 40) when the centre is the origin. And similarly for the *hyperbola.*

Next in the equation of the *parabola* (Art. 38), write $x + \alpha$ for x and $y + \beta$ for y;

then $(y + \beta)^2 = 4 m (x + \alpha)$, or $y^2 + 2 \beta y + \beta^2 = 4 m x + 4 m \alpha.$

Now the coefficient of x cannot vanish, for then $m = \mathrm{A\,S} = 0$, and consequently, the parabola has not a centre.

DIAMETERS OF CURVES.

45. *To find the equation of a diameter to any curve of the second order.*

Definition.—The diameter of a curve is the locus of the middle points of a series of parallel chords.

Let $$y^2 = 4 m x \quad . \quad . \quad . \quad . \quad . \quad (1),$$

be the equation of a parabola, the vertex being the origin (Art. 38),

and $$y = p x + q \quad . \quad . \quad . \quad . \quad (2),$$

the equation of a given straight line; then (Art. 12) the equation

$$y = p x + q' \quad . \quad . \quad . \quad . \quad (3),$$

represents any straight line parallel to (2), q' admitting of all possible values.

Eliminating y between (1) and (3), there results for the points of intersection of the parabola (1), and the line (3), the equation

$$x^2 + \frac{2}{p^2} (p q' - 2 m) x + \frac{q'^2}{p^2} = 0 \quad . \quad . \quad . \quad . \quad (4).$$

Hence if x_1, x_2 be the roots of this equation, and x', y' the coordinates of the middle point of the chord which joins the points of intersection of (1) and (3), we have, by Art. 23, and Art. 125, *Algebra*,

$$x' = \tfrac{1}{2}(x_1 + x_2) = -\frac{pq' - 2m}{p^2} \quad . \quad . \quad . \quad . \quad (5).$$

But (x', y') being a point in (3),

$$y' = px' + q' \quad . \quad . \quad . \quad . \quad (6).$$

Eliminating q' which varies for different positions of the parallel chords, between (5) and (6), there results for the locus of the middle points of the parallel chords, the equation

$$y' = \frac{2m}{p}, \text{ or (suppressing the } dash) \ y = \frac{2m}{p} \quad . \quad . \quad . \quad . \quad (7),$$

which (Art. 12) is a straight line parallel to the axis of x. Hence all diameters of the parabola are parallel.

Similarly, the equation of a diameter to the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2,$$

which bisects all chords parallel to (2),

is $$y = -\frac{b^2}{a^2p}x \quad . \quad . \quad . \quad . \quad (8).$$

Also, $$y = \frac{b^2}{a^2p}x \quad . \quad . \quad . \quad . \quad (9),$$

is the analogous equation for the hyperbola.

Hence all the diameters of the ellipse and hyperbola pass through the centre.

Cor. 1. Let a diameter of the parabola (1) pass through the point (h, k) in the curve. Then because it is parallel to the axis of x its equation (Art. 12) is

$$y = k \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

But if $$y = \alpha x + \beta \quad . \quad . \quad . \quad . \quad (11),$$

be one of the chords which this diameter bisects, its equation by (7) is also

$$y = \frac{2m}{\alpha} \quad . \quad . \quad . \quad . \quad (12).$$

Hence $k = \dfrac{2m}{\alpha}$, or $\alpha = \dfrac{2m}{k}$. Consequently (11) becomes

$$y = \frac{2m}{k}x + \beta \quad . \quad . \quad . \quad . \quad (13).$$

Comparing this with the equation of the tangent at the point (h, k), Art. 43, it is easily seen that the tangent and the chord (13) are parallel; and similarly for the ellipse and hyperbola. Hence *the chords bisected by any diameter are parallel to the tangent at the extremity of that diameter.*

Definition.—Two diameters of the ellipse and hyperbola so related that each bisects the chords parallel to the other, are called *conjugate diameters.*

Cor. 2. Since the angular coefficients of the equations of tangents to

the ellipse and hyperbola at the point (h, k), Art. 43, remain the same when $-h$ and $-k$ are written for h and k, it follows *that the tangents at the extremities of any diameter are parallel.*

MISCELLANEOUS EXERCISES ON THE CONIC SECTIONS AND OTHER CURVES.

1. Prove that if a circle be described on the radius vector of a parabola, the tangent at the vertex is a tangent to the circle.

2. Find the condition that the line $y = \alpha x + \beta$ may be a tangent to the parabola $y^2 = 4mx$. *Ans.* $\alpha\beta = m$.

3. Show how you can find all the geometric elements of a conic section (focus, vertex, directrix, etc.) from a given portion of its arc.

4. Find the locus of the centre of a circle which constantly touches a fixed straight line, and passes through a given point not in the line. *Ans.* A parabola.

5. Prove that the locus of the centre of a circle which touches a given straight line and a given circle is a parabola.

6. Prove that if P be any point in an ellipse whose foci are S and H, $\tan \frac{1}{2} \text{PSH} \tan \frac{1}{2} \text{PHS} = \dfrac{1-e}{1+e}$.

7. The length of the perpendicular upon the tangent from the centre of an ellipse is equal to $a(1 - e^2 \cos^2 \theta)^{\frac{1}{2}}$, in which θ is the inclination of the tangent to the major axis.

8. Prove that the locus of the foci of all the parabolas having the same directrix, and a point common to all, is a circle.

9. If in an ellipse there be taken three abscissas in arithmetical progression, the radii vectores drawn from the focus to the extremities of the ordinates at those points will also be in arithmetical progression.

10. What is the locus of the vertices of parabolas having the same tangent and the same directrix? *Ans.* A straight line.

11. Given the major axis of an ellipse, and also a fixed point on the minor axis, from which a normal is drawn to the curve, what is the locus of the points of intersection of the normal and ellipse? *Ans.* The locus is a circle.

12. Prove that two tangents to a parabola drawn from the same point in the directrix are at right angles to each other.

13. Prove that the tangent at any point of a parabola meets the directrix and latus rectum produced in two points equally distant from the focus.

14. Draw in a parabola a chord of given length, and find the locus of its middle point. *Ans.* A curve of the fourth order.

15. At what point in the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ does the tangent make an angle of 45° with the radius vector drawn to the same point?

$$Ans.\ x = a\left(\frac{a^2 - 2b^2}{a^2 - b^2}\right)^{\frac{1}{2}},\ y = \frac{b^2}{(a^2-b^2)^{\frac{1}{2}}}.$$

16. Find the condition that two tangents drawn from the same point (α, β) to the parabola $y^2 = 4mx$, may be equal. *Ans.* $\beta = 0$.

17. Let A C B be a semicircle of which A B $(2r)$ is the diameter, B D an indefinite straight line perpendicular to A B, and A C D a straight line meeting the semicircle in C and B D in D; then if P be a point in A C such that A P = C D, the locus of P is the *Cissoid of Diocles.* Find its equation, and thence show that B D is an asymptote to the curve. *Ans.* The equation is $y^2(2r - x) - x^3 = 0$.

18. The *Conchoid of Nicomedes* is thus generated: A B is an indefinite straight line, and C a given point without it; from C a perpendicular C D P is drawn to A B meeting it in D, and straight lines C D′ P′, C D″ P″, etc., are drawn in such manner that D P, D′ P′, D″ P″, etc., are all equal, D′, D″, etc., being in A B, then the locus of the points P, P′, etc., is the *Conchoid.* Find its equation, when C D = a, and D P = b, and thence trace the curve. *Ans.* $x^2 y^2 = (b^2 - y^2)(a + y)^2$.

19. If M Q be an ordinate to the semicircle A Q B (radius = r), and it be produced to P so that M P : M Q :: A B : A M; then the locus of P is a curve called the *Witch of Agnesi.* Find its equation, and thence deduce some of its geometrical properties.

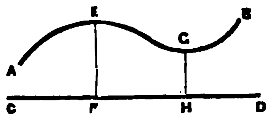
Ans. Its equation is $y^2 x = 4r^2(2r - x)$.

ELEMENTS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS.

I. THE DIFFERENTIAL CALCULUS.

ELEMENTARY PRINCIPLES AND ILLUSTRATIONS.

1. THE algebraic processes which have been already investigated enable us to solve a large number of the questions which arise in practice, but for the solution of others they are inconvenient, and often inadequate. In the annexed figure, let A B represent a curved line, supposed to be described in accordance with some given law, by which its distance from the straight line C D is constantly varied. It may be desirable to determine the point in C D at which the distance, as E F, is the greatest, or as G H, the least, and the measure of these distances, and also to determine the area of E F G H, and consequently the side of a rectangle which would give the same area, or the measure of the average distance between the two lines; but to effect these, in every case, exceeds the power of algebra, and we must have recourse to a new calculus adapted to inquiries of this nature.



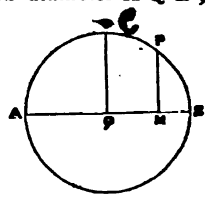
The difference between the case which is here assumed and the cases to which the usual processes of algebra are applicable, may be easily traced. By these processes we are enabled to ascertain the values of particular quantities which are dependent on each other, when as many distinct conditions with reference to some known quantity or quantities are given, from which as many equations can be obtained as there are unknown quantities, while the cases for which a new calculus is required appear to be those in which *variable* quantities, as the ordinate F E, and the area E F G H, dependent on the value of another variable quantity, as C F, have to be dealt with. The general principles on which a calculus adapted to such investigations has been formed will be explained presently.

2. The study of algebra leads to that of variable quantities, in which a letter is considered as assuming all possible values between given *limits*, and the results of that supposition are developed in the differential and integral calculus in which all quantities are considered either *constant* or *variable*.

A *constant* quantity is one whose value or magnitude remains unchanged throughout the operation or investigation in which it is employed.

A *variable* quantity is one which admits of an unlimited number of

values. Let A C B be a circle whose centre is Q and diameter A Q B; draw Q C at right angles to A B; take any point P in the circumference, and draw P M at right angles to A B. Now if we conceive the point P to move from B to C, the line P M will *vary* in magnitude from 0 to Q C, the radius. Thus A M, B M, Q M, and M P are all variable quantities, and the radius Q C or Q B is a constant quantity. Constant quantities are usually represented by the earlier letters, a, b, c, d; and those that are variable by the later ones, as u, v, x, y, z. In the expression $a + bx + cx^2$, a, b, c, are constant quantities, and x is the variable.



3. A *function* of a variable quantity is an expression involving that variable, and usually one or more constant quantities. Thus the expressions

$$a x^2 + b, \quad a \sin x, \quad a + a^x - \log (b + x)$$

are severally functions of the variable quantity x.

If the value of a quantity depend on that of another which is variable, the former is termed a *function* of the latter. Thus if

$$u = a x^2 + b, \; u = a \sin x, \; u = a + a^x - \log (b + x);$$

then, in each of these equations, u is a function of x. It is not to be understood that u is the *same* function of the variable x in all these equations, and as the term "function" may be denoted by a single letter, as f, ϕ, ψ, F, etc., the preceding equations may be written symbolically, thus

$$u = f x \quad , \qquad u = \phi x \quad , \qquad u = \mathrm{F} x:$$

where the different letters f, ϕ, F denote different functions of the variable. The quantity x is called the *independent* variable, and u the *dependent* variable.

A quantity may be a function of several independent variables, as in the expression

$$u = a x^2 + b x y + c y^2 + m x + n y + p.$$

Here u is a function of the independent variables x and y, and may be written $u = f(x, y)$.

Functions are distinguished into *algebraic* and *transcendental*. An *algebraic* function is one that may be expressed in a finite number of terms, and in which the variable is subjected to some of the elementary operations of algebra, as addition, subtraction, multiplication, division, involution and evolution. Thus, if m and n be finite, $u = a x^m + (b x^2 - 2 c x)\,(a^2 + x^2)^n$ is an algebraic function. A *transcendental* function is one that cannot be expressed in a finite number of terms, as

$$u = \log (1 - x), \; u = a^x, \; u = \sin x.$$

Functions are likewise either *explicit* or *implicit*. An *explicit* function is one in which the value of the dependent variable is exhibited in terms of the independent variable and constants, as

$$u = a x^2 + b x + c.$$

A function is *implicit* when some operation is necessary for exhibiting its value in terms of the independent variable or variables. Thus in the equation

$$u^2 + 2 x u + a^2 = 0,$$

u is an implicit function of x. Resolving this equation for u, gives $u = -x \pm \sqrt{(x^2 - a^2)}$, and in this form u is an explicit function of x.

4. A *variable* is said to be *continuous* when in passing from one assigned magnitude to another it passes through all the intermediate ones, and a *function* of a quantity which varies continuously is said to be *continuous* between two assigned values, if in passing from one of them to the other it passes through all the intermediate values. Thus an angle may be made to vary continuously, increasing or diminishing by degrees, minutes, or seconds, or even the minutest fractions of seconds.

A *discontinuous* variable, and a *discontinuous function* of a variable, do not fulfil the conditions of passing through all the intermediate values. Thus the tangent of an angle is continuous when the angle varies from 0° to 90°; but it is discontinuous at 90°, passing at once from a positive value to a negative one. In the differential and integral calculus all quantities are supposed to vary continuously, and to the student who is acquainted only with the ordinary processes of algebra, the principles of the calculus may, at first, appear somewhat unintelligible, especially as abstract quantities are considered as admitting of continuous change, and of taking certain *finite* ratios as they approach the limits of zero or of infinity. As the principle involved in a limiting ratio must be thoroughly understood, we shall first advert to some illustrations of it.

ILLUSTRATIONS OF THE PRINCIPLE OF LIMITS.

5. *Definition.* *The limiting value* of an expression is the quantity towards which it continually approaches, by making the variable continually approach a certain value.

Thus a circle is the limit of the area of an inscribed regular polygon. For by continually increasing the number of the sides of the polygon, its area will approach more and more to the area of the circle, and their difference may be made less than any quantity that can be assigned. The circle is consequently said to be the limit of the inscribed polygon when the number of sides approaches infinity. The following examples will tend to illustrate the important subject of limits.

Ex. 1. Find the limit of $\dfrac{a^2 - x^2}{a - x}$, when x approaches to the value of a.

Let $x = a - h$, then $\dfrac{a^2 - x^2}{a - x} = \dfrac{a^2 - (a - h)^2}{a - (a - h)} = \dfrac{2\,a\,h - h^2}{h} = 2\,a - h$; hence, as x approaches to a, h is successively diminished, and the limit is evidently $2\,a$. This will appear also when we consider that $\dfrac{a^2 - x^2}{a - x} = a + x$, for all values of x, and therefore as x approaches to a, the more will $a + x$ approach to $2\,a$. Thus,

$$\text{if } a = 10, \text{ and } x = 8, \text{ then } \frac{a^2 - x^2}{a - x} = \frac{10^2 - 8^2}{10 - 8} = 18,$$

$$\text{if } a = 20, \text{ and } x = 19, \text{ then } \frac{a^2 - x^2}{a - x} = \frac{20^2 - 19^2}{20 - 19} = 39, \text{ and so on.}$$

Ex. 2. Find the limit of $\dfrac{3\,x + 7}{6\,x - 5}$, when x approaches ∞.

Dividing numerator and denominator by x, gives $\dfrac{3x+7}{6x-5} = \dfrac{3+\dfrac{7}{x}}{6-\dfrac{5}{x}}$;

but as the value of x approaches ∞, the values of $\dfrac{7}{x}$ and $\dfrac{5}{x}$ approach to evanescence; therefore the limit required is $\dfrac{3}{6} = \dfrac{1}{2}$.

Ex. 3. Find the limit of $\dfrac{ax}{x+a}$, when x approaches ∞. *Ans.* a.

Ex. 4. Find the limit of $\dfrac{a^3-x^3}{a^2-x^2}$, when x approaches a. *Ans.* $\dfrac{3}{2}a$.

Ex. 5. Find the limit of $\dfrac{2ax+x^2}{x}$, when x approaches 0. *Ans.* $2a$.

6. The elementary principles of the differential calculus may be illustrated by considering the changes produced on certain functions corresponding to changes in the variable.

Let u denote a function of the variable x, and u' the same function of $x+h$, that is, let $u = fx$, and $u' = f(x+h)$, the latter function having the same form with respect to $x+h$ which the former has with respect to x; then $u'-u$ will be the change produced on the function, corresponding to h, the change of the variable, and the relation between these simultaneous changes made on the function and the variable will be best exhibited by dividing $u'-u$ by h. Thus, if $u = ax^3$, and $u' = a(x+h)^3 = ax^3 + 3ax^2h + 3axh^2 + ah^3$; then will

$$u'-u = 3ax^2h + 3axh^2 + ah^3 \ldots\ldots (1),$$

$$\text{and } \frac{u'-u}{h} = 3ax^2 + 3axh + ah^2 \ldots\ldots (2).$$

Again, if $u = \dfrac{a}{x^2}$, and $u' = \dfrac{a}{(x+h)^2} = \dfrac{a}{x^2+2xh+h^2}$, or dividing,

$$u' = \frac{a}{x^2} - \frac{2ah}{x^3} + \frac{3ah^2+2ah^3}{x^3(x+h)^2}; \text{ then will}$$

$$u'-u = -\frac{2ah}{x^3} + \frac{3ah^2+2ah^3}{x^3(x+h)^2} \ldots\ldots\ldots\ldots (1'),$$

$$\text{and } \frac{u'-u}{h} = -\frac{2a}{x^3} + \frac{3ah+2ah^2}{x^3(x+h)^2} \ldots\ldots\ldots\ldots (2').$$

In both these instances the variable x receives the increment h, and the corresponding changes, or increments, of the functions are the expressions marked (1) and (1′). It will be observed that the *first* terms of the second members of (2) and (2′) are independent of the increment h, and consequently if h be continually diminished down towards zero, the second members of (2) and (2′) will tend to become simply their first terms, and by giving to h a sufficiently small value, each of them may be made to differ from its first term by a quantity less than anything that can be assigned. These first terms are the *limits* to which the second members, and consequently the first, tend when the increment h approaches more and more to zero. Thus in equation (2) the limit to

which the quotient of $u' - u$ divided by h approaches, as h is diminished towards zero, is $3\,a\,x^2$; and though the terms of the fraction in the first member, viz., $u' - u$ and h suffer diminution by the continued diminution of h, yet the fraction itself does not suffer unlimited diminution, but tends eventually to take as its value the first term, $3\,a\,x^2$, of the second member of the equation. This will be obvious when we consider that the magnitude of a fraction does not depend on the *actual* magnitude of its terms, but on their *comparative* magnitudes. Thus the value of the fraction $\frac{600}{700}$ is the same as that of the fraction $\frac{6}{7}$, and also the same as that of the fraction $\frac{\cdot000000006}{\cdot000000007}$. Hence though, in the case of any proposed function, the increments of the variable and the function, h and $u' - u$, may be made smaller and smaller, so as to become as nearly evanescent as we choose, it does not necessarily follow that the fraction arising from dividing $u' - u$ by h will become evanescent. It will be constant if the function be $u = a\,x$, and it will be diminishing towards a limit if the function be $u = a\,x^3$. Take the latter function, $u = a\,x^3$, and let $a = 10$, and $x = 2$, then by equation (2), the limit is $3\,a\,x^2 = 3 \times 10 \times 2^2 = 120$. Now if

$$h = 1;\ \text{then}\ \frac{u' - u}{h} = \frac{10\,(3^3 - 2^3)}{1} = \frac{190}{1} = 190,$$

$$h = \cdot1;\ \text{then}\ \frac{u' - u}{h} = \frac{10\,(2\cdot1^3 - 2^3)}{\cdot1} = \frac{12\cdot61}{\cdot1} = 126\cdot1,$$

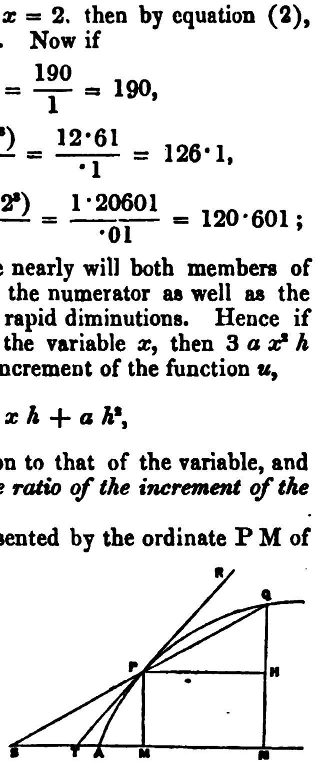
$$h = \cdot01;\ \text{then}\ \frac{u' - u}{h} = \frac{10\,(2\cdot01^3 - 2^3)}{\cdot01} = \frac{1\cdot20601}{\cdot01} = 120\cdot601;$$

and thus the smaller h is taken the more nearly will both members of equation (2) approach to the limit 120, the numerator as well as the denominator of the first member suffering rapid diminutions. Hence if $u = a\,x^3$, and if h be the increment of the variable x, then $3\,a\,x^2\,h + 3\,a\,x\,h^2 + a\,h^3$ is the corresponding increment of the function u,

$$\frac{u' - u}{h} = 3\,a\,x^2 + 3\,a\,x\,h + a\,h^2,$$

is the ratio of the increment of the function to that of the variable, and the first term, $3\,a\,x^2$, is the *limit of the ratio of the increment of the function to that of the variable.*

7. In general u may always be represented by the ordinate P M of some curve A P Q, whose abscissa A M $= x$. Let R P T be a line touching the curve at P, and Q P S a line cutting it in Q and P; then if the point Q be conceived to approach P, the line Q P S will approach to coincidence with the line R P T. Draw Q N parallel to P M, and P H parallel to the abscissa A M. Let M N $= h$, then Q N $= u' = f\,(x + h)$,



and $\frac{u' - u}{h} = \frac{\text{QN} - \text{PM}}{\text{MN}} = \frac{\text{QH}}{\text{PH}} = \frac{\text{PM}}{\text{MS}}$.

But as Q approaches P, the intersection S will approach T, and therefore the limit of $\frac{u' - u}{h} = \frac{\text{PM}}{\text{MT}} = \tan \text{PTM}$. (Plane Trig. Art. 20).

These illustrations will enable the student to form a correct notion of what is meant by a *limit* to the value of a *variable function* and a *variable ratio.*

8. It has been seen that if $u = a\,x^3$, and $u' = a\,(x + h)^3$, then

$$u' - u = 3\,a\,x^2 h + 3\,a\,x\,h^2 + a\,h^3.$$

This is termed the *difference* of the function $u = a\,x^3$, and whenever the difference of a function can be expressed in terms containing successively h, h^2, h^3, etc., as multipliers, *the first term of the difference* is called the *differential* of the original function.

Now if $u = x$ be the proposed function, then $u' = x + h$, and we have $u' - u = h$, so that h is the differential, as well as the difference of the function $u = x$.

The differential of any quantity or function of any quantity, is indicated by prefixing to it the letter d, the initial of the word differential. Thus $d\,u$ is called the differential of u, $d\,x$ the differential of x, and so on; but when $u = x$, then h is the differential of u or x; hence $h = d\,x$.

Again, if $u = a\,x^2$, then $u' = a\,(x + h)^2$, and $u' - u = 2\,a\,x\,h + a\,h^2$; hence, $d\,u$ or $d\,(a\,x^2) = 2\,a\,x\,d\,x$.

The coefficient $2\,a\,x$, or the multiplier of $d\,x$ is called the *differential coefficient* of the proposed function, and since in this case we have

$$\frac{u' - u}{h} = 2\,a\,x + a\,h,$$

it is obvious that the differential coefficient, $2\,a\,x$, is the limit of the ratio of the increment of the function to that of the variable.

9. From the preceding observations it is evident that the limit of the ratio of the simultaneous increments of a function, and of the variable on which it depends, will be different for different functions, and that there exists such a connexion between the function and the limit of the ratio that the one may be derived from the other. This connexion gives rise to an extensive and important analytical theory, consisting of the two following parts:—

I. *A function of a variable quantity being given, to determine the limit of the ratio of the increment of the function to the increment of the variable*, and conversely,

II. *Having given the limit of the ratio of the corresponding increments of a function and its variable, to determine the function.*

The former of these divisions is called the *differential calculus*, and the latter the *integral calculus.*

The *differential calculus* is that branch of analysis whose object is to determine the limit of the ratio of the increment of any function to the increment of the variable, and to explain some of the principal uses which may be made of the limit of this ratio in pure mathematics.

The differential calculus will consequently be founded on the following *definition.* If a variable quantity, x, be increased by a quantity h, and if any function of x be taken from the same function of $x + h$, and the remainder be divided by h, the limit to which the quotient so obtained will continually approach, and from which it may be made to differ by a quantity less than any that can be assigned, is called the *differential*

coefficient of the function of x, and the product of this by dx is called the *differential* of that function.

The elementary functions whose differentials or differential coefficients are to be investigated, may be included in the four forms

$$(fx)^n,\ a^{(fx)},\ \log_a(fx),\ \text{and } \sin(fx);$$

and to one or other of these all other functions may be reduced, and their differentials or differential coefficients will be obtained by means of those of the four forms above.

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

10. The process of finding the differential, or the differential coefficient of a function, may be considered as a particular operation performed on quantity somewhat analogous to the ordinary operations of algebra; and to perform this process on a function, is to *differentiate* the function: the process itself is called *differentiation*, and the result of the process is the differential of the function.

11. Let it be required to differentiate $u = ax^2 + bx + c$.

Change x into $x + h$; then we have

$u' = a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + 2axh + bh + ah^2$;

but $u = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = ax^2 + bx + c$;

$$\therefore \frac{u'-u}{h} = \frac{(2ax+b)h + ah^2}{h} = 2ax + b + ah;$$

and when h approaches 0, we shall have ultimately (Art. 8)

$$\frac{du}{dx} = 2ax + b, \text{ and } du = 2ax\,dx + b\,dx.$$

Generally, if $u = afx - \frac{1}{b}\phi x \pm c$; then we have

$$u' = af(x+h) - \frac{1}{b}\phi(x+h) \pm c;$$

$$\therefore \frac{u'-u}{h} = a \cdot \frac{f(x+h) - fx}{h} - \frac{1}{b} \cdot \frac{\phi(x+h) - \phi x}{h}.$$

Now when h approaches 0, we have (Art. 8)

$$\frac{du}{dx} = \frac{a\,dfx}{dx} - \frac{1}{b} \cdot \frac{d\phi x}{dx}, \text{ and } du = a\,dfx - \frac{1}{b}d\phi x \quad . \quad . \quad . \quad (1).$$

Hence it follows tnat *the differential of a quantity composed of several functions of the same variable, connected by addition or subtraction, is equal to the differentials of the several functions connected by the signs belonging to those functions.* Also, that *a constant quantity connected with a function by addition or subtraction disappears in the differential; but that a constant quantity connected with a function by multiplication or division is retained as a multiplier or divisor in the differential.*

12. Let it be required to differentiate $u = x^n$.

By the binomial theorem (Algebra, 136) we have $u' = (x+h)^n =$

$$x^n + nx^{n-1}h + \frac{n(n-1)}{1\,.\,2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}h^3 + \text{etc.};$$

$$\therefore \frac{u' - u}{h} = n x^{n-1} + \frac{n(n-1)}{1 \,.\, 2} x^{n-2} h + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} x^{n-3} h^2 + \text{etc.};$$

$$\therefore \frac{du}{dx} = n x^{n-1}, \text{ and } du = n x^{n-1} dx \quad . \quad . \quad . \quad . \quad (2).$$

This result is true, whether n be an integer or a fraction, positive or negative.* Hence *the differential of any power of a variable quantity*

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* This is the most important of all the functions in the calculus, and though the expansion of the binomial $(x + h)^n$ is true for all values of n, whether they be integral or fractional, positive or negative, it may be useful to determine the differential of x^n independently of the binomial theorem. Let $u = x^n$, and $u' = (x + h)^n$; then the first term of the expansion of $(x + h)^n$ is x^n, the second $n x^{n-1} h$, and the succeeding terms contain successively h^2, h^3, etc., as multipliers. If n is a positive integer, we have, by multiplication,

$$(x + h)^2 = x^2 + 2 x h + h^2, \text{ and } (x + h)^3 = x^3 + 3 x^2 h + 3 x h^2 + h^3,$$

in each of which the property holds. But if it be true for the m^{th} power, so that

$$(x + h)^m = x^m + m x^{m-1} h + \text{A} h^2 + \text{B} h^3 + \text{etc.},$$

where A, B, etc., are functions of x independent of h; then multiplying by $x + h$,

$$(x+h)^{m+1} = x^{m+1} + (m + 1)\, x^m h + (\text{A} x + m x^{m-1})\, h^2 + \text{etc.},$$

which is still of the same form, the index being now $m + 1$. But the property holds for the second and third powers, therefore it must hold for the fourth, and if true for the fourth, it is true for the fifth, and so on; consequently it holds universally. We have, therefore, when n is any positive integer whatever,

$$(x + h)^n = x^n + n x^{n-1} h + \text{A} h^2 + \text{B} h^3 + \text{etc.};$$

$$\therefore \frac{u' - u}{h} = n x^{n-1} + \text{A} h + \text{B} h^2 + \text{C} h^3 + \text{etc.};$$

$$\therefore \frac{du}{dx} = n x^{n-1}, \text{ and } du = n x^{n-1} dx.$$

If $n = \frac{p}{q}$, then p and q being whole positive numbers, we have $u = x^{\frac{p}{q}}$, and therefore $u^q = x^p$. Differentiating, we get $q u^{q-1} du = p x^{p-1} dx$; but since $u = x^{\frac{p}{q}}$, therefore $u^{q-1} = x^{\frac{p}{q}(q-1)} = x^{p - \frac{p}{q}} = x^{p-n}$; and by substitution in the equation $q u^{q-1} du = p x^{p-1} dx$, we have (since $p = nq$)

$$q x^{p-n} du = n q x^{p-1} dx, \text{ or } du = n x^{n-1} dx.$$

Again, if $n = -p$, where p is a whole number, then $u = x^{-p} = \frac{1}{x^p}$, and $u' = \frac{1}{(x + h)^p}$;

hence $u' = \dfrac{1}{x^p + p x^{p-1} h + \text{A} h^2 + \text{etc.}} = x^{-p} - p x^{-p-1} h + \text{A}' h^2 + \text{etc.}$;

$$\therefore \frac{u' - u}{h} = -p x^{-p-1} + \text{A}' h + \text{etc.}; \text{ consequently when } h \text{ approaches } 0,$$

$$\frac{du}{dx} = -p x^{-p-1} = n x^{n-1}; \text{ and } du = n x^{n-1} dx.$$

Lastly, if $n = -\frac{p}{q}$, then p and q being whole positive numbers, we have

$$u = x^{-\frac{p}{q}} \text{ or } u^q = x^{-p}; \text{ hence } q u^{q-1} du = -p x^{-p-1} dx.$$

But $u^{q-1} = x^{-\frac{p}{q}(q-1)} = x^{-p+\frac{p}{q}} = x^{-p-n}$, and $-p = nq$; therefore by substitution,

$$q x^{-p-n} du = n q x^{-p-1} dx, \text{ and } du = n x^{n} \quad x.$$

The truth of the result, in all cases, is therefore established independently of the binomial theorem.

having its index constant is equal to the continued product of the index of the variable, the power of the quantity whose index is less by unity than the given index, and the differential of the variable.

13. Let it be required to differentiate the product $u = yz$, where y and z are both functions of the same variable x.

Differentiating by formulas (1) and (2), both members of the identical equation, $y^2 + 2yz + z^2 = (y+z)^2$, we get

$$2y\,dy + 2d(yz) + 2z\,dz = 2(y+z)(dy+dz)$$
$$= 2y\,dy + 2y\,dz + 2z\,dy + 2z\,dz.$$

Rejecting the quantities common to both members, and dividing by 2,

$$du \text{ or } d(yz) = y\,dz + z\,dy \quad . \quad . \quad . \quad . \quad (3).$$

Hence *the differential of the product of two functions is equal to the sum of the products of each function by the differential of the other.*

If we divide each member of (3) by yz, we get

$$\frac{d(yz)}{yz} = \frac{y\,dz}{yz} + \frac{z\,dy}{yz} = \frac{dy}{y} + \frac{dz}{z};$$

and if $yz = u$; then $tyz = tu$, and $d(tyz) = d(tu)$

but $\dfrac{du}{u} = \dfrac{d(yz)}{yz} = \dfrac{dy}{y} + \dfrac{dz}{z}$; and, consequently,

$$\frac{d(tyz)}{tyz} = \frac{d(tu)}{tu} = \frac{dt}{t} + \frac{du}{u} = \frac{dt}{t} + \frac{dy}{y} + \frac{dz}{z};$$

$$\therefore d(tyz) = yz\,dt + tz\,dy + ty\,dz \quad . \quad . \quad . \quad . \quad (4).$$

Similarly $d(vtyz) = tyz\,dv + vyz\,dt + vtz\,dy + vty\,dz$; and, therefore, whatever may be the number of factors, *the differential of the product is equal to the sum of the products of the differential of each factor by the product of all the others.*

14. To find the differential of the quotient $u = \dfrac{y}{z}$, where y and z are both functions of the same variable.

Here we have $y = uz$, and $dy = d(uz) = u\,dz + z\,du$; therefore

$$du = \frac{dy}{z} - u\frac{dz}{z} = \frac{dy}{z} - \frac{y}{z}\cdot\frac{dz}{z} = \frac{z\,dy - y\,dz}{z^2} \quad . \quad . \quad . \quad . \quad (5).$$

Hence *the differential of a fraction whose terms are functions of the same variable, is equal to the differential of the numerator multiplied by the denominator, diminished by the differential of the denominator multiplied by the numerator, and the difference divided by the square of the denominator.*

Examples on the Differentiation of Algebraic Functions.

15. Find the differentials, or the differential coefficients of the following algebraic functions.

1. Let $u = 5x^4$.

This is the product of a function x^4, and a constant 5; hence (Art. 11)

$$du = 5\,dx^4 = 5 . 4x^{4-1}\,dx = 20x^3\,dx \text{ (Art. 12)};$$

and $\dfrac{du}{dx} = 20x^3$, the differential coefficient.

2. Let $u = x^3 + 2x^2 + 3x + c$.

Here $du = 3x^2 dx + 4x dx + 3 dx$, and therefore

$$\frac{du}{dx} = 3x^2 + 4x + 3.$$

3. Let $u = (a + bx)^2$.

Considering $a + bx$ as one quantity, we have, by (2) Art. 12,

$$du = 2(a + bx) d(a + bx) = 2(a + bx) . b dx = 2b(a + bx) dx;$$

$$\therefore \frac{du}{dx} = 2b(a + bx).$$

4. Let $u = (a + bx^2)^n$.

Here $du = n(a + bx^2)^{n-1} d(a + bx^2) = n(a + bx^2)^{n-1} . 2bx dx$;

$$\therefore \frac{du}{dx} = 2bnx(a + bx^2)^{n-1}.$$

5. Let $u = \sqrt{(a^2 - x^2)}$.

Here $du = d(a^2 - x^2)^{\frac{1}{2}} = \frac{1}{2}(a^2 - x^2)^{\frac{1}{2}-1} d(a^2 - x^2)$

$$= \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x dx = -\frac{x dx}{\sqrt{(a^2 - x^2)}}; \quad \therefore \frac{du}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}}.$$

6. Let $u = (a + x)(b + 2x^2)$.

By Art. 13 equation (3), we have

$$\begin{aligned} du &= (a + x) d(b + 2x^2) + (b + 2x^2) d(a + x) \\ &= (a + x) . 4x\, dx + (b + 2x^2)\, dx; \end{aligned}$$

$$\therefore \frac{du}{dx} = 4x(a + x) + b + 2x^2 = 6x^2 + 4ax + b.$$

7. Let $u = \dfrac{ax}{a^2 + x^2}$.

By Art. 14 equation (5), we have

$$du = \frac{(a^2 + x^2) d(ax) - ax\, d(a^2 + x^2)}{(a^2 + x^2)^2} = \frac{a(a^2 + x^2) dx - 2ax^2 dx}{(a^2 + x^2)^2};$$

$$\therefore \frac{du}{dx} = \frac{a(a^2 + x^2) - 2ax^2}{(a^2 + x^2)^2} = \frac{a(a^2 - x^2)}{(a^2 + x^2)^2}.$$

8. Let $u = \sqrt{\{x + \sqrt{(a^2 + x^2)}\}}$.

Here $du = d\{x + \sqrt{(a^2 + x^2)}\}^{\frac{1}{2}} = \frac{1}{2}\{x + \sqrt{(a^2 + x^2)}\}^{-\frac{1}{2}} \times d\{x + \sqrt{(a^2 + x^2)}\}$;

but $d\{x + \sqrt{(a^2 + x^2)}\} = dx + \dfrac{x\, dx}{\sqrt{(a^2 + x^2)}} = \dfrac{x + \sqrt{(a^2 + x^2)}}{\sqrt{(a^2 + x^2)}} dx$;

$$\therefore du = \frac{1}{2\{x + \sqrt{(a^2 + x^2)}\}^{\frac{1}{2}}} \cdot \frac{x + \sqrt{(a^2 + x^2)}}{\sqrt{(a^2 + x^2)}} dx$$

$$= \frac{\sqrt{\{x + \sqrt{(a^2 + x^2)}\}}}{2\sqrt{(a^2 + x^2)}} \cdot dx; \quad \therefore \frac{du}{dx} = \frac{\sqrt{\{x + \sqrt{(a^2 + x^2)}\}}}{2\sqrt{(a^2 + x^2)}}.$$

EXERCISES.

1. $u = ax^3 - bx^2 + cx - 4$. *Ans.* $\dfrac{du}{dx} = 3ax^2 - 2bx + c$.

2. $u = a x^{\frac{5}{2}} - b x^{\frac{3}{2}} \pm c.$ *Ans.* $\dfrac{du}{dx} = \frac{1}{2} x^{\frac{1}{2}} (5ax - 3b).$

3. $u = a x^{-\frac{5}{2}} - b x^{-\frac{3}{2}} \pm c.$ *Ans.* $\dfrac{du}{dx} = -\dfrac{1}{2x^{\frac{5}{2}}}\left(\dfrac{5a}{x} - 3b\right).$

4. $u = (a^2 - x^2)^2.$ *Ans.* $\dfrac{du}{dx} = -4x(a^2 - x^2).$

5. $u = (2ax - x^2)^3.$ *Ans.* $\dfrac{du}{dx} = 6(a - x)(2ax - x^2)^2.$

6. $u = (a + bx^n)^m.$ *Ans.* $\dfrac{du}{dx} = mnbx^{n-1}(a + bx^n)^{m-1}.$

7. $u = (2ax + x^2)^n.$ *Ans.* $\dfrac{du}{dx} = 2n(a + x)(2ax + x^2)^{n-1}.$

8. $u = x(a + x)(a^2 + x^2).$ *Ans.* $\dfrac{du}{dx} = a^3 + 2a^2x + 3ax^2 + 4x^3.$

9. $u = (a+x)^m (b+x)^n.$ *Ans.* $\dfrac{du}{dx} = (a+x)^m (b+x)^n \left\{\dfrac{m}{a+x} + \dfrac{n}{b+x}\right\}.$

10. $u = \dfrac{a - x}{a + x}.$ *Ans.* $\dfrac{du}{dx} = -\dfrac{2a}{(a + x)^2}.$

11. $u = \dfrac{(x + 4)^2}{x + 3}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{(x + 2)(x + 4)}{(x + 3)^2}.$

12. $u = \dfrac{x^2 - x + 1}{x^2 + x - 1}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{2x(x - 2)}{(x^2 + x - 1)^2}.$

13. $u = \sqrt{(a + x)}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{(a + x)}}.$

14. $u = \sqrt{(1 + x^2)}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{x}{\sqrt{(1 + x^2)}}.$

15. $u = \sqrt{(ax^2 + bx + c)}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{2ax + b}{2\sqrt{(ax^2 + bx + c)}}.$

16. $u = (a - x)\sqrt{(a + x)}.$ *Ans.* $\dfrac{du}{dx} = -\dfrac{a + 3x}{2\sqrt{(a + x)}}.$

17. $u = (a + x)\sqrt{(a - x)}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{a - 3x}{2\sqrt{(a - x)}}.$

18. $u = (1 - x^{\frac{1}{2}} + x^{\frac{2}{3}})^{\frac{3}{4}}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{1}{8x^{\frac{1}{2}}} \cdot \dfrac{4x^{\frac{1}{6}} - 3}{(1 - x^{\frac{1}{2}} + x^{\frac{2}{3}})^{\frac{1}{4}}}.$

19. $u = (a^2 - x^2)\sqrt{(a + x)}.$ *Ans.* $\dfrac{du}{dx} = \frac{1}{2}(a - 5x)\sqrt{(a + x)}.$

20. $u = \dfrac{a + x}{\sqrt{(a - x)}}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{3a - x}{2(a - x)^{\frac{3}{2}}}.$

21. $u = \dfrac{\sqrt{(a + x)}}{\sqrt{(a - x)}}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{a}{(a - x)\sqrt{(a^2 - x^2)}}.$

22. $u=\sqrt{\{x-\sqrt{(a^2-x^2)}\}}$. *Ans.* $\dfrac{du}{dx}=\dfrac{x+\sqrt{(a^2-x^2)}}{2\sqrt{(a^2-x^2)}\{x-\sqrt{(a^2-x^2)}\}^{\frac{1}{2}}}$

23. $u=x(1+x^2)\sqrt{(1-x^2)}$. *Ans.* $\dfrac{du}{dx}=\dfrac{1+x^2-4x^4}{\sqrt{(1-x^2)}}$.

24. $u=\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$. *Ans.* $\dfrac{du}{dx}=-\dfrac{1}{x^2\sqrt{(1-x^2)}}-\dfrac{1}{x^2}$.

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

16. The transcendental functions may be comprised in two classes; first, the logarithmic, including the exponential; and, secondly, the trigonometrical.

I. *Logarithmic and Exponential Functions.*

17. To find the differentials of $u = \log_a x$ and $u = \log_e x$.

$$u'=\log_a(x+h)=\log_a\left\{x\left(1+\frac{h}{x}\right)\right\}=\log_a x+\log_a\left(1+\frac{h}{x}\right);$$

$$\therefore\ \frac{u'-u}{h}=\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)=\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)=\frac{1}{x}\log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}}.$$

But by the binomial theorem,

$$\left(1+\frac{h}{x}\right)^{\frac{x}{h}}=1+\frac{x}{h}\cdot\frac{h}{x}+\frac{\frac{x}{h}\left(\frac{x}{h}-1\right)}{1.2}\cdot\frac{h^2}{x^2}+\frac{\frac{x}{h}\left(\frac{x}{h}-1\right)\left(\frac{x}{h}-2\right)}{1.2.3}\cdot\frac{h^3}{x^3}+\text{etc.}$$

$$=1+1+\frac{1-\frac{h}{x}}{1.2}+\frac{\left(1-\frac{h}{x}\right)\left(1-\frac{2h}{x}\right)}{1.2.3}+\text{etc.};$$

and when h is continually diminished towards zero, the second member of this tends to become

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{etc.}=2\cdot7182818=e.$$

$$\text{Hence the limit of }\frac{u'-u}{h}\text{ or }\frac{1}{x}\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}=\frac{1}{x}\log_a e;$$

consequently $\qquad \dfrac{du}{dx}=\dfrac{1}{x}\log_a e=\dfrac{1}{x}\cdot\dfrac{1}{\log_e a}$ (Algebra, Art. 138),

and $\qquad du=\dfrac{dx}{x}\cdot\dfrac{1}{\log_e a}$ = differential of $\log_a x$.

The multiplier $\dfrac{1}{\log_e a}$ is called the *modulus* of the system of logarithms whose base is a, and therefore *the differential of the logarithm of a quantity is found by dividing the differential of the quantity by the quantity itself, and multiplying the quotient by the modulus of the system.*

If the base of the system be e, then $\log_e e = 1$, and if $u = \log_e x$,

$$du=\frac{dx}{x}=\text{differential of }\log_e x.$$

Hence *the differential of the Napierian logarithm of a quantity is equal to the differential of the quantity divided by the quantity itself.*

18. To find the differentials of $u = a^x$ and $u = e^x$.

By the principles of logarithms, we have $\log_e u = x \log_e a$, and differentiating both members, we get

$$\frac{du}{u} = dx \cdot \log_e a, \text{ or } du = u\,dx \,.\, \log_e a = a^x\,dx \,.\, \log_e a;$$

$$\therefore \frac{du}{dx} = a^x \log_e a.$$

Hence to differentiate a variable power of a constant quantity, *multiply that power by the differential of the index, and the result by the Napierian logarithm of the constant quantity.*

If $a = e$; then we have $\log_e e = 1$, and therefore if $u = e^x$,

$$\frac{du}{dx} = e^x, \text{ and } du = e^x\,dx.$$

Hence *the differential of e^x is the product of e^x, and the differential of the exponent.*

Examples.

1. Let $u = \log_e \dfrac{a+x}{a-x} = \log_e (a+x) - \log_e (a-x)$.

$$\text{Here } du = \frac{d\,(a+x)}{a+x} - \frac{d\,(a-x)}{a-x} = \frac{dx}{a+x} + \frac{dx}{a-x};$$

$$\therefore \frac{du}{dx} = \frac{1}{a+x} + \frac{1}{a-x} = \frac{2a}{a^2 - x^2}.$$

2. Let $u = e^{e^x}$.

If we put $z = e^x$; then will $u = e^z$, and differentiating these, we get

$$dz = e^x\,dx \text{ and } du = e^z\,dz = e^z\,e^x\,dx = e^{e^x}\,e^x\,dx;$$

$$\therefore \frac{du}{dx} = e^{e^x}\,e^x.$$

3. Let $u = \log_e \left(\dfrac{1+x}{1+x}\right)^{\frac{1}{2}} = \frac{1}{2}\log_e (1+x) - \frac{1}{2}\log_e (1-x)$.

$$du = \frac{1}{2}\cdot\frac{d\,(1+x)}{1+x} - \frac{1}{2}\cdot\frac{d\,(1-x)}{1-x} = \frac{dx}{2\,(1+x)} + \frac{dx}{2\,(1-x)};$$

$$\therefore \frac{du}{dx} = \frac{1}{2\,(1+x)} + \frac{1}{2\,(1-x)} = \frac{1}{1-x^2}.$$

4. Let $u = \log_e (\log_e x)$ or $u = \log_e^2 x$.

If $z = \log_e x$, then will $u = \log_e z$, and differentiating these, we get

$$dz = \frac{dx}{x} \text{ and } du = \frac{dz}{z} = \frac{dx}{zx} = \frac{dx}{x \log_e x};\ \therefore \frac{du}{dx} = \frac{1}{x \log_e x}.$$

19. The student will now be in possession of principles to enable him to differentiate the more complex logarithmic and exponential functions; and whenever the base of the system of logarithms is not indicated, it will be understood that the Napierian system is employed.

Exercises.

1. $u = x \log (a + x)$. *Ans.* $\dfrac{du}{dx} = \log (a + x) + \dfrac{x}{a + x}$.

2. $u = \log_a c\,(a + x)^2$. *Ans.* $\dfrac{du}{dx} = \dfrac{2}{(a + x) \log_e a}$.

3. $u = e^x (x - 1)$. *Ans.* $\dfrac{du}{dx} = x e^x$.

4. $u = e^x (x^2 - 2x + 2)$. *Ans.* $\dfrac{du}{dx} = x^2 e^x$.

5. $u = e^x \log x$. *Ans.* $\dfrac{du}{dx} = e^x (\log x + x^{-1})$.

6. $u = \dfrac{e^x - 1}{e^x + 1}$. *Ans.* $\dfrac{du}{dx} = \dfrac{2 e^x}{(e^x + 1)^2}$.

7. $u = \log \dfrac{e^x - 1}{e^x + 1}$. *Ans.* $\dfrac{du}{dx} = \dfrac{2 e^x}{e^{2x} - 1}$.

8. $u = 2 e^{\sqrt{x}} \left(x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\right)$. *Ans.* $\dfrac{du}{dx} = x e^{\sqrt{x}}$.

9. $u = \log \{x + a + \sqrt{(2ax + x^2)}\}$. *Ans.* $\dfrac{du}{dx} = \dfrac{1}{\sqrt{(2ax + x^2)}}$.

10. $u = \log \{x \sqrt{-1} - \sqrt{(a^2 - x^2)}\}$. *Ans.* $\dfrac{du}{dx} = \dfrac{1}{\sqrt{(x^2 - a^2)}}$.

11. $u = \log \dfrac{x}{\sqrt{(x^2 + 1)} + x}$. *Ans.* $\dfrac{du}{dx} = \dfrac{1}{x} - \dfrac{1}{\sqrt{(x^2 + 1)}}$.

12. $u = \log \left\{\dfrac{(x + 2)^2}{\sqrt{(x + 1)}\,(x + 3)^{\frac{3}{2}}}\right\}$. *Ans.* $\dfrac{du}{dx} = \dfrac{x}{x^3 + 6x^2 + 11x + 6}$.

13. $u = \log \dfrac{\sqrt{(x^2 + 1)} - 1}{\sqrt{(x^2 + 1)} + 1}$. *Ans.* $\dfrac{du}{dx} = \dfrac{2}{x \sqrt{(x^2 + 1)}}$.

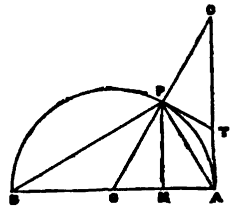
14. $u = e^x \sqrt{\dfrac{1 + x}{1 - x}}$. *Ans.* $\dfrac{du}{dx} = e^x . \dfrac{2 - x^2}{1 - x^2} . \sqrt{\dfrac{1 + x}{1 - x}}$.

II. *Trigonometrical or Circular Functions.*

20. Before proceeding to the differentiation of the trigonometrical or circular functions, it will be necessary to prove the following properties:—

The limiting ratio of the arc, chord, tangent, and sine of a circular arc is a ratio of equality.

Let A P be an arc of a circle whose centre is O, and diameter A B. Draw the tangents A C and P T; then C P T is a right angle, and C T is greater than P T; hence A C is greater than A T and P T together, and the arc A P, which is greater than the chord A P, but less than A T and T P together, is also greater than the chord A P, but less than the tangent A C. Now by similar triangles, O A C and O M P, we have



$$\frac{\text{A C}}{\text{P M}} = \frac{\text{O A}}{\text{O M}}; \text{ hence limit of } \frac{\text{A C}}{\text{PM}} = 1;$$

because when P approaches to A, then M also approaches to A.

Also, since $\frac{\text{A P}}{\text{PM}} = \frac{\text{A B}}{\text{B P}}$, the limit of the ratio $\frac{\text{A P}}{\text{P M}} = 1$, because as P approaches A, the magnitude of the line B P approaches to that of A B.

Hence, also, the limit of the ratio $\frac{\text{arc A P}}{\text{PM}} = 1$.

21. To find the differential of $u = \sin x$.

Here $u' = \sin (x + h) = \sin x + \sin (x + h) - \sin x$
$= \sin x + 2 \sin \frac{1}{2} h \cos (x + \frac{1}{2} h)$ . . . . (Trigonometry, Art. 16);

$$\therefore \frac{u' - u}{h} = \frac{2 \sin \frac{1}{2} h}{h} \cos (x + \frac{1}{2} h) = \frac{\sin \frac{1}{2} h}{\frac{1}{2} h} \cos (x + \frac{1}{2} h).$$

But (20) the limit of the ratio $\frac{\sin \frac{1}{2} h}{\frac{1}{2} h}$ is unity; hence

$$\frac{d u}{d x} = \cos x, \text{ and } d u = \cos x \, d x.$$

22. To find the differential of $u = \cos x$.

$$\begin{aligned} \text{Here } u' &= \cos (x + h) = \cos x + \cos (x + h) - \cos x \\ &= \cos x - \{\cos x - \cos (x + h)\} \end{aligned}$$

$= \cos x - 2 \sin \frac{1}{2} h \sin (x + \frac{1}{2} h)$ . . . . (Trigonometry, Art. 16);

$$\therefore \frac{u' - u}{h} = - \frac{2 \sin \frac{1}{2} h}{h} \sin (x + \frac{1}{2} h) = - \frac{\sin \frac{1}{2} h}{\frac{1}{2} h} \sin (x + \frac{1}{2} h).$$

Taking the limits of the ratios, we get, as in (21),

$$\frac{d u}{d x} = - \sin x, \text{ and } d u = - \sin x \, d x.$$

23. To find the differential of $u = \tan x$.

Since $\sin x = \tan x \cos x$, by the principles of trigonometry, we have

$$d \sin x = \cos x \, . \, d \tan x + \tan x \, . \, d \cos x,$$
$$\text{or } \cos x \, d x = \cos x \, . \, d \tan x - \tan x \sin x \, d x.$$

Transposing and dividing by $\cos x$, gives

$$d \tan x = (1 + \tan^2 x) \, d x = \sec^2 x \, d x = \frac{d x}{\cos^2 x}.$$

Similarly, $d \cot x = d \frac{1}{\tan x} = - \operatorname{cosec}^2 x \, d x = - \frac{d x}{\sin^2 x}$.

24. To find the differential of $u = \sec x$.

Since by trigonometry $\sec x \cos x = 1$; therefore, by differentiating, $\cos x \, d \sec x + \sec x \, d \cos x = 0$, or $\cos x \, d \sec x - \sec x \sin x \, d x = 0$; therefore, by transposing and dividing by $\cos x$, we have

$$d \sec x = \sec x \tan x \, d x.$$

Similarly, $\quad d \operatorname{cosec} x = - \operatorname{cosec} x \cot x \, d x.$

25. To find the differential of $u = \operatorname{vers} x$.

Since $\operatorname{vers} x = 1 - \cos x$; therefore $d u = d (1 - \cos x) = \sin x \, a x$;

$$\therefore \frac{d u}{d x} = \sin x, \text{ and } d u = \sin x \, d x.$$

Inverse Trigonometrical Functions.

26. In all these investigations respecting trigonometrical functions, the arc has been considered as the independent variable, and the sine, cosine, tangent, etc., as functions of it. But an arc may also be regarded as a function of its sine, cosine, tangent, etc., and the following very convenient notation, due to Sir John Herschel, is now generally employed to designate such inverse functions. Thus if $y = \sin x$, the inverse notation will be $x = \sin^{-1} y$, the former expressing the idea that y is equal to the sine of the arc denoted by x, and the latter that x *is equal to the arc whose sine is* y. In like manner, if $u = \tan^{-1} x$, then u is the arc whose tangent is equal to x.

27. Before proceeding to the differentiation of these inverse trigonometrical functions, it will be necessary to prove the following

Relation between Inverse Differential Coefficients.

If $y = f x$, then will $\dfrac{dy}{dx} \cdot \dfrac{dx}{dy} = 1$, or $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$.

For if y be a function of x, then inversely x will also be a function of y; now if h and k be the simultaneous increments of x and y, then as h approaches 0, k will also approach 0, and by common algebra,

$$\frac{y' - y}{x' - x} \cdot \frac{x' - x}{y' - y} = 1, \text{ or } \frac{y' - y}{x' - x} = \frac{1}{\frac{x' - x}{y' - y}}.$$

Now the limiting value of $\dfrac{y' - y}{x' - x}$ or $\dfrac{dy}{dx}$ may be found when h, the increment of x, approaches 0. In like manner, since x is a function of y, the limiting value of $\dfrac{x' - x}{y' - y}$ or $\dfrac{dx}{dy}$ may be found when k, the increment of y, approaches 0, which it does when h approaches 0; hence, taking the limiting value of both ratios, we get

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1, \text{ or } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Ex. Let $y = \dfrac{x}{x+1}$; then will $x = \dfrac{y}{1-y}$. Now, when x is changed into $x + h$, then

$$y' - y = \frac{x+h}{x+h+1} - \frac{x}{x+1} = \frac{h}{(x+1)(x+h+1)};$$

$$\frac{y' - y}{h} = \frac{1}{(x+1)(x+h+1)}; \text{ and } \therefore \frac{dy}{dx} = \frac{1}{(x+1)^2} = \frac{y^2}{x^2}.$$

Again, since $x = \dfrac{y}{1-y}$, and y is changed into $y + k$;

$$\therefore x' - x = \frac{y+k}{1-(y+k)} - \frac{y}{1-y} = \frac{k}{(1-y)(1-y-k)};$$

$$\therefore \frac{x' - x}{k} = \frac{1}{(1-y)(1-y-k)}, \text{ and hence } \frac{dx}{dy} = \frac{1}{(1-y)^2} = \frac{x^2}{y^2}.$$

Hence $$\frac{dy}{dx} \cdot \frac{dx}{dy} = \frac{y^2}{x^2} \cdot \frac{x^2}{y^2} = 1.$$

28. Let $u = \sin^{-1}\frac{x}{a}$.

The direct function is $\sin u = \frac{x}{a}$, and differentiating both members,

$$\frac{dx}{a} = \cos u\, du = (1 - \sin^2 u)^{\frac{1}{2}} du = \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}.du = \frac{\sqrt{(a^2 - x^2)}}{a} du;$$

$$\therefore \frac{dx}{du} = \sqrt{(a^2 - x^2)}, \text{ and } \frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{\sqrt{(a^2 - x^2)}}.$$

29. Let $u = \cos^{-1}\frac{x}{a}$.

The direct function is $\frac{x}{a} = \cos u$; hence, by differentiating,

$$\frac{dx}{a} = -\sin u\, du = -(1 - \cos^2 u)^{\frac{1}{2}}.du = -\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}.du$$

$$= -\frac{\sqrt{(a^2 - x^2)}}{a}.du;$$

$$\therefore \frac{du}{dx} = -\frac{1}{\sqrt{(a^2 - x^2)}}, \text{ and } du = -\frac{dx}{\sqrt{(a^2 - x^2)}}.$$

30. Let $u = \tan^{-1}\frac{x}{a}$.

The direct function is $\frac{x}{a} = \tan u$; hence, differentiating, we have

$$\frac{dx}{a} = \sec^2 u\, du = (1 + \tan^2 u)\, du = \left(1 + \frac{x^2}{a^2}\right) du = \frac{a^2 + x^2}{a^2} du;$$

$$\therefore \frac{du}{dx} = \frac{a}{a^2 + x^2}, \text{ and } du = \frac{a\,dx}{a^2 + x^2}.$$

31. Let $u = \cot^{-1}\frac{x}{a}$.

Here $\frac{x}{a} = \cot u$, $\frac{dx}{a} = -\operatorname{cosec}^2 u\, du = -\left(1 + \frac{x^2}{a^2}\right) du = -\frac{a^2 + x^2}{a^2} du$;

$$\therefore \frac{du}{dx} = -\frac{a}{a^2 + x^2}, \text{ and } du = -\frac{a\, dx}{a^2 + x^2}.$$

32. Let $u = \sec^{-1}\frac{x}{a}$.

Here $\frac{x}{a} = \sec u$, and $\frac{dx}{a} = \sec u \tan u\ du = \frac{x}{a}\sqrt{\left(\frac{x^2}{a^2} - 1\right)} du$;

$$\therefore \frac{du}{dx} = \frac{a}{x\sqrt{(x^2 - a^2)}}, \text{ and } du = \frac{a\, dx}{x\sqrt{(x^2 - a^2)}}.$$

Similarly, if $u = \operatorname{cosec}^{-1}\frac{x}{a}$; then $du = -\frac{a\,dx}{x\sqrt{(x^2 - a^2)}}$.

33. Let $u = \operatorname{vers}^{-1}\frac{x}{a}$.

Here $\frac{x}{a} = \operatorname{vers} u$, and $\frac{dx}{a} = \sin u\,du = \sqrt{(1 - \cos^2 u)}\,du$;

or $\frac{dx}{a} = \sqrt{\{1 - (1 - \operatorname{vers} u)^2\}}\,du = \sqrt{(2\operatorname{vers} u - \operatorname{vers}^2 u)}\,du$

$$= \sqrt{\left(\frac{2x}{a} - \frac{x^2}{a^2}\right)}\,du = \frac{\sqrt{(2ax - x^2)}}{a}\,du;$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{(2ax - x^2)}}, \text{ and } du = \frac{dx}{\sqrt{(2ax - x^2)}}.$$

34. In the preceding differentiations of the inverse trigonometrical functions, the radius of the circle is a, and they may all be expressed to radius unity, by making $a = 1$ in the several differentials.

Differentiation of a function of a function.

35. If $u = fv$, and $v = \phi x$, then will $\frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx}$.

Let x be changed into $x + h$, and let the corresponding changes on v and u be denoted by $v + k$ and u'; then by common algebra, we have

$$\frac{u' - u}{h} = \frac{u' - u}{k}\cdot\frac{k}{h}.$$

But when h, and consequently k, approaches 0, the preceding equation becomes

$$\frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx}.$$

Ex. Let $u = \log\sin x$.
Assume $v = \sin x$, then $u = \log v$, and differentiating these,

$$\frac{dv}{dx} = \cos x, \text{ and } \frac{du}{dv} = \frac{1}{v} = \frac{1}{\sin x};$$

$$\therefore \frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx} = \frac{1}{\sin x}\cdot\cos x = \cot x.$$

36. The principles which have been investigated for the differentiation of transcendental and compound functions will be exemplified presently, and we may now collect the various formulas for the differentiation of all functions into the following table.

TABLE OF THE DIFFERENTIALS OF FUNCTIONS.

$dx^n = nx^{n-1}\,dx$,	$d(x^n \pm c) = nx^{n-1}\,dx$.
$d(yz) = y\,dz + z\,dy$,	$d.\frac{y}{z} = \frac{z\,dy - y\,dz}{z^2}$.
$d\log_a x = \frac{dx}{\log_e a . x}$.	$d\log_e x = \frac{dx}{x.}$.
$da^x = a^x\,dx\log_e a$.	$de^x = e^x\,dx$.

$$d\sin x = \cos x\, dx, \qquad d\cos x = -\sin x\, dx.$$
$$d\tan x = \sec^2 x\, dx, \qquad d\cot x = -\operatorname{cosec}^2 x\, dx.$$
$$d\sec x = \sec x \tan x\, dx, \qquad d\operatorname{cosec} x = -\operatorname{cosec} x \cot x\, dx.$$
$$d\operatorname{vers} x = \sin x\, dx. \qquad d\operatorname{covers} x = -\cos x\, dx.$$
$$d\sin^{-1}\frac{x}{a} = \frac{dx}{\sqrt{(a^2-x^2)}} \qquad d\cos^{-1}\frac{x}{a} = -\frac{dx}{\sqrt{(a^2-x^2)}}.$$
$$d\tan^{-1}\frac{x}{a} = \frac{a\,dx}{a^2+x^2}. \qquad d\cot^{-1}\frac{x}{a} = -\frac{a\,dx}{a^2+x^2}.$$
$$d\sec^{-1}\frac{x}{a} = \frac{a\,dx}{x\sqrt{(x^2-a^2)}}. \qquad d\operatorname{cosec}^{-1}\frac{x}{a} = -\frac{a\,dx}{x\sqrt{(x^2-a^2)}}.$$
$$d\operatorname{vers}^{-1}\frac{x}{a} = \frac{dx}{\sqrt{(2ax-x^2)}}, \qquad d\operatorname{covers}^{-1}\frac{x}{a} = -\frac{dx}{\sqrt{(2ax-x^2)}}.$$

37. When the radius of the circle is unity, the last eight differentials become

$$d\sin^{-1}x = \frac{dx}{\sqrt{(1-x^2)}}, \qquad d\cos^{-1}x = -\frac{dx}{\sqrt{(1-x^2)}},$$
$$d\tan^{-1}x = \frac{dx}{1+x^2}, \qquad d\cot^{-1}x = -\frac{dx}{1+x^2}.$$
$$d\sec^{-1}x = \frac{dx}{x\sqrt{(x^2-1)}}, \qquad d\operatorname{cosec}^{-1}x = -\frac{dx}{x\sqrt{(x^2-1)}}.$$
$$d\operatorname{vers}^{-1}x = \frac{dx}{\sqrt{(2x-x^2)}}, \qquad d\operatorname{covers}^{-1}x = -\frac{dx}{\sqrt{(2x-x^2)}}.$$

Examples.

1. Let $u = \sin 3x - \cos 2x + \sin x$.

Here $du = \cos 3x\, d(3x) + \sin 2x\, d(2x) + \cos x\, dx$

$= 3\cos 3x\, dx + 2\sin 2x\, dx + \cos x\, dx;$

$$\therefore \frac{du}{dx} = 3\cos 3x + 2\sin 2x + \cos x.$$

2. Let $u = \sin 2x \cos x$.

$$\frac{du}{dx} = 2\cos 2x\cos x - \sin 2x \sin x = \cos 2x \cos x + \cos(2x+x),$$
$$= \cos 2x \cos x + \cos 3x.$$

3. Let $u = x - \sin x \cos x$.

Here $\dfrac{du}{dx} = 1 - \cos^2 x + \sin^2 x = 2\sin^2 x.$

4. Let $u = \log\sqrt{\left(\dfrac{1+\cos x}{1-\cos x}\right)} = \frac{1}{2}\log(1+\cos x) - \frac{1}{2}\log(1-\cos x).$

Here $\dfrac{du}{dx} = \dfrac{1}{2}\cdot\dfrac{-\sin x}{1+\cos x} - \dfrac{1}{2}\cdot\dfrac{\sin x}{1-\cos x} = -\dfrac{\sin x}{2}\left(\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x}\right),$

or, $\dfrac{du}{dx} = -\dfrac{\sin x}{2}\cdot\dfrac{2}{1-\cos^2 x} = -\dfrac{\sin x}{\sin^2 x} = -\dfrac{1}{\sin x}.$

5. Let $u = \sin^{-1}\dfrac{x}{\sqrt{(1+x^2)}}$.

The direct function is $\sin u = \dfrac{x}{\sqrt{(1+x^2)}}$ and $\sin^2 u = \dfrac{x^2}{1+x^2}$;

therefore, $\cos u = \sqrt{(1-\sin^2 u)} = \sqrt{\left(1 - \dfrac{x^2}{1+x^2}\right)} = \dfrac{1}{\sqrt{(1+x^2)}}$;

consequently, $\tan u = \dfrac{\sin u}{\cos u} = \dfrac{x}{\sqrt{(1+x^2)}} \times \sqrt{(1+x^2)} = x$.

Differentiating this, we get $\sec^2 u\, du = dx$; hence

$$\frac{du}{dx} = \frac{1}{\sec^2 u} = \cos^2 u = \frac{1}{1+x^2}, \text{ and } du = \frac{dx}{1+x^2}.$$

The inverse trigonometrical functions may be put in a variety of forms; for if $\sin u = 2x\sqrt{(1-x^2)}$; then $\cos u = \sqrt{(1-\sin^2 u)} = 1 - 2x^2$, $\text{vers}\, u = 1 - \cos u = 2x^2$, $\tan u = \dfrac{2x\sqrt{(1-x^2)}}{1-2x^2}$, $\sec u = \dfrac{1}{1-2x^2}$, etc. Hence $u = \sin^{-1} 2x\sqrt{(1-x^2)}$, $u = \cos^{-1}(1-2x^2)$, $u = \text{vers}^{-1} 2x^2$,

$$u = \tan^{-1}\frac{2x\sqrt{(1-x^2)}}{1-2x^2},\quad u = \sec^{-1}\frac{1}{1-2x^2}, \text{ etc.}$$

are only different forms of the same inverse function, and all these have the same differential coefficient. In a similar manner we see, that if $u = \sin^{-1}\dfrac{x}{\sqrt{(1+x^2)}}$; then $u = \cos^{-1}\dfrac{1}{\sqrt{(1+x^2)}}$, and $u = \tan^{-1} x$, as in the present example; and it is much easier to differentiate the latter form than either of the others.

6. Let $u = \sin^{-1} 2x\sqrt{(1-x^2)}$.

Here $\sin u = 2x\sqrt{(1-x^2)}$, $\cos u = 1 - 2x^2$, and differentiating this, we get $\quad -\sin u\, du = -4x\, dx$;

$$\therefore \frac{du}{dx} = \frac{4x}{\sin u} = \frac{4x}{2x\sqrt{(1-x^2)}} = \frac{2}{\sqrt{(1-x^2)}}, \text{and } du = \frac{2\,dx}{\sqrt{(1-x^2)}}.$$

7. Let $u = \cos^{-1}\dfrac{b + a\cos x}{a + b\cos x}$.

Here $\cos u = \dfrac{b + a\cos x}{a + b\cos x}$, and $\sin u = \sqrt{\left\{1 - \left(\dfrac{b + a\cos x}{a + b\cos x}\right)^2\right\}}$,

or, $\quad \sin u = \sqrt{\left\{\dfrac{(a + b\cos x)^2 - (b + a\cos x)^2}{(a + b\cos x)^2}\right\}} = \dfrac{\sqrt{(a^2-b^2)}\sin x}{a + b\cos x}$.

Differentiating the value of $\cos u$, we have

$$-\sin u\, du = \frac{(a+b\cos x)\, d(b+a\cos x) - (b + a\cos x)\, d(a+b\cos x)}{(a + b\cos x)^2},$$

$$= \frac{-a\sin x\,(a+b\cos x)\, dx + b\sin x\,(b+a\cos x)\, dx}{(a + b\cos x)^2},$$

$$= \frac{-(a^2-b^2)\sin x\, dx}{(a + b\cos x)^2} = \frac{-\sqrt{(a^2-b^2)}\, dx}{a + b\cos x}\,.\sin u;$$

consequently, $\dfrac{du}{dx} = \dfrac{\sqrt{(a^2-b^2)}}{a + b\cos x}$, and $du = \dfrac{\sqrt{(a^2-b^2)}\, dx}{a + b\cos x}$.

8. Let $y = \sqrt{(2ax - x^2)} + a \operatorname{vers}^{-1}\dfrac{x}{a}$, which is the equation of the cycloid.

Here, $dy = \dfrac{(a-x)\,dx}{\sqrt{(2ax - x^2)}} + \dfrac{a\,dx}{\sqrt{(2ax - x^2)}} = \dfrac{\sqrt{(2a-x)}\,dx}{\sqrt{x}}$;

$$\therefore \frac{dy}{dx} = \sqrt{\left(\frac{2a-x}{x}\right)}, \text{ and } dy = \sqrt{\left(\frac{2a-x}{x}\right)}\,dx.$$

9. Let $u = \cos^{-1}(4x^3 - 3x)$.

Here, $\cos u = 4x^3 - 3x$, and $\sin u = \sqrt{\{1 - (4x^3 - 3x)^2\}}$,

or, $\sin u = \sqrt{\{(1 + 4x^3 - 3x)(1 - 4x^3 + 3x)\}}$

$= \sqrt{\{1 + x)(2x-1)^2(1-x)(2x+1)^2\}}$

$= (2x-1)(2x+1)\sqrt{\{(1+x)(1-x)\}}$

$= (4x^2 - 1)\sqrt{(1-x^2)}$.

But since $\cos u = 4x^3 - 3x$, we get, by differentiation,

$$-\sin u\,du = 12x^2\,dx - 5\,dx = 3(4x^2 - 1)\,dx;$$

$$\therefore \frac{du}{dx} = \frac{-3(4x^2-1)}{\sin u} = \frac{-3(4x^2-1)}{(4x^2-1)\sqrt{(1-x^2)}} = -\frac{3}{\sqrt{(1-x^2)}}.$$

Or thus:—Since by trigonometry $\cos 3u = 4\cos^3 u - 3\cos u$,

or, $\cos u = 4\cos^3\dfrac{u}{3} - 3\cos\dfrac{u}{3}$; hence, $x = \cos\dfrac{u}{3}$, and $u = 3\cos^{-1}x$;

$$\therefore \frac{du}{dx} = 3\,d\cos^{-1}x = -\frac{3}{\sqrt{(1-x^2)}}, \text{ as before.}$$

10. Let $u = \tan^m x \sec^n x$.

Take the logarithms of both members and differentiate,

then, $\log u = m \log \tan x + n \log \sec x$,

$$\therefore \frac{du}{u} = m\frac{\sec^2 x\,dx}{\tan x} + n\frac{\sec x \tan x\,dx}{\sec x},$$

$$= \frac{m(1 + \tan^2 x)\,dx + n\tan^2 x\,dx}{\tan x}.$$

Hence, $\dfrac{du}{dx} = \dfrac{u}{\tan x}\{m + (m+n)\tan^2 x\}$

$= \tan^{m-1}x \sec^n x\,\{m + (m+n)\tan^2 x\}$.

Thus we see, that when a function consists of products and quotients of roots and powers, it is generally advantageous first to take the logarithms of both members, and then to differentiate the result by the principles investigated for the differentiation of logarithmic functions.

Additional Exercises.

1. $u = x\sin x + \cos x$. *Ans.* $\dfrac{du}{dx} = x\cos x$.

2. $u = 2x\sin x + (2 - x^2)\cos x$. *Ans.* $\dfrac{du}{dx} = x^2 \sin x$.

3. $u = e^{\sin x}$. *Ans.* $\dfrac{du}{dx} = e^{\sin x}\cos x$.

4. $u = \tan x - x$. *Ans.* $\dfrac{du}{dx} = \tan^2 x$.

5. $u = e^{\cos x}\sin x.$ *Ans.* $\dfrac{du}{dx} = e^{\cos x}(\cos x - \sin^2 x).$

6. $u = \dfrac{\sin x}{1 + \tan x}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{\cos^3 x - \sin^3 x}{1 + \sin 2x}.$

7. $u = 2\log\sin x + \operatorname{cosec}^2 x.$ *Ans.* $\dfrac{du}{dx} = -2\cot^3 x.$

8. $u = \sqrt{(a^2 - x^2)} + a\sin^{-1}\dfrac{x}{a}.$ *Ans.* $\dfrac{du}{dx} = \left(\dfrac{a - x}{a + x}\right)^{\frac{1}{2}}.$

9. $u = e^{ax}\sin rx.$ *Ans.* $\dfrac{du}{dx} = e^{ax}(a\sin rx + r\cos rx).$

10. $u = \tan^{-1}\dfrac{2x}{1 - x^2}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{2}{1 + x^2}.$

11. $u = \log\left(\dfrac{1 + x}{1 - x}\right)^{\frac{1}{4}} + \dfrac{1}{2}\tan^{-1}x.$ *Ans.* $\dfrac{du}{dx} = \dfrac{1}{1 - x^4}.$

12. $u = \dfrac{1}{2\sqrt{ab}}\sin^{-1}\dfrac{2x\sqrt{ab}}{a + bx^2}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{1}{a + bx^2}.$

13. $u = \sin^{-1}(3x - 4x^3).$ *Ans.* $\dfrac{du}{dx} = \dfrac{3}{\sqrt{(1 - x^2)}}.$

14. $u = \tan^{-1}\dfrac{\sqrt{(1 + x^2)} - 1}{x}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{1}{2(1 + x^2)}.$

15. $u = \log\left(\dfrac{1 + \sqrt{-1}\tan x}{1 - \sqrt{-1}\tan x}\right)^n.$ *Ans.* $\dfrac{du}{dx} = 2n\sqrt{-1}.$

16. $u = \log(\cos x + \sqrt{-1}\sin x).$ *Ans.* $\dfrac{du}{dx} = \sqrt{-1}.$

17. $u = \log\left(\dfrac{1 + \sin x}{1 - \sin x}\right)^{\frac{1}{2}}.$ *Ans.* $\dfrac{du}{dx} = \sec x.$

18. $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right).$ *Ans.* $\dfrac{du}{dx} = \sec x.$

19. $u = \sin^{-1}\dfrac{1 - x^2}{1 + x^2}.$ *Ans.* $\dfrac{du}{dx} = -\dfrac{2}{1 + x^2}.$

20. $u = x^m e^{\sin x}.$ *Ans.* $\dfrac{du}{dx} = x^{m-1}e^{\sin x}(m + x\cos x).$

21. $u = \log\left(\dfrac{1 - \cos mx}{1 + \cos mx}\right)^{\frac{1}{2}}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{m}{\sin mx}.$

22. $u = \log(xe^{\cos x}).$ *Ans.* $\dfrac{du}{dx} = \dfrac{1}{x} - \sin x.$

23. $u = \cot^{-1}\sqrt{\dfrac{1 - x}{x}}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{(x - x^2)}}.$

24. $u = \dfrac{1 - \cos x}{\cos^2 x}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{\sin x(2 - \cos x)}{\cos^3 x}.$

25. $u = \log(x\sin x + \cos^2 x).$ *Ans.* $\dfrac{du}{dx} = \dfrac{\sin x - \sin 2x + x\cos x}{x\sin x + \cos^2 x}.$

26. $u = \frac{2}{\sqrt{b}} \tan^{-1} \frac{x\sqrt{b}}{\sqrt{(ax - bx^2)}}$. *Ans.* $\frac{du}{dx} = \frac{1}{\sqrt{(ax - bx^2)}}$.

27. $u = \frac{1}{\sqrt{ab}} \cos^{-1} \frac{a - bx}{a + bx}$. *Ans.* $\frac{du}{dx} = \frac{1}{(a + bx)\sqrt{x}}$.

28. $u = \tan^{-1} \frac{b + 2cx}{\sqrt{(4ac - b^2)}}$. *Ans.* $\frac{du}{dx} = \frac{\sqrt{(4ac - b^2)}}{2(a + bx + cx^2)}$.

29. $u = \frac{\sin 5x}{5} + \frac{\sin 7x}{7} - \frac{\sin x}{1} - \frac{\sin 11x}{11}$.

Ans. $\frac{du}{dx} = 4 \sin 2x \sin 3x \cos 6x$.

30. $u = \frac{1}{4\sqrt{2}} \log \frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1 - x^2}$.

Ans. $\frac{du}{dx} = \frac{1}{1 + x^4}$.

SUCCESSIVE DIFFERENTIATION.

38. If the differential coefficient of a function be variable, it may be submitted to the process of differentiation, and its differential coefficient, if still variable, may also be differentiated, and so on, as far as we choose. But if by successive differentiations we arrive at a *constant* differential coefficient, the process will terminate, since the differential of a constant quantity is zero. The differential dx, of the independent variable x, is regarded as constant in all these differentiations, and *the differential of the differential of a function* is called the *second differential of the function,* and denoted by d^2 prefixed to the function. Thus $d\,du$, the differential of du, is written d^2u, and is called the *second* differential of the original function u. In a similar manner the *third, fourth, fifth,* and in general the n^{th} differentials of u are denoted by d^3u, d^4u, d^5u, and d^nu, respectively. The powers of dx are written dx^2, dx^3, dx^4, ... dx^n, and must be carefully distinguished from $d(x^2)$, $d(x^3)$, etc., or $d.x^2$, $d.x^3$, ... $d.x^n$, which indicate the differentials of x^2, x^3, ... x^n.

EXAMPLES.

1. Let $u = ax^n$.

Here $du = nax^{n-1}dx$, and $\frac{du}{dx} = nax^{n-1}$,

$d^2u = n(n-1)ax^{n-2}dx^2$, $\quad \frac{d^2u}{dx^2} = n(n-1)ax^{n-2}$,

$d^3u = n(n-1)(n-2)ax^{n-3}dx^3$, $\quad \frac{d^3u}{dx^3} = n(n-1)(n-2)ax^{n-3}$,

etc. etc.

$d^nu = n(n-1)(n-2) \ldots . 3.2.1\,a\,dx^n$,

or $\frac{d^nu}{dx^n} = n(n-1)(n-2) \ldots . 3.2.1\,a$ = constant.

2. Find the fourth differential coefficient of $u = ax^4 + bx^3 + cx^2 + ex + f$. *Ans.* $\frac{d^4u}{dx^4} = 24a$.

3. Find the n^{th} differential coefficient of $u = e^{ax}$. *Ans.* $\frac{d^n u}{d x^n} = a^n e^{ax}$.

4. Find the fourth differential coefficient of $u = \sin(ax + b)$.

Ans. $\frac{d^4 u}{d x^4} = a^4 \sin(ax + b)$.

5. Find the fourth and fifth differential coefficients of $x^4 - \frac{1}{x^4}$.

Ans. $\frac{d^4 u}{d x^4} = 2.3.4 - \frac{4.5.6.7}{x^8}, \frac{d^5 u}{d x^5} = \frac{4.5.6.7.8}{x^9}$.

6. Find the fourth and fifth differential coefficients of $u = \sin nx$.

Ans. $\frac{d^4 u}{d x^4} = n^4 \sin nx, \frac{d^5 u}{d x^5} = n^5 \cos nx$.

7. Find the n^{th} differential coefficient of $u = x e^x$.

Ans. $\frac{d^n u}{d x^n} = (n + x) e^x$.

MACLAURIN'S THEOREM.

39. If $u = fx$ be any function of x, which can be expanded in a series of the form

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots,$$

where A, B, C, etc., are constants independent of x; then will

$$u = U_0 + U_1 x + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{1.2.3} + U_4 \frac{x^4}{1.2.3.4} + \text{etc.},$$

where U_0, U_1, U_2, etc., are the values of $u, \frac{du}{dx}, \frac{d^2 u}{dx^2}$, etc., when $x = 0$.

Since

$$\begin{aligned}
u &= A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \\
\therefore \frac{du}{dx} &= B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots \\
\frac{d^2 u}{dx^2} &= 2C + 2.3Dx + 3.4Ex^2 + \ldots \\
\frac{d^3 u}{dx^3} &= 2.3D + 2.3.4Ex + \ldots \\
\text{etc.} & \qquad\qquad \text{etc.}
\end{aligned}$$

Now make $x = 0$ in all these equations, then we get

$$U_0 = A,\ U_1 = B,\ U_2 = 2C,\ U_3 = 2.3D, \text{ etc.},$$

$$\therefore A = U_0,\ B = U_1,\ C = \frac{U_2}{1.2},\ D = \frac{U_3}{1.2.3}, \text{ etc.},$$

$$\therefore u = U_0 + U_1 x + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{1.2.3} + U_4 \frac{x^4}{1.2.3.4} + \ldots$$

EXAMPLES.

1. Expand $\sin x$ and $\cos x$ in terms of x.

Let $u = \sin x$,	Let $u = \cos x$,
$\frac{du}{dx} = \cos x$,	$\frac{du}{dx} = -\sin x$,

$$\frac{d^2 u}{d x^2} = -\sin x, \qquad \frac{d^2 u}{d x^2} = -\cos x,$$

$$\frac{d^3 u}{d x^3} = -\cos x, \qquad \frac{d^3 u}{d x^3} = \sin x,$$

$$\frac{d^4 u}{d x^4} = \sin x, \qquad \frac{d^4 u}{d x^4} = \cos x,$$

etc. etc.

In these functions and their successive differential coefficients, let $x = 0$, then for $\sin x$, $U_0 = 0$, $U_1 = 1$, $U_2 = 0$, $U_3 = -1$, $U_4 = 0$, etc., for $\cos x$, $U_0 = 1$, $U_1 = 0$, $U_2 = -1$, $U_3 = 0$, $U_4 = 1$, etc.,

$$\therefore \sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3\ldots7} + \text{etc.},$$

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3\ldots6} + \text{etc.}$$

Since $d \sin x = \cos x\, d x$, and $d \cos x = -\sin x\, d x$, the series for $\cos x$ may be derived from that for $\sin x$ by the formula $\cos x = \frac{d \sin x}{d x}$, or the series for $\sin x$ from that for $\cos x$ by the formula, $\sin x = -\frac{d \cos x}{d x}$.

2. Let $u = \cos^3 x$.

By trigonometry, $\cos^3 x = \frac{1}{4}(3 \cos x + \cos 3 x)$, from which the successive differential coefficients may be readily determined. If we differentiate the proposed function in the usual manner, we get

$$\frac{d u}{d x} = -3 \cos^2 x \sin x, \text{ and therefore}$$

$$\frac{d^2 u}{d x^2} = 2.3 \cos x \sin^2 x - 3 \cos^3 x = 2.3 \cos x - 9 \cos^3 u = 6 \cos x - 9 u.$$

We may consequently find the successive differential coefficients by either of these processes as follows:

$$u = \cos^3 x, \qquad \text{or } u = \tfrac{1}{4}(3 \cos x + \cos 3 x),$$

$$\frac{d u}{d x} = -3 \cos^2 x \sin x, \qquad \frac{d u}{d x} = -\tfrac{3}{4}(\sin x + \sin 3 x),$$

$$\frac{d^2 u}{d x^2} = 6 \cos x - 9 u, \qquad \frac{d^2 u}{d x^2} = -\tfrac{3}{4}(\cos x + 3 \cos 3 x),$$

$$\frac{d^3 u}{d x^3} = -6 \sin x - 9 \frac{d u}{d x}, \qquad \frac{d^3 u}{d x^3} = \tfrac{3}{4}(\sin x + 3^2 \sin 3 x),$$

$$\frac{d^4 u}{d x^4} = -6 \cos x - 9 \frac{d^2 u}{d x^2}, \qquad \frac{d^4 u}{d x^4} = \tfrac{3}{4}(\cos x + 3^3 \cos 3 x),$$

etc. etc.

If $x = 0$ in all these, the values of the odd differential coefficients will be zero, and hence by the formula

$$\cos^3 x = 1 - \tfrac{3}{4}\left\{\frac{1+3}{1.2} x^2 - \frac{1+3^3}{1.2.3.4} x^4 + \frac{1+3^5}{1.2\ldots6} x^6 - \frac{1+3^7}{1.2\ldots8} x^8 + \ldots (-1)^{\frac{n}{2}} \cdot \frac{1+3^{n-1}}{1.2\ldots n} x^n\right\}$$

$$= 1 - \tfrac{1}{2} x^2 + \frac{21}{24} x^4 - \frac{61}{240} x^6 + \frac{547}{13440} x^8 -, \text{ etc.}$$

3. Let $u = \sin^{-1} x$.

Then

$$\frac{du}{dx} = \frac{1}{\sqrt{(1-x^2)}} = (1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2} x^2 + \frac{1.3}{2.4} x^4 + \frac{1.3.5}{2.4.6} x^6 + \ldots . (1),$$

$$\frac{d^2 u}{dx^2} = x + \frac{1.3}{2} x^3 + \frac{1.3.5}{2.4} x^5 + \frac{1.3.5.7}{2.4.6} x^7 + \ldots .$$

$$\frac{d^3 u}{dx^3} = 1 + \frac{1.3^2}{2} x^2 + \frac{1.3.5^2}{2.4} x^4 + \frac{1.3.5.7^2}{2.4.6} x^6 + \ldots .$$

$$\frac{d^4 u}{dx^4} = 1.3^2 x + \frac{1.3.5^2}{2} x^3 + \frac{1.3.5.7^2}{2.4} x^5 + \ldots .$$

$$\frac{d^5 u}{dx^5} = 1^2.3^2 + \frac{1.3^2.5^2}{2} x^2 + \frac{1.3.5^2.7^2}{2.4} x^4 + \ldots .$$

etc. etc.

Let $x = 0$ in all these, then we have

$$\sin^{-1} x = x + \frac{x^3}{1.2.3} + \frac{3^2.x^5}{1.2.3.4.5} + \frac{3^2.5^2.x^7}{1.2.\ldots.7} + \text{etc.} \ldots .$$

The simplest way of obtaining this development is to assume

$$u = \sin^{-1} x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \text{etc.} \ldots\ldots\ldots\ldots (2);$$

then $\dfrac{du}{dx} = c_1 + 2 c_2 x + 3 c_3 x^2 + 4 c_4 x^3 + 5 c_5 x^4 + \text{etc.}$

Comparing this with (1), we get

$$c_1 = 1,\ 2 c_2 = 0,\ 3 c_3 = \frac{1}{2},\ 4 c_4 = 0,\ 5 c_5 = \frac{1.3}{2.4}, \text{ etc.}$$

Hence $\quad c_1 = 1,\ c_2 = 0, c_3 = \dfrac{1}{2.3},\ c_4 = 0,\ c_5 = \dfrac{1.3}{2.4.5}, \text{ etc.};$

and when $x = 0$, then $c_0 = \sin^{-1} 0 = 0$. Substituting these values in (2),

$$\sin^{-1} x = x + \frac{x^3}{1.2.3} + \frac{3^2.x^5}{1.2.3.4.5} + \text{etc.}$$

Let $x = \sin 30^\circ = \dfrac{1}{2}$; then $\sin^{-1} x = \text{arc } 30^\circ = \dfrac{\pi}{6}$; hence

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{1.2.3}.\frac{1}{2^3} + \frac{3^2}{1.2.3.4.5}.\frac{1}{2^5} + \text{etc.} = \cdot 5235987756;$$

$\therefore \pi = 3.1415926536$, which expresses the ratio of the semi-circumference of a circle to its radius, or the ratio of the circumference to the diameter. Two right angles, or 180°, are also represented by π.

4. Let $u = \tan^{-1} x$.

Here $\quad \dfrac{du}{dx} = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \text{etc.}$

$$\therefore \quad \frac{d^2 u}{dx^2} = -2x + 4x^3 - 6x^5 + 8x^7 - 10x^9 + \ldots .$$

$$\frac{d^3 u}{dx^3} = -2 + 3.4x^2 - 5.6x^4 + 7.8x^6 - 9.10x^8 + \ldots.$$

$$\frac{d^4 u}{dx^4} = 2.3.4x - 4.5.6x^3 + 6.7.8x^5 - 8.9.10x^7 + \ldots.$$

etc. etc.

If $x = 0$, then $U_0 = 0$, $U_1 = 1$, $U_2 = 0$, $U_3 = -2$, $U_4 = 0$, $U_5 = 2.3.4$, etc.

$$\therefore \qquad \tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \text{etc.}$$

Let $\qquad x = \tan 45° = 1$, then $\tan^{-1}x = \text{arc } 45° = \frac{\pi}{4}$;

$$\text{hence} \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

$$= \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} + \frac{2}{17.19} + \text{etc.} = \cdot 785398.$$

This series is not so convergent as the former, and it is therefore not so well adapted for computation.

Examples for Practice.

1. $u = (a + x)^n$.
2. $u = \log_e(1 + x)$.
3. $u = a^x$.
4. $u = \sqrt{(1 + x^2)}$.
5. $u = \sec x$.
6. $u = e^{\sin x}$.

Answers.

1. $a^n + n a^{n-1} x + \frac{n(n-1)}{1.2} a^{n-2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} a^{n-3} x^3 + \text{etc.}$

2. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$

3. $1 + \frac{\log_e a}{1} x + \frac{(\log_e a)^2}{1.2} x^2 + \frac{(\log_e a)^3}{1.2.3} x^3 + \frac{(\log_e a)^4}{1.2.3.4} x^4 + \text{etc.}$

4. $1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \text{etc.}$

5. $1 + \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} + \frac{61x^6}{1.2.3\ldots6} + \frac{1385x^8}{1.2.3\ldots8} + \frac{50521x^{10}}{1.2.3\ldots10} + \text{etc.}$

6. $1 + x + \frac{x^2}{2} - \frac{3x^4}{2.3.4} - \frac{8x^5}{2.3.4.5} - \frac{3x^6}{1.2.3\ldots6} + \frac{47x^7}{1.2.3\ldots7} + \text{etc.}$

TAYLOR'S THEOREM.

40. Preparatory to the investigation of this important theorem, it is necessary to establish the following proposition :—

The differential coefficient of $u = f(x+y)$ is the same, whichever of the parts, x and y, is supposed to vary, and x and y are independent of each other.

For let $x + y$ receive the increment h, then will $u = f(x + y)$, $u' = f(x + y + h)$, and $\frac{u' - u}{h} = \frac{f(x + y + h) - f(x + y)}{h}$.

Now if h be continually diminished towards zero, the first member $\frac{u'-u}{h}$ will tend to become $\frac{du}{dx}$ or $\frac{du}{dy}$, according as h is regarded as the increment of x or of y; while the other member tends to the same limit in each case; consequently the two differential coefficients are identical, and $\frac{du}{dx}=\frac{du}{dy}$.

Let $u=(x+y)^3$; then supposing x to be variable and y constant, we have

$$\frac{du}{dx}=3\,(x+y)^2.$$

And if we regard y as variable, and x constant, then we get

$$\frac{du}{dy}=3\,(x+y)^2=\frac{du}{dx}.$$

41. Let u denote a function of x, and u' the new value of u when x becomes $x+h$, then will

$$u'=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3}+\frac{d^4u}{dx^4}\cdot\frac{h^4}{1.2.3.4}+\text{etc.}$$

For suppose that the development of $f(x+h)$ is of the form

$$f(x+h)=fx+\mathrm{A}\,h^a+\mathrm{B}\,h^b+\mathrm{C}\,h^c+\mathrm{E}\,h^e+\ldots.\ (1),$$

where the quantities A, B, C, E, etc., are functions of x independent of h, and a, b, c, e, etc., are constant indices to be determined. It will be readily seen that none of the indices a, b, c, etc., can be negative, for then the second member of (1) would involve a term of the form $\mathrm{Q}\,h^{-q}=\frac{\mathrm{Q}}{h^q}$, which when $h=0$, would be *infinite*, while the first member would become fx. Neither can the first term of (1) be different from fx, for when $h=0$, we have the identical equation $fx=fx$. Now suppose that the terms of (1) are arranged in the ascending order of their positive indices, and let us differentiate (1), first with respect to x, and then with respect to h; then

$$\frac{df(x+h)}{dx}=\frac{dfx}{dx}+\frac{d\mathrm{A}}{dx}h^a+\frac{d\mathrm{B}}{dx}h^b+\frac{d\mathrm{C}}{dx}h^c+\ldots.$$

$$\frac{df(x+h)}{dh}=a\mathrm{A}\,h^{a-1}+b\mathrm{B}\,h^{b-1}+c\mathrm{C}\,h^{c-1}+e\mathrm{E}\,h^{e-1}+\ldots$$

But by (40) the first members of these equations are identical, and therefore the series in the second members must also be identical; whence equating the indices of the several powers of h, and likewise the coefficients of the corresponding terms, we get

$$a-1=0,\ b-1=a,\ c-1=b,\ \text{etc.};\ \therefore a=1,\ b=2,\ c=3,\ \text{etc.}$$

Also, since $h^{a-1}=h^0=1$, we have, by equating the coefficients of the like powers of h,

$$a\mathrm{A}=\frac{dfx}{dx},\ b\mathrm{B}=\frac{d\mathrm{A}}{dx},\ c\mathrm{C}=\frac{d\mathrm{B}}{dx},\ e\mathrm{E}=\frac{d\mathrm{C}}{dx},\ \text{etc.};$$

$$\therefore \mathrm{A}=\frac{dfx}{dx},\ \mathrm{B}=\frac{1}{2}\frac{d\mathrm{A}}{dx}=\frac{1}{2}\frac{d^2fx}{dx^2},\ \mathrm{C}=\frac{1}{3}\frac{d\mathrm{B}}{dx}=\frac{1}{2.3}\frac{d^3fx}{dx^3},\ \text{etc.}$$

Hence

$$f(x+h) = fx + \frac{dfx}{dx}h + \frac{d^2fx}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3fx}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.}$$

$$\text{or } u' = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \frac{d^4u}{dx^4}\cdot\frac{h^4}{1.2.3.4} + \text{etc.}$$

This theorem is, perhaps, one of the most important in the whole range of pure mathematics. It was first discovered by the celebrated analyst Dr. Brook Taylor, and is known by the name of Taylor's Theorem. If h be negative, or $u' = f(x - h)$, then will

$$u' = u - \frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \frac{d^4u}{dx^4}\cdot\frac{h^4}{1.2.3.4} - \text{etc.}$$

Taylor's theorem is applicable to the general development of all functions of $x + h$, while no particular value is assigned to x; but it is evident that if a value be assigned to the variable x, such as to render any of the differential coefficients infinite, the theorem will not apply to such a case, and will not give the particular development, which must therefore be sought for by the ordinary operations of algebra.

42. Maclaurin's theorem may be readily deduced from Taylor's in the following manner. Let $x = 0$, then the coefficients $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, etc., which are functions of x, will become constants if $x = 0$, and denoting them as in Maclaurin's theorem, we have

$$fh = \text{U}_0 + \text{U}_1\frac{h}{1} + \text{U}_2\frac{h^2}{1.2} + \text{U}_3\frac{h^3}{1.2.3} + \text{etc.}$$

But this equation is true for all values of h, and U_0, U_1, U_2, etc., are independent of h; therefore writing x for h, for the sake of uniformity of notation,

$$fx = \text{U}_0 + \text{U}_1\frac{x}{1} + \text{U}_2\frac{x^2}{1.2} + \text{U}_3\frac{x^3}{1.2.3} + \text{etc.}$$

which is Maclaurin's theorem, and therefore it is only a particular case of the theorem of Taylor.

Examples.

1. To expand $\sin(x + h)$.

Here $u = \sin x$, $\frac{du}{dx} = \cos x$, $\frac{d^2u}{dx^2} = -\sin x$, $\frac{d^3u}{dx^3} = -\cos x$, $\frac{d^4u}{dx^4}$ $= \sin x$, etc.; hence, by Taylor's theorem, $\sin(x + h)$

$$= \sin x + \cos x\frac{h}{1} - \sin x\frac{h^2}{1.2} - \cos x\frac{h^3}{1.2.3} + \sin x\frac{h^4}{1.2.3.4} + \text{etc.}$$

$$= \sin x\left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \text{etc.}\right) + \cos x\left(h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \text{etc.}\right)$$

Let $x = 0$, then $\sin h = h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \text{etc.}$, as in Art. 39.

2. If $u = \tan x$, it is required to expand $u' = \tan(x + h)$ in a series.

Here $\frac{du}{dx} = \sec^2 x$,

$$\frac{d^2u}{dx^2} = 2\sec x \frac{d \sec x}{dx} = 2\sec^2 x \tan x = 2(\tan x + \tan^3 x),$$

$$\frac{d^3u}{dx^3} = 2(\sec^4 x + 3\tan^2 x \sec^2 x) = 2(1 + 4\tan^2 x + 3\tan^4 x),$$

$$\frac{d^4u}{dx^4} = 2^3(2\tan x + 5\tan^3 x + 3\tan^5 x),$$

etc. etc.

$$\therefore\ u' = \tan(x+h) = \tan x + \sec^2 x \,.\, h + \tan x \sec^2 x \,.\, h^2 + \sec^2 x(1 + 3\tan^2 x) \,.\, \frac{h^3}{3} + \text{etc.}$$

3. Expand $u' = \log_e (x + h)$ in a series.

Ans. $\log_e x + \frac{h}{x} - \frac{1}{2} \cdot \frac{h^2}{x^2} + \frac{1}{3} \cdot \frac{h^3}{x^3} - \frac{1}{4} \cdot \frac{h^4}{x^4} + \text{etc.}$

4. Expand $u' = \cos(x + h)$ in a series.

Ans. $\cos x - \sin x \,.\, h - \cos x \,.\, \frac{h^2}{1.2} + \sin x \,.\, \frac{h^3}{1.2.3} + \cos x \,.\, \frac{h^4}{1.2.3.4} - \text{etc.}$

5. Expand $u' = \tan^{-1}(x + h)$ in a series.

Ans. $u + \cos^2 u \,.\, h - \frac{1}{2} \sin 2u \cos^2 u \,.\, h^2 - \frac{1}{3} \cos 3u \cos^3 u \,.\, h^3 + \frac{1}{4} \sin 4u \cos^4 u \cdot h^4 + \text{etc.}$, where $u = \tan^{-1} x$.

6. Expand $u' = a^{x+h}$ in a series.

Ans. $a^x \left(1 + ch + \frac{c^2 h^2}{1.2} + \frac{c^3 h^3}{1.2.3} + \frac{c^4 h^4}{1.2.3.4} + \text{etc.}\right)$;

where $c = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}$

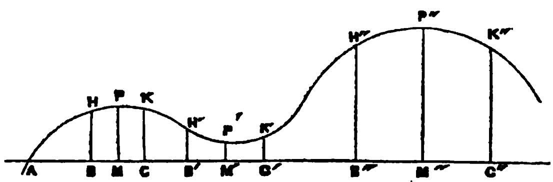
MAXIMA AND MINIMA FUNCTIONS OF ONE VARIABLE.

43. The value of a function of a variable quantity may either increase or diminish with the increase of the variable; and after increasing to a certain limit, it may begin to decrease, while the variable increases; or after decreasing to a certain limit, it may begin to increase with the increase of the variable. The value of a function is therefore said to be a *maximum* when that value is *greater* than either its immediately preceding or succeeding values, and a *minimum* when that value is *less* than any of the values which immediately precede or follow it.

44. A function which increases continually with the increase of the variable does not admit of a maximum or minimum value; but if, after passing a maximum value, the function decreases to a certain limit, and then begins to increase, it admits of both a maximum and minimum value; and if, after passing this minimum value, the function increases to a certain limit, and then begins to decrease, it will admit of a *second maximum* value; hence a *maximum* value of a function is not necessarily the *greatest* value of which the function may be susceptible, nor a *minimum* value the *least* value. If the value of the function increase and diminish alternately several times, the maximum and minimum values may succeed each other to any extent. The distinguishing cha-

racteristic of a *maximum* value of a function is that it is *greater* than any of the immediately preceding or succeeding values; and of a *minimum*, that it is *less* than any of the values which immediately precede or follow it.

45. Let A P P′ P″ be a curve referred to the line A B C″ as axis, and let the variable abscissa $AB = x$; then the corresponding ordinate B H is a function of the variable abscissa A B. If $BH = u$, then $u = fx$; and while the variable abscissa A B, or x, increases, the ordinate B H, or u, increases till it attains its maximum position M P. After passing the maximum position M P, the ordinate decreases till it comes into the position M′ P′, when it is a minimum; and after passing the minimum position M′ P′, it increases till it attains a *second* maximum position M″ P″, and so on.



46. The determination of these maximum and minimum values of a function constitutes one of the principal applications of the Differential Calculus; but before proceeding to the investigation of the principles for determining such values, the following lemma must be premised:—

LEMMA.—*If, while a variable quantity increases, a function of it likewise increase, the differential coefficient of the function is positive; but if the function decrease, its differential coefficient is negative.*

For the differential coefficient of fx is the limit to which the value of the fraction $\dfrac{f(x+h) - fx}{h}$ continually approaches, as h, the increment of x, approaches zero. Now the preceding fraction, as well as the limit $\dfrac{du}{dx}$, to which it tends, will be positive or negative according as $f(x+h)$ is greater or less than fx; that is, according as fx is increased or diminished by the addition of h. Hence if $u = fx$, and x increase continually, then $\dfrac{du}{dx}$ will be *positive* or *negative* according as u or fx is increasing or decreasing.

The same relation will exist between any differential coefficient and the one immediately following it, because the latter is the differential coefficient of the former. Thus $\dfrac{d^2u}{dx^2}$ is the differential coefficient of $\dfrac{du}{dx}$, which is generally a new function of the variable; and $\dfrac{d^3u}{dx^3}$ is the differential coefficient of $\dfrac{d^2u}{dx^2}$, and so on.

47. From the last Article, and the definitions of maximum and minimum values of functions, it follows that *the differential coefficient of a function of a variable quantity changes its sign when that function becomes a maximum or a minimum.*

In the case of a maximum, the function first increases and then dimi-

nishes; and in the case of a minimum, it first diminishes and then increases.

Now a varying quantity changes its sign only in becoming nothing or infinite; hence $u = fx$ can be a maximum or a minimum only when $\frac{du}{dx} = 0$, or $\frac{du}{dx} = \infty$.

Since, after passing a maximum value, the function begins to diminish, therefore the value of its *first* differential coefficient $\frac{du}{dx}$ is *decreasing;* consequently (46) the differential coefficient of this quantity will be *negative;* that is, $\frac{d^2u}{dx^2} < 0$. In a similar manner, if the value of $\frac{du}{dx}$ is *increasing*, its differential coefficient $\frac{d^2u}{dx^2}$ will be *positive*, or $\frac{d^2u}{dx^2} > 0$. It may happen that $\frac{d^2u}{dx^2}$, instead of being positive or negative, may be zero, as well as $\frac{du}{dx}$. When such a case occurs, it may be shown by similar reasoning that if a value of x, derived from the equation $\frac{du}{dx} = 0$, render also $\frac{d^2u}{dx^2} = 0$, there can be no maximum or minimum, unless that value of x makes also $\frac{d^3u}{dx^3} = 0$, and $\frac{d^4u}{dx^4}$ *negative* or *positive* respectively; that is, $\frac{d^4u}{dx^4} < 0$ indicates a maximum, and $\frac{d^4u}{dx^4} > 0$ a minimum value of the function. We can therefore have maximum and minimum values only when the first differential coefficient that does not vanish is of an even order.*

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* The following investigation of the principles for determining the maximum and minimum values of a function may be interesting to the student.

Let u be any function of x in which the variable has attained a value which renders the function either a maximum or a minimum; then if x be increased or diminished by an indefinitely small quantity h, the developments of $f(x+h)$ and $f(x-h)$ will exhibit the values of u immediately preceding and succeeding the maximum or minimum values of u; hence in the case of a *maximum*, the values of u corresponding to $x+h$ and $x-h$ will be both less than the maximum value; and, in the case of a *minimum*, they will be both greater than the minimum value.

If, then, $u = fx$, $u_1 = f(x+h)$, and $u_2 = f(x-h)$, then, by Taylor's Theorem,

$$u_1 = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.}$$

$$u_2 = u - \frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.}$$

or

$$u_1 - u = \frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots \quad (\alpha),$$

$$u_2 - u = -\frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots \quad (\beta).$$

Now, if the value of u is a maximum, it must be greater than either the immediately preceding or succeeding values u_1 or u_2; and if the value of u is a minimum, it must be less than either u_1 or u_2. In either case, $u_1 - u$ and $u_2 - u$, or their

48. Hence we have the following principles for determining the maximum and minimum values of a function fx.

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equivalent values (α) and (β) must have the same algebraical sign. But while $\frac{du}{dx}$ is not zero, and a value a of x does not render any of the differential coefficients infinite, h may always be taken so small that the first term $\frac{du}{dx}h$ shall exceed the sum of all the other terms of the series, and therefore the sign of the aggregate of all the terms can be made to depend on the sign of the first term; hence the series (α) will be positive and (β) negative, or *vice versâ*, according as the value of $\frac{du}{dx}$ is positive or negative; consequently, when u is a maximum, the value of $\frac{du}{dx}$ must be zero, or $\frac{du}{dx} = 0$. When, therefore, u admits of a maximum, $\frac{du}{dx} = 0$, and

$$u_1 - u = \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots$$

$$u_2 - u = \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots$$

Now if when $x = a$, $\frac{d^2u}{dx^2}$ is not zero, the sign of $u_1 - u$ and $u_2 - u$ will depend on that of $\frac{d^2u}{dx^2}$, since h^2 is positive; but when u is a maximum, and therefore greater than either u_1 or u_2, the sign of $\frac{d^2u}{dx^2}$ must be *negative;* consequently, when u is a maximum,

$$\frac{du}{dx} = 0, \text{ and } \frac{d^2u}{dx^2} \text{ is } \textit{negative}.$$

Again, when u is a *minimum*, u is less than u_1 or u_2, and therefore both

$$u_1 - u = \frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots$$

and

$$u_2 - u = -\frac{du}{dx}h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots$$

must be *positive*. This cannot be the case unless $\frac{du}{dx} = 0$, and therefore both

$$u_1 - u = \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots$$

and

$$u_2 - u = \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \dots$$

must be *positive*, and hence, as before, when u is a minimum,

$$\frac{du}{dx} = 0, \text{ and } \frac{d^2u}{dx^2} \text{ is } \textit{positive}.$$

Hence if $u = fx$ be either a maximum or a minimum, then $\frac{du}{dx} = 0$; and if the value or values of x found from this equation be substituted in $\frac{d^2u}{dx^2}$, the signs of the result will indicate the corresponding maxima or minima values.

If the result is negative, the value of x will give u a maximum.
If the result is positive, the value of x will give u a minimum.

Should, however, a value of x, derived from $\frac{du}{dx} = 0$, render also $\frac{d^2u}{dx^2} = 0$, then,

(1). Find the differential coefficient $\frac{du}{dx}$ of the proposed function, and determine the real value or values of x in the equation $\frac{du}{dx} = 0$.

(2). Find whether $\frac{du}{dx}$ changes its sign when x passes through each of these values, and those values which give a change from $+$ to $-$, render fx a maximum; but those which give a change from $-$ to $+$ make fx a minimum. If there is no change of sign, fx does not admit either of a maximum or a minimum value.

(3). The maximum and minimum values may also be distinguished by finding the second differential coefficient, $\frac{d^2u}{dx^2}$, of the function; then those values of x, derived from the equation $\frac{du}{dx} = 0$, which, when substituted for x in $\frac{d^2u}{dx^2}$, give negative results, indicate maximum values of fx, and those which give positive results indicate minimum values. If $\frac{d^2u}{dx^2}$ should vanish for any such values of x, find additional differential coefficients; then if the first of these coefficients that does not vanish be of an even order, as $\frac{d^4u}{dx^4}$, the value of x, which produces it, will render fx a maximum, if that coefficient, $\frac{d^4u}{dx^4}$, be negative; otherwise, fx will be a minimum.

49. The process of finding the maximum and minimum values of functions may sometimes be simplified by the following obvious expedients:

(1). A constant multiplier or divisor may be omitted, and a constant quantity, connected by addition or subtraction, may be likewise omitted.

(2). Suppress any factor of $\frac{du}{dx}$, which is always positive, for it is only the sign, and not the actual magnitude of $\frac{du}{dx}$ with which we are concerned.

(3). The reciprocal of a maximum is a minimum, and *vice versâ.*

(4). When a function is a maximum or minimum, its logarithm is generally a maximum or minimum, and when a function con-

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reasoning as above, so long as $\frac{d^3u}{dx^3}$ exists there can be no maximum or minimum; but when the value of x, derived from $\frac{du}{dx} = 0$, makes also $\frac{d^2u}{dx^2} = 0$, $\frac{d^3u}{dx^3} = 0$, and gives $\frac{d^4u}{dx^4}$ *negative*, it indicates a maximum, while u is a minimum when it makes $\frac{d^4u}{dx^4}$ *positive.*

sists of products or quotients of roots and powers, the differentiation is much facilitated by first taking its logarithm, and then considering the result as the function under consideration.

(5). The square or cube of a positive maximum or minimum is also a maximum or minimum.

(6). In testing by the sign of $\frac{d^2 u}{d x^2}$ whether the value of u is a maximum or a minimum, the process may be simplified when $\frac{d u}{d x}$ is of the form $x_1 . x_2 . x_3 . . . . .$, and the value $x = a$ causes one of the factors, as x_2, to vanish. For when $x = a$ makes x_2 vanish, all the others vanish with it, and $\frac{d^2 u}{d x^2}$ is reduced to one term.

EXAMPLES.

1. Let $u = x^3 - 3x^2 - 9x + 30$, to find those values of x, which render u a maximum or a minimum.

Here $\frac{d u}{d x} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 0$; consequently $x^2 - 2x - 3 = 0$, or $x^2 - 2x = 3$, and hence $x = 3$ or $x = -1$.

Now $\frac{d^2 u}{d x^2} = 2x - 2$, suppressing the factor 3, which does not affect the *sign* of the result; hence when

$$x = \quad 3, \frac{d^2 u}{d x^2} = +, \text{ and } x = 3 \text{ gives } u \text{ a minimum,}$$

$$x = -1, \frac{d^2 u}{d x^2} = -, \text{ and } x = -1 \text{ gives } u \text{ a maximum.}$$

2. Find the fraction which shall exceed its square by the greatest possible quantity.

Let x denote the fraction, then $u = x - x^2 =$ a maximum. Differentiating, we get $\frac{d u}{d x} = 1 - 2x = 0$; hence $x = \frac{1}{2}$.

The second differential coefficient is evidently a negative quantity, showing that the value of x renders u a maximum.

3. Find the maximum and minimum values of the function

$$u = x^4 - 4x^3 + 6x^2 - 4x + 16.$$

Here $\frac{d u}{d x} = 4x^3 - 12x^2 + 12x - 4 = 0$; consequently, dividing by 4,

$$x^3 - 3x^2 + 3x - 1 = 0, \text{ or } (x - 1)^3 = 0; \therefore x = 1.$$

Now $\frac{d^2 u}{d x^2} = 3x^2 - 6x + 3 = 3(x - 1)^2 = 0$, when $x = 1$.

Differentiating again, we get $\frac{d^3 u}{d x^3} = 6(x - 1) = 0$, when $x = 1$.

Also $\frac{d^4 u}{d x^4} = +$; and therefore the value $x = 1$ renders the function u a minimum.

4. Let $u = x \tan e - \dfrac{x^2}{4\,a \cos^2 e}$; to find x when u is a maximum.

$$\frac{d\,u}{d\,x} = \tan e - \frac{x}{2\,a \cos^2 e} = 0; \text{ hence we get}$$

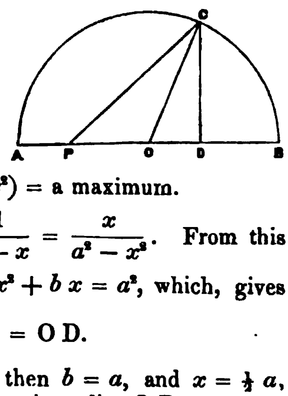
$$x = 2\,a \tan e \cos^2 e = 2\,a \sin e \cos e = a \sin 2\,e.$$

Also $\dfrac{d^2 u}{d\,x^2} = -\dfrac{1}{2\,a \cos^2 e}$, which is negative; hence u is a maximum when $x = a \sin 2\,e$. Substituting this value of x in the given function, we get $u = a \tan e \sin 2\,e - \dfrac{a^2 \sin^2 2\,e}{4\,a \cos^2 e} = 2\,a \sin^2 e - a \sin^2 e = a \sin^2 e.$

This example is the equation of the path of a projectile in free space, and the maximum value of u, viz., $a \sin^2 e$ is the greatest height to which the projectile will rise above the horizontal plane through the point of projection.

5. Let A B C be a semicircle, and P any point in the diameter A B; find the position of the ordinate C D when the triangle P C D, formed by drawing the line P C, is a maximum.

Let O be the centre of the semicircle, and draw the radius O C. Put A O $= a$, P O $= b$, and O D $= x$; then C D $= \sqrt{(a^2 - x^2)}$, and P D $= b + x$; hence the area of the triangle P C D $= \frac{1}{2}(b + x)\sqrt{(a^2 - x^2)}$. By Article 49 we may reject the factor $\frac{1}{2}$ and take the logarithm of the function to be made a maximum; then we have



$$u = \log(b + x) + \tfrac{1}{2}\log(a^2 - x^2) = \text{a maximum}.$$

Hence $\dfrac{d\,u}{d\,x} = \dfrac{1}{b + x} - \dfrac{x}{a^2 - x^2} = 0$, or $\dfrac{1}{b + x} = \dfrac{x}{a^2 - x^2}$. From this equation we get $a^2 - x^2 = b\,x + x^2$, or $2\,x^2 + b\,x = a^2$, which, gives

$$x = \frac{-\,b \pm \sqrt{(8\,a^2 + b^2)}}{4} = \text{O D}.$$

If the given point P coincide with A, then $b = a$, and $x = \frac{1}{2}\,a$, showing that, in this case, the point D bisects the radius O B.

If the given point P coincide with O, then $b = 0$, and $x = \dfrac{a}{2}\sqrt{2}$.

6. Find the maximum and minimum values of the function

$$\frac{x^2 - 3\,x + 2}{x^2 + 3\,x + 2} \text{ or } \frac{(x - 1)\,(x - 2)}{(x + 1)\,(x + 2)}.$$

Dividing the numerator by the denominator, gives $1 - \dfrac{6\,x}{x^2 + 3\,x + 2}$; hence if the given fraction be a maximum, then the fraction $\dfrac{6\,x}{x^2 + 3\,x + 2}$, and consequently (49) $\dfrac{x}{x^2 + 3x + 2}$ will be a minimum; hence the reciprocal of this $\dfrac{x^2 + 3x + 2}{x}$ or $x + 3 + \dfrac{2}{x}$ will be a minimum (49).

Rejecting the constant quantity 3, we get $u = x + \frac{2}{x}$ = a maximum or minimum when the proposed fraction is a maximum or a minimum. Differentiating, we get $\frac{du}{dx} = 1 - \frac{2}{x^2} = 0$; hence $x = \pm \sqrt{2}$.

Again, $\frac{d^2 u}{dx^2} = + \frac{4}{x^3}$ which is *positive* when $x = +\sqrt{2}$ and *negative* when $x = -\sqrt{2}$; hence if

$$x = \sqrt{2}, \text{ then } \frac{x^2 - 3x + 2}{x^2 + 3x + 2} = 12\sqrt{2} - 17 = -\cdot 02943728 = \text{min.}$$

$$x = -\sqrt{2}, \text{ then } \frac{x^2 - 3x + 2}{x^2 + 3x + 2} = -12\sqrt{2} - 17 = -33\cdot 9705627 = \text{max.}$$

We have here an instance in which a maximum value is *less* than a minimum one, because the value $x = -1$, which lies between $x = \sqrt{2}$ and $x = -\sqrt{2}$, renders the function infinite.*

Otherwise thus:—Taking the logarithm of the function, we get

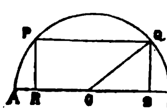
$$u = \log(x-1) + \log(x-2) - \log(x+1) - \log(x+2).$$

Now $\frac{du}{dx} = \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x+1} - \frac{1}{x+2} = \frac{2}{x^2-1} + \frac{4}{x^2-4} = 0$;

hence $(x^2 - 4) + 2(x^2 - 1) = 3x^2 - 6 = 0$; and $x = \pm\sqrt{2}$, as before.

7. To inscribe the greatest rectangle in a given semicircle.

4 Let A P Q B be the given semicircle, O its centre, A B its diameter, and P Q S R the required maximum incribed rectangle. Let $AO = a = OQ$, and $OS = OR = x$, the rectangle being symmetrical with respect to the diameter through the centre O at right angles to A B; hence $QS = \sqrt{(OQ^2 - OS^2)} = \sqrt{(a^2 - x^2)}$; and the area of the rectangle is $= PQ \cdot QS = 2x\sqrt{(a^2 - x^2)}$. Rejecting the constant factor 2, and squaring the result, we get



$$u = x^2(a^2 - x^2) = a^2x^2 - x^4; \text{ hence } \frac{du}{dx} = 2a^2x - 4x^3 = 0;$$

therefore $x = 0$, or $x = \sqrt{\frac{a^2}{2}} = \frac{a}{2}\sqrt{2}$.

Also $\frac{d^2u}{dx^2} = 2(a^2 - 6x^2) = -$, when $x = \frac{a}{2}\sqrt{2}$; hence the rectangle P S is a maximum when $RS = 2x = a\sqrt{2}$, and $QS = \sqrt{(a^2 - x^2)} = \frac{a}{2}\sqrt{2}$.

The value of R S is double that of Q S; therefore $RS = 2QS$.

8. Let $u = x^5(x-3)^3$.

Here $\frac{du}{dx} = 5x^4(x-3)^3 + 3x^5(x-3)^2 = x^4(x-3)^2(8x - 15) = 0$.

Now $x^4(x-3)^2$ is always positive whatever be the value of x; hence

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* The student will find it interesting to attempt graphical delineations of the curves of which the proposed functions are the equations, by giving values to x, and then computing the corresponding values of the ordinates.

the sign of $\frac{du}{dx}$ must depend on the factor $8x - 15$. From the equation $8x - 15 = 0$, we get $x = 1\frac{7}{8}$, and taking two values of x, the one a little less than $1\frac{7}{8}$, and the other a little greater, we find that $\frac{du}{dx}$ will change its sign from $-$ to $+$ in passing through the value $1\frac{7}{8}$. Thus

if $x = 1\frac{3}{4}$, then $8x - 15$, and $\frac{du}{dx}$ are *negative;*

if $x = 2$, then $8x - 15$, and $\frac{du}{dx}$ are *positive.*

Hence the value $x = 1\frac{7}{8}$ renders the proposed function a minimum; and differentiating $8x - 15$, gives $\frac{d^2u}{dx^2} = +$, indicating also a minimum.

9. Find the values of x which render y a maximum or minimum in the implicit equation $y^3 - 3axy + x^3 = 0$.

Differentiating, $3y^2\frac{dy}{dx} - 3ay - 3ax\frac{dy}{dx} + 3x^2 = 0$ . . . . (1).

But when y is a maximum or minimum, $\frac{dy}{dx} = 0$; hence (1) reduces to $x^2 = ay$. Eliminating y between this and the given equation, we get $x^3(x^3 - 2a^3) = 0$; hence $x = 0$, or $x = a\sqrt[3]{2}$. Substitute these values separately in the equation $ay = x^2$; and we get the corresponding values of y, viz., $y = 0$ or $y = a\sqrt[3]{4}$.

Differentiating (1) a second time, recollecting that every term which contains $\frac{dy}{dx}$ is zero, we obtain $3y^2\frac{d^2y}{dx^2} - 3ax\frac{d^2y}{dx^2} + 6x = 0$;

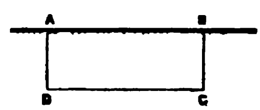
$$\text{whence } \frac{d^2y}{dx^2} = \frac{2x}{ax - y^2} = \frac{2a^2x}{a^2x - a^2y^2} = \frac{2a^2x}{a^3x - x^4} = \frac{2a^2}{a^3 - x^3}.$$

If $x = 0$, then $\frac{d^2y}{dx^2} = \frac{2}{a}$, which gives y a minimum, and

if $x = a\sqrt[3]{2}$, then $\frac{d^2y}{dx^2} = -\frac{2}{a}$, which makes y a maximum.

10. A rectangular court is to be formed so as to contain a given area, and a wall already built is available for one of its sides; find its dimensions so that the least expense may be incurred.

Let A B C D be the rectangular court of which the side A B is already formed; then the expense will be the least when the sum of the three remaining sides is the least. Let the given area $= a^2$, and the side $BC = x$; then will $CD = \frac{a^2}{x}$;



hence $u = 2x + \frac{a^2}{x}$ = a minimum; consequently

$$\frac{du}{dx} = 2 - \frac{a^2}{x^2} = 0, \text{ from which } x = \frac{a}{2}\sqrt{2} = AD \text{ or } BC;$$

and $CD = AB = \frac{a^2}{x} = a\sqrt{2}$ = twice the breadth A D or B C.

The second differential coefficient is positive, and the walling B C D A is a minimum.

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11. Construct a cylinder whose total surface may be equal to a given area, and its volume a maximum.

Let x = the radius of the base of the cylinder, y = its height, and a^2 = the given surface; then

πx^2 = area of the base, and $2\pi x$ = the circumference of base;

$$\therefore\ 2\pi x^2 + 2\pi x y = a^2, \text{ and } y = \frac{a^2 - 2\pi x^2}{2\pi x}.$$

Now the volume of the cylinder is

$$\pi x^2 . y = \pi x^2 . \frac{a^2 - 2\pi x^2}{2\pi x} = \tfrac{1}{2} x\,(a^2 - 2\pi x^2);$$

$$\therefore\ u = x\,(a^2 - 2\pi x^2) = a^2 x - 2\pi x^3 = \text{a maximum}.$$

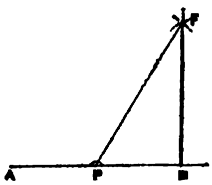
Hence $\frac{du}{dx} = a^2 - 6\pi x^2 = 0$, which gives $x = \frac{a}{\sqrt{6\pi}} = \frac{a}{6\pi}\sqrt{6\pi}$;

and $y = \frac{a^2 - 2\pi x^2}{2\pi x} = \frac{a^2 - \frac{1}{3}a^2}{2x} \cdot \frac{\sqrt{6\pi}}{a} = \frac{a}{3\pi}\sqrt{6\pi} = 2x.$

Consequently the cylinder is equilateral, having its altitude equal to the diameter of its base, and its volume is obviously a maximum, since $\frac{d^2u}{dx^2}$ is *negative.*

12. A small plane surface is placed horizontally upon a table, and illuminated by a lamp placed at a given horizontal distance; find the height of the flame so that the plane shall receive the greatest illumination from it.

Let a = the given distance P B of the small plane or disk from the foot of the lamp B F, and let θ = the variable angle F P B; therefore we have



$\text{F P} = \frac{\text{P B}}{\cos \text{F P B}} = \frac{a}{\cos\theta} = a \sec\theta$ = the distance from F to P.

Now the degree of illumination varies inversely as the square of the distance F P, and directly as the sine of the inclination F P B; hence we have

$$\sin\theta \times \frac{\cos^2\theta}{a^2} = \frac{1}{a^2}\sin\theta\cos^2\theta = \text{a maximum}.$$

Let $u = \sin\theta\cos^2\theta$; then $\frac{du}{d\theta} = \cos^3\theta - 2\cos\theta\sin^2\theta = 0$; therefore $\cos\theta = 0$, or $\cos^2\theta - 2\sin^2\theta = 0$, which gives $\tan\theta = \pm\frac{1}{2}\sqrt{2}$.

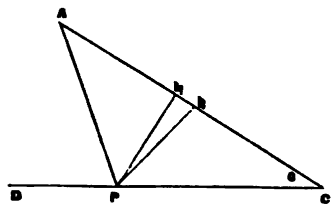
Differentiating a second time, we get

$$\frac{d^2u}{d\theta^2} = \sin\theta\cos^3\theta\,(2\tan^2\theta - 7) \text{ which vanishes for } \cos\theta = 0.$$

But if $\tan\theta = \frac{1}{2}\sqrt{2}$, then both $\sin\theta$ and $\cos\theta$ are positive, and $2\tan^2\theta - 7$ is *negative;* hence this value renders the function a maximum; and $\text{B F} = a\tan\theta = \frac{a}{2}\sqrt{2}$ = the height of the flame F.

13. A and B are two known objects in the straight line A B C; find a point P in the given line C D from which these objects shall subtend the greatest angle.

Let A C = a, B C = b, C P = x, and draw P A, P B, and also P M perpendicular to A B C; then by trigonometry we have (angle A C P being $= \theta$)



$\text{P M} = x \sin \theta$, $\text{C M} = x \cos \theta$, $\text{A M} = a - x \cos \theta$, $\text{B M} = x \cos \theta - b$;

hence $\tan \text{APM} = \dfrac{\text{A M}}{\text{P M}} = \dfrac{a - x \cos \theta}{x \sin \theta}$.

and $\tan \text{B P M} = \dfrac{x \cos \theta - b}{x \sin \theta}$;

but $\tan \text{APB} = \tan (\text{APM} + \text{BPM}) = \dfrac{\tan \text{APM} + \tan \text{BPM}}{1 - \tan \text{APM} . \tan \text{BPM}}$

$$= \frac{(a-b) x \sin \theta}{x^2 \sin^2 \theta - (a - x \cos \theta)(x \cos \theta - b)} = \frac{(a-b) x \sin \theta}{x^2 + (a-b) x \cos \theta + a b}.$$

Now if the angle A P B be a maximum, its tangent will also be a maximum;* hence suppressing the constant factor $(a - b) \sin \theta$, and taking the reciprocal of the expression, we get

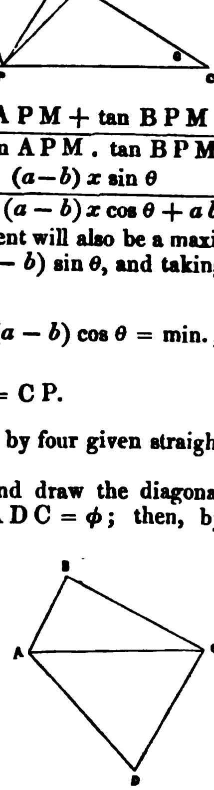
$$u = \frac{x^2 + (a-b) x \cos \theta + a b}{x} = x + \frac{a b}{x} + (a - b) \cos \theta = \text{min.};$$

$$\therefore \frac{d u}{d x} = 1 - \frac{a b}{x^2} = 0, \text{ which gives } x = \sqrt{a b} = \text{C P}.$$

14. Find the greatest area that can be included by four given straight lines.

Let A B = a, B C = b, C D = c, D A = d, and draw the diagonal A C; let also angle A B C = θ, and angle A D C = ϕ; then, by trigonometry, $a^2 + b^2 - 2 a b \cos \theta = \text{A C}^2$

$= c^2 + d^2 - 2 c d \cos \phi$; . . . . . (1).



Again, the double area of the quadrilateral gives

$u = ab \sin \theta + c d \sin \phi =$ a maximum . (2).

Differentiating (1) and (2) with respect to θ as the independent variable, we get

$$a b \sin \theta - c d \sin \phi \frac{d \phi}{d \theta} = 0 \quad . \ . \ . \ (3);$$

$$\frac{d u}{d \theta} = a b \cos \theta + c d \cos \phi \frac{d \phi}{d \theta} = 0 \quad . \ . \ (4).$$

Multiply (3) by $\cos \phi$, and (4) by $\sin \phi$, and add; then we get

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* This is not generally true, but if the angle be less than 90°, it may be employed with much advantage. The following method is preferable, though not quite so simple:

Since $\text{A P M} = \tan^{-1}\left(\dfrac{a \operatorname{cosec} \theta}{x} - \cot \theta\right)$ and $\text{B P M} = \tan^{-1}\left(\cot \theta - \dfrac{b \operatorname{cosec} \theta}{x}\right)$.

$\therefore u = \tan^{-1}\left(\dfrac{a \operatorname{cosec} \theta}{x} - \cot \theta\right) + \tan^{-1}\left(\cot \theta - \dfrac{b \operatorname{cosec} \theta}{x}\right) =$ a maximum.

Differentiating this we get the same result, *viz.* $x = \sqrt{ab}$.

$$a b (\sin \theta \cos \phi + \cos \theta \sin \phi) = 0, \text{ or } \sin (\theta + \phi) = 0 = \sin \pi;$$

hence $\theta + \phi = \pi$, and the quadrilateral may be inscribed in a circle.

Also the area of the quadrilateral is found by (1) and (2) to be

$$u = \tfrac{1}{2}(a b + c d) \sin \theta = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

$$\text{where } s = \tfrac{1}{2}(a + b + c + d).$$

4

Exercises in Maxima and Minima.

1. Divide 21 into two parts so that the less multiplied by the square of the greater may be a maximum. *Ans.* 14 and 7.

2. Inscribe the greatest rectangle in a triangle whose base is a feet and perpendicular b feet. *Ans.* The sides are $\frac{1}{2} a$ and $\frac{1}{2} b$.

3. When will $u = 4x^3 - x^2 - 2x + 5$ be a maximum or minimum?
Ans. When $x = \frac{1}{2}$, $u = 4\frac{1}{4}$, a minimum,
when $x = -\frac{1}{3}$, $u = 5\frac{11}{27}$, a maximum.

4. Of all triangles upon the same base, and having the same perimeter, the isosceles has the greatest area.

5. Inscribe the greatest isosceles triangle in a circle whose radius is a.
Ans. The triangle is equilateral,
side $= a\sqrt{3}$, area $= \frac{3}{4} a^2 \sqrt{3}$.

6. Of all the squares inscribed in a square whose side is a, which is the least? *Ans.* The side is $\frac{1}{2} a \sqrt{2}$.

7. To divide the number a into two parts, such that the product of the m^{th} power of the one by the n^{th} power of the other shall be a maximum.
Ans. $\dfrac{m a}{m+n}$ and $\dfrac{n a}{m+n}$.

8. To describe the least isosceles triangle about a circle whose radius is a. *Ans.* The triangle is equilateral,
side $= 2 a \sqrt{3}$, area $= 3 a^2 \sqrt{3}$.

9. Bisect a triangle by the shortest line.
Ans. The length of the line $= \sqrt{\{\frac{1}{2}(a+b-c)(a-b+c)\}}$.

10. The edges of a rectangular piece of tin are to be turned up so as to form the greatest rectangular vessel; the length of the piece of tin is a inches and the breadth b inches; find how much of the edge must be turned up. *Ans.* $\frac{1}{6}\{a + b - \sqrt{(a^2 - ab + b^2)}\}$.

11. Find the dimensions of the greatest cone which can be cut out of a sphere whose radius is a.
Ans. Radius of base $= \dfrac{2 a \sqrt{2}}{3}$, height $= \frac{4}{3} a$.

12. Find the dimensions of the greatest cylinder that can be formed out of a cone whose altitude $= a$ and radius of base $= r$.
Ans. Radius of base $= \frac{2}{3} r$, height $= \frac{1}{3} a$.

13. A cistern, open at top, is to be formed with a square base; find its dimensions so that a superficial feet of lead may cover it.
Ans. Side of base $= \dfrac{a}{3}\sqrt{3}$, height $= \dfrac{a}{6}\sqrt{3}$.

14. What is the height of the greatest cylinder that can be cut out of a sphere whose radius is a. *Ans.* Height $= \frac{2}{3} a \sqrt{3}$.

15. Through a given point, between the lines containing a given angle, to draw a straight line which shall cut off the least triangle, the distances of the point from the lines being a and b.
Ans. The line is bisected in the given point.

16. Inscribe the greatest rectangle in an ellipse, the major axis being $= 2a$ and the minor axis $= 2b$.

Ans. The length $= a\sqrt{2}$, and breadth $= b\sqrt{2}$.

17. Through a given point between the lines containing a right angle, to draw the shortest line terminated by the given lines, the distances of the point from the lines being a and b.

Ans. The segments of the lines cut off by the shortest line are

$a^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})$ and $b^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})$; and the line itself is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

18. Cut the greatest parabola from a given cone.

Ans. Axis is $\frac{3}{4}$ths of the slant side of the cone.

19. Inscribe the greatest parabola in a given isosceles triangle.

Ans. Axis is $\frac{3}{4}$ths of the altitude of the triangle.

20. Given the whole surface of a cylinder, to find its form when its volume is a maximum. *Ans.* Altitude = the diameter of the base.

21. Given the length of a circular arc $= 2a$, to find what part of a circle it must be that the area of the corresponding segment may be a maximum.

Ans. It must constitute a semicircle, whose radius is $= \dfrac{2a}{\pi}$.

22. Of all the cones whose convex surface is $= a$, find that whose volume is a maximum.

Ans. The radius of the base $= \frac{1}{3}\sqrt{\dfrac{3a}{\pi}}$.

23. If the whole surface of a cone be $= a$, what are its dimensions when the volume is a maximum.

Ans. Radius of base $= \frac{1}{2}\sqrt{\dfrac{a}{\pi}}$, and slant height $= \frac{3}{2}\sqrt{\dfrac{a}{\pi}}$.

24. To determine the maximum and minimum values of y in the equation $y^2 - 2axy + x^2 = m^2$.

Ans. When $x = \dfrac{am}{\sqrt{(1-a^2)}},\ y = \dfrac{m}{\sqrt{(1-a^2)}}$. a maximum,

$x = \dfrac{-am}{\sqrt{(1-a^2)}},\ y = \dfrac{-m}{\sqrt{(1-a^2)}}$, a minimum.

25. Given $u = \sin^m\theta \sin^n(\alpha - \theta)$, to find θ that u may be a maximum or minimum.

Ans. $\dfrac{\sin(\alpha - 2\theta)}{\sin\alpha} = \dfrac{n-m}{n+m}$.

26. The height of an inclined plane is a feet; find its length, so that a given weight P, acting freely upon another weight W, in a direction parallel to the plane, may draw it up the plane in the least time.

Ans. Length $= \dfrac{2a\text{W}}{\text{P}}$.

27. P and Q are two weights suspended over a fixed pulley, where P is given and greater than Q; find the weight of Q, so that P, acting

freely upon Q, may communicate to it the greatest momentum in a given time. *Ans.* $Q = P(\sqrt{2} - 1)$.

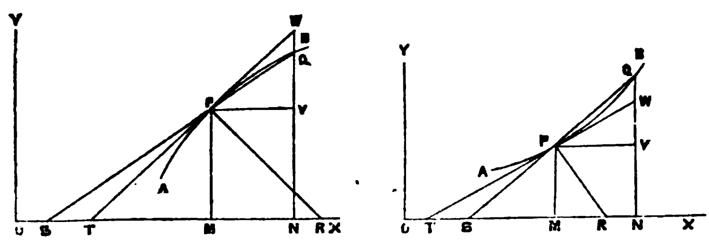
28. A, B, C are three elastic bodies in a straight line of which A and C are given; find B so that the motion communicated by A to C through B may be a maximum. *Ans.* $B = (A\,C)^{\frac{1}{2}}$.

29. Find the straight line of quickest descent from a given point to a given circle in the same vertical plane.

TANGENTS TO CURVES.

50. Let A P B be a curve referred to rectangular coordinates O X and O Y, and let it be required to draw a tangent to the curve at a given point P in the curve.

Let P and Q be two points of the curve, and S P Q an indefinite straight line drawn through these two points. Through P and Q draw P M, Q N parallel to the axis of y, and P V parallel to the axis of x, meeting Q N in V. Conceive the point Q to move towards P; then the direction of the secant S P Q will approach a certain limiting position as Q approaches P. Let P T be the limiting position, then P T is called the ***tangent to the curve at the point*** P. Draw P R at right angles to P T, then P R is called the ***normal to the curve at the point*** P. Also M T is called the ***subtangent***, and M R the ***subnormal***.



Let $y = f x$ be the equation to the curve, $O M = x$, $M P = y$, $O N = x + h$, $N Q = y'$; then will $\tan PSX = \dfrac{QV}{VP} = \dfrac{y' - y}{h}$;

and since PT is the limiting position of QPS, when Q approaches P, that is, when $y' - y$ and h approach zero; therefore

$$\tan PSX = \frac{y' - y}{h} \text{ will be changed into } \tan PTX = \frac{dy}{dx}.$$

$$\text{Hence the subtangent } MT = \frac{PM}{\tan PTM} = \frac{y}{\frac{dy}{dx}} = \frac{y\,dx}{dy} \quad \ldots \quad (1).$$

Angle T P R being a right angle, we have angle M P R = angle P T X;

$$\text{hence the subnormal } MR = PM \tan PTM = y\frac{dy}{dx} = \frac{y\,dy}{dx} \quad . \quad . \quad (2).$$

The lengths of the portions of the tangent and normal intercepted between the point of contact and the axis of x are given by the equations

$$PT = \sqrt{(PM^2 + MT^2)} = \sqrt{\left(y^2 + y^2\frac{dx^2}{dy^2}\right)} = y\sqrt{\left(1 + \frac{dx^2}{dy^2}\right)} \quad . \quad . \quad (3).$$

$$P R = \sqrt{(P M^2 + M R^2)} = \sqrt{\left(y^2 + y^2 \frac{d y^2}{d x^2}\right)} = y \sqrt{\left(1 + \frac{d y^2}{d x^2}\right)} \quad . \; . \; (4).$$

51. Let N Q, produced if necessary, meet the tangent T P in W, and if we put O N $= x'$, and N W $= y'$; then $\frac{W V}{V P} = \tan P T M$; that is

$$\frac{y' - y}{x' - x} = \frac{d y}{d x}, \text{ therefore } y' - y = \frac{d y}{d x}(x' - x) \quad . \; . \; (5),$$

which is the *equation of the tangent* to the curve at the point whose co-ordinates are $x\, y$; the coordinates of any point in the tangent being $x'\, y'$.

By the principles of coordinate geometry (Art. 17) the *equation of the normal* is $y' - y = -\frac{d x}{d y}(x' - x)$. . . . . (6).

The inclinations of the tangent and the normal at any point of a curve to the coordinate axes being determined from (5) and (6), the angle (α) which a curve makes with its axis is known, since the curve cuts the axis at the same angle as its tangent; hence

$$\tan \alpha = \frac{d y}{d x}, \text{ or } \alpha = \tan^{-1} \frac{d y}{d x}. \quad . \; . \; . \; . \; . \; (7).$$

Examples.

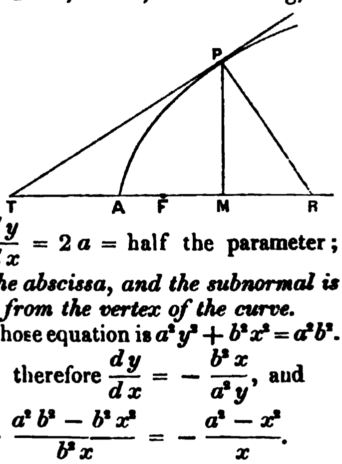
4 1. To draw a tangent to the parabola at any given point in the curve.

The equation to the parabola is $y^2 = 4 a x$; hence, differentiating,

$$2 y \, d y = 4 a \, d x, \text{ and } \frac{d y}{d x} = \frac{2 a}{y};$$

consequently (1) $M T = \frac{y}{\frac{d y}{d x}} = \frac{y^2}{2a}$

$$= \frac{4 a x}{2 a} = 2 x = 2 \, A M.$$



Again, the subnormal $M R = y \frac{d y}{d x} = 2 a =$ half the parameter; hence *the subtangent is equal to twice the abscissa, and the subnormal is equal to twice the distance of the focus from the vertex of the curve.*

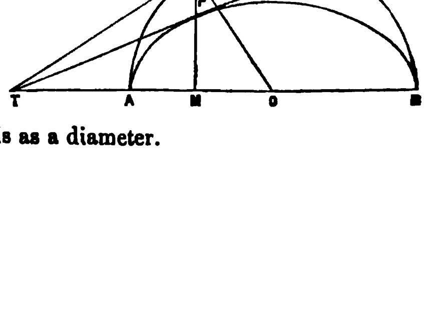
2. Find the subtangent in the ellipse whose equation is $a^2 y^2 + b^2 x^2 = a^2 b^2$.

Here $2 a^2 y \, d y + 2 b^2 x \, d x = 0$; therefore $\frac{d y}{d x} = -\frac{b^2 x}{a^2 y}$, and

$$M T = y \div \frac{d y}{d x} = -\frac{a^2 y^2}{b^2 x} = -\frac{a^2 b^2 - b^2 x^2}{b^2 x} = -\frac{a^2 - x^2}{x}.$$

The negative sign in the result, supposing x and y to be positive, shows that the angle which the tangent makes with the axis is *obtuse* instead of acute.

The value of the subtangent is independent of the minor axis, and will therefore be the same for all ellipses having the same major axis, as well as for the circle described on that axis as a diameter.



3. Find the subtangent, and the equation of the tangent in the hyperbola, its equation being $a^2 y^2 - b^2 x^2 = -a^2 b^2$.

Here $\dfrac{dy}{dx} = \dfrac{b^2 x}{a^2 y}$, and $\mathrm{M\,T} = \dfrac{y}{\dfrac{dy}{dx}} = \dfrac{a^2 y^2}{b^2 x} = \dfrac{b^2 x^2 - a^2 b^2}{b^2 x} = \dfrac{x^2 - a^2}{x}$.

Also the equation of the tangent at the point $x'y'$ is (5)

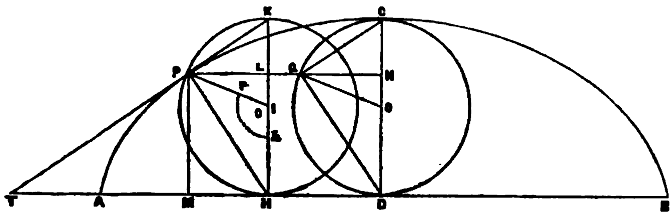
$$y' - y = \frac{b^2 x}{a^2 y}(x' - x),$$

or $a^2 y y' - a^2 y^2 = b^2 x x' - b^2 x^2$; whence $a^2 y y' - b^2 x x' = a^2 y^2 - b^2 x^2$; but $a^2 y^2 - b^2 x^2 = -a^2 b^2$ by the equation to the curve; therefore

$a^2 y y' - b^2 x x' = -a^2 b^2$, is the equation required.

4. Let the curve be the Cycloid.

If a circle H P K be always in the same plane, and roll, without sliding, along the straight line A B, the curve described by any point P in its circumference is called a *cycloid*.



Let $\mathrm{A\,M} = x$, $\mathrm{M\,P} = y$, be the coordinates of P in any of its positions; and let $r = \mathrm{P\,I}$, the radius of the generating circle. Draw the diameter H I K perpendicular to A B; and let angle $\mathrm{P\,I\,H} = \theta$: draw also P L parallel to A B; then since all the points in the circumference are applied successively to A B, the arc $\mathrm{P\,H} = \mathrm{A\,H}$; consequently

arc $\mathrm{P\,H} = \mathrm{A\,H} = r\theta$, $\mathrm{P\,L} = \mathrm{M\,H} = r\sin\theta$, $\mathrm{I\,L} = -r\cos\theta$;

$$\therefore\ x = \mathrm{A\,H} - \mathrm{H\,M} = r\theta - r\sin\theta = r(\theta - \sin\theta) \quad . \ . \ (1),$$

$$y = \mathrm{H\,I} + \mathrm{I\,L} = r - r\cos\theta = r(1 - \cos\theta) \quad . \ . \ (2).$$

These two equations (1) and (2), taken simultaneously, represent this interesting curve, and its properties may be readily deduced from them.

From (2) we have $y = r \operatorname{vers}\theta$, or $\theta = \operatorname{vers}^{-1}\dfrac{y}{r}$, and by the circle $\mathrm{P\,L} = \sqrt{(\mathrm{H\,L}\,.\,\mathrm{L\,K})}$, that is, $r\sin\theta = \sqrt{(2ry - y^2)}$. Substituting these in (1), we get

$$x = r\operatorname{vers}^{-1}\frac{y}{r} - \sqrt{(2ry - y^2)} \quad . \ . \ . \ . \ . \ . \ (3).$$

If the highest point C of the curve be taken as the origin of coordinates, and if $\mathrm{C\,N} = x$, $\mathrm{N\,P} = y$ be the coordinates of P, and angle $\mathrm{C\,O\,Q} = \theta$, we should find

$$x = r(1 - \cos\theta) \quad , \quad y = r(\theta + \sin\theta),$$

and

$$y = r\operatorname{vers}^{-1}\frac{x}{r} + \sqrt{(2rx - x^2)} \quad . \ . \ . \ . \ . \ . \ . \ . \ (3').$$

Differentiating (1) and (2), we get $\dfrac{d\theta}{dx} = \dfrac{1}{r(1-\cos\theta)} = \dfrac{1}{y}$, and $\dfrac{dy}{d\theta} = r\sin\theta$;

therefore $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{r \sin\theta}{y}$, and $y \cdot \frac{dy}{dx} = r \sin\theta = MH$.

Hence M H is the subnormal; therefore P K, which is perpendicular to the normal P H, is the tangent, and it is obviously parallel to C Q, if P Q N be parallel to A B. To draw a tangent to the curve, we have then to draw P Q N parallel to A B, cutting the circle described on the axis C D as a diameter in Q; join C Q, and draw T P K parallel to C Q; T P is the tangent.

The student may find it useful to verify this result by means of either of the equations to the curve (3) or (3′).

5. If A C B be a semicircle, and C D any ordinate; and if D C be produced to P, so that D P is a fourth proportional to A D, A B and D C, the locus of P is a curve called the *Witch* of Agnesi. Find its equation and draw a tangent to it.

Ans. If $AD = x$, $DC = y$, $AB = 2r$; then the equation to the curve is $xy = 2r\sqrt{(2rx - x^2)}$, and subtangent $= -\frac{2rx - x^2}{r}$.

6. If A D, a chord of a given circle be produced to cut B C, a tangent at the extremity of the diameter A B, in C; and if, towards A in the line C A, C P be taken equal to A D, the locus of the point P is a curve called the *Cissoid* of Diocles. Find the equation of the curve, and draw a tangent to it.

Ans. If A be the origin, x, y the coordinates of P, and $AB = 2r$; then the equation is $y^2(2r - x) = x^3$, and subtangent $= \frac{x(2r - x)}{3r - x}$.

7. The straight line A X and the point O, without it, being given in position, if from O any straight line A P be drawn cutting A X in C, and if C P be taken always equal to a given line, the locus of P is a curve called the *Conchoid* of Nicomedes. Find its equation, and draw a tangent to it.

Ans. If O A B be the line perpendicular to A X, and if $OA = a$, $AB = CP = b$, and x, y the coordinates of P referred to the axis A X (A being the origin); then the equation of the curve is $x^2 y^2 = (b^2 - y^2)(a + y)^2$, and the subtangent $= -\frac{ab^2 + y^3}{y\sqrt{(b^2 - y^2)}}$.

8. The *logarithmic curve* is a line of such a nature that the abscissa has a constant ratio to the logarithm of the ordinate. Expressing this ratio by 1 to log a, we have $1 : \log a :: x : \log y$; consequently

$$\log y = x \log a, \text{ or } y = a^x.$$

Draw a tangent to this curve.

Ans. Subtangent = the modulus of the system whose base is a, and it is therefore a *constant* quantity.

9. Find the subnormal in the curve whose equation is

$$3ay^2 = 2x^3 - a^3. \qquad \textit{Ans. } \frac{x^2}{a}.$$

10. Find the subtangent in the curve whose equation is

$$y^2(a - x) = (a + x)^2. \qquad \textit{Ans. } \frac{2(a^2 - x^2)}{3a - x}.$$

11. The equation of the curve is $y = (a^2 - x^2)\sqrt{(a + x)}$; find its subtangent. *Ans.* $\dfrac{2(a^2 - x^2)}{a - 5x}$.

12. Find the length of the normal at the point $(x = a)$ of the curve whose equation is $x^2 + 4y^2 = 4a^2$. *Ans.* $\dfrac{a}{4}\sqrt{13}$.

ASYMPTOTES TO CURVES.

52. If a curve have an infinite branch such that there is a straight line which it can never meet, but to which, if it be sufficiently extended, it can approach nearer than any distance, however small, that can be assigned, the straight line is said to be an *asymptote to the curve.* An asymptote is therefore to be considered as a line with which a tangent to the curve would continually tend to coincide, when one of the coordinates is increased without limit.

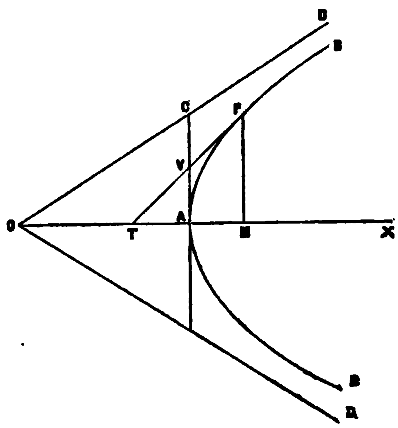
Asymptotes parallel to either of the coordinate axes may be at once determined, by observing whether any finite value of one of the coordinates renders the other infinite. Thus if the equation of the curve be

$y^2(2r - x) = x^3$, we have $y = \dfrac{x^{\frac{3}{2}}}{\sqrt{(2r - x)}}$, which is infinite when

$x = 2r$; therefore the curve has an asymptote parallel to the axis of y at the distance $x = 2r$ from the origin. If the value $x = 0$ renders $y = \pm\infty$, then the axis of y is an asymptote, and if $y = 0$ gives $x = \pm\infty$, the axis of x is an asymptote.

Ex. Find whether the hyperbola has asymptotes; and, if so, determine their position.

The equation to the hyperbola is $a^2y^2 - b^2x^2 = -a^2b^2$, the origin being at the centre, and its subtangent $MT = \dfrac{x^2 - a^2}{x} = x - \dfrac{a^2}{x}$; therefore $AT = a - \dfrac{a^2}{x}$, which, when x is infinite, is $= a$. The hyperbola, therefore, has asymptotes, and they pass through the centre of the curve. To determine their position, we have, by similar triangles, PTM, VTA; MT : MP :: AT : AV; that is,



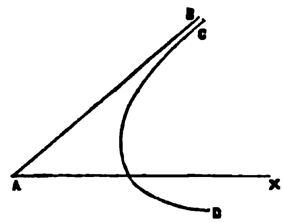
$$\frac{x^2 - a^2}{x} : y :: \frac{ax - a^2}{x} : AV = \frac{ay}{x + a} = \frac{\pm b\sqrt{(x^2 - a^2)}}{x + a};$$

or, $AV = \pm b\sqrt{\left(\dfrac{x - a}{x + a}\right)} = \pm b \div \left\{\dfrac{1 + \frac{a}{x}}{1 - \frac{a}{x}}\right\}^{\frac{1}{2}} = \pm b$, when x is infinite.

Hence take $AC = AC' = b$, and through the centre, O, draw the two asymptotes OD and OD'.

53. The asymptotes of a curve may be found in the following manner:

Let A B be an asymptote of the branch of a curve C D, ***which is not parallel to the axis of y;*** then if $y = fx$ be the equation of the curve, the equation of the asymptote will be of the form



$$y = \alpha x + \beta \quad . \quad . \quad . \quad . \quad (1).$$

Now the difference between $\alpha x + \beta$ and the ordinate of the curve corresponding to the same abscissa, must necessarily be a certain function of x, which may be called V, and which approaches zero, when x approaches $\pm \infty$, according as the branch under consideration extends indefinitely on the positive or negative side of the axis of y. The equation of this branch of the curve will be, therefore,

$$y = \alpha x + \beta + V \quad . \quad . \quad . \quad . \quad (2),$$

from which we have only to determine α and β, when x approaches $\pm \infty$. Dividing by x gives $\frac{y}{x} = \alpha + \frac{\beta + V}{x}$, and the term $\frac{\beta + V}{x}$ will tend towards zero according as x increases, and will vanish when $x = \infty$, so that α is the *limit* of the second member of the equation, and consequently that of the first also; hence

$$\alpha = \text{limit of } \frac{y}{x} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Having found α, we get from (1), $y - \alpha x = \beta + V$; hence β is the limit of the second member, since $V = 0$ when $x = \infty$; therefore

$$\beta = \text{limit of } (y - \alpha x) \quad . \quad . \quad (4).$$

The equation of the asymptote is thus completely determined, and thence its position is known.

Let the equation of the curve be reduced, if possible, to the form

$$y = \alpha x + \beta + V x^{-1} + \delta x^{-2} + \text{etc.};$$

then when x is taken indefinitely great, the terms involving negative indices will vanish, and the equation to the infinite branch of the curve will be simply

$$y = \alpha x + \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

which is the asymptotic equation, and designates a rectilinear asymptote.

EXAMPLES.

1. Find the asymptotes of the curve $a^2 y^2 - b^2 x^2 = -a^2 b^2$.

Here $\frac{y}{x} = \pm \frac{b}{a} \cdot \frac{\sqrt{(x^2 - a^2)}}{x} = \pm \frac{b}{a} \sqrt{\left(1 - \frac{a^2}{x^2}\right)}$; which, when $x = \infty$, gives (3) $\qquad \alpha = \pm \frac{b}{a}$.

Again, $y - \alpha x = \pm \frac{b}{a} \sqrt{(x^2 - a^2)} \mp \frac{b}{a} x$, which, when $x = \infty$, gives (4)

$$\beta = \pm \frac{b}{a} . \infty \mp \frac{b}{a} . \infty = 0.$$

Hence the equations of the asymptotes are

$$y = \frac{b}{a}x, \text{ and } y = -\frac{b}{a}x.$$

Otherwise. The equation of the curve can be reduced to the form

$$y = \pm\frac{b}{a}\sqrt{(x^2 - a^2)} = \pm\frac{b}{a}\left(x - \frac{a^2}{2x} - \frac{a^4}{8x^3} - \text{etc.}\right)$$

or $$y = \pm\frac{b}{a}x \mp \left(\frac{1}{2}abx^{-1} + \frac{1}{8}a^3bx^{-3} + \text{etc.}\right),$$

therefore $y = \pm\frac{b}{a}x$ is the equation of the asymptotes, as before.

2. Let the equation of the curve be $y^2 = x^2 + x$.

Here $\alpha = \text{limit of } \frac{y}{x} = \text{limit of } \frac{\sqrt{(x^2+x)}}{x} = \text{limit of } \sqrt{\left(1+\frac{1}{x}\right)} = \pm 1,$

and $\beta = \text{limit of } (y - \alpha x) = \text{limit of } \{\sqrt{(x^2+x)} - x\}$; but

$$\sqrt{(x^2+x)} - x = \frac{x}{\sqrt{(x^2+x)}+x} = \frac{1}{\sqrt{\left(1+\frac{1}{x}\right)}+1} = \frac{1}{2}, \text{ when } x = \infty;$$

consequently $\beta = \frac{1}{2}$, and the equations of the asymptotes are

$$y = x + \tfrac{1}{2}, \text{ and } y = -x + \tfrac{1}{2}.$$

3. Let the equation of the curve be $y^2 = \dfrac{x^2(x+a)}{x-a}$.

If $x = a$, then $y = \infty$; hence there is an asymptote parallel to the axis of y at the distance a.

Again, $$y = \pm x\sqrt{\frac{x+a}{x-a}} = x\left\{\frac{1-\frac{a}{x}}{1+\frac{a}{x}}\right\}^{\frac{1}{2}}$$

$$= \pm x\sqrt{\left(1 + \frac{2a}{x} + \frac{2a^2}{x^2} + \dots\right)} = \pm x\left(1 + \frac{a}{x} + \frac{a^2}{x^2} + \text{etc.}\right)$$

$$= \pm\,(x + a + a^2x^{-1} + \text{etc.})$$

Hence the equations to the other two asymptotes are

$$y = x + a, \text{ and } y = -x - a.$$

4. Find the asymptotes of the curve $yx^2 + a^3 = x^3$.

Ans. The axis of y continued in the negative direction is one of the asymptotes, and the other passes through the origin, and makes an angle of 45° with the axis of x.

5. Find the asymptote of the curve $y^3 = x^3 + ax^2$. *Ans.* $y = x + \frac{a}{3}$.

6. Find the asymptotes to the curve $y^3 = 3x^2 - x^3$. *Ans.* $y = -x + 1$.

7. Find the equations of the asymptotes to the curve whose equation is $y(x^2 - 3bx + 2b^2) = x^3$. *Ans.* $x = b$, $x = 2b$, and $y = x + 3b$.

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES.

54. In many inquiries in physical science, functions depend on two or more variables, which are independent of one another; thus the path described by a planet depends not only on its distance from the sun and

its projectile motion, but it is also influenced by the attractions of the other planets.

Let $u = f(x, y)$ denote a function of two independent variables, x and y; and let x be increased by h, and y by k; then if $u' = f(x + h, y + k)$, we have

$$u' - u = f(x + h, y + k) - f(x, y)$$
$$= \frac{f(x+h, y) - f(x, y)}{h} h + \frac{f(x+h, y+k) - f(x+h, y)}{k} k;$$

the latter form being obtained by adding $f(x + h, y)$ and subtracting it, and introducing the multipliers and divisors h and k. Now if h be diminished towards zero, the first fraction in the second member will approach to the limit $\frac{df(x,y)}{dx} dx$, or $\frac{du}{dx} dx$; and if both h and k be diminished towards zero, the other fraction will approach to the limit $\frac{df(x,y)}{dy} dy$, or $\frac{du}{dy} dy$. Also $u' - u$ will be changed to du; hence

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy \quad . \quad . \quad . \quad . \quad (1).$$

The first member, du, is called the *total differential* of u; $\frac{du}{dx} dx$ and $\frac{du}{dy} dy$ are called the *partial differentials* of u; and $\frac{du}{dx}$ and $\frac{du}{dy}$ are called the *partial differential coefficients* of u.*

In a similar manner, if $u = f(x, y, z,)$ then will

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dz} dz \; . \; . \; . \; (2).$$

Since the position of a point in space is in general referred to three rectangular coordinate planes, the equations of surfaces, etc., are of the forms

$$f(x, y, z) = 0, \text{ or } z = f(x, y);$$

and in the application of the differential calculus it is seldom necessary to develop a function of more than two variables.

Hence, to find the differential of a function of two independent variables, *differentiate the given function first with respect to one of the variables, and then with respect to the other; the sum of these partial differentials will give the total differential*; thus,

if $u = x^n \sin y$, then $du = n x^{n-1} dx \sin y + x^n \cos y \, dy$;

if $u = xy$, then $du = y\,dx + x\,dy$, as in (13);

if $u = \frac{x}{y}$, then $du = \frac{y\,dx - x\,dy}{y^2}$, as in (14).

55. If a function of two variables be differentiated successively, first

---

* The denominators dx and dy are not to be considered as mere divisors of du; but that $\frac{du}{dx}$ is the differential coefficient obtained by regarding x alone as variable, and $\frac{du}{dy}$ the one found by supposing y alone as variable.

with respect to one of the variables, and the result with respect to the other, the final result will be the same in whatever order the processes succeed one another. Thus $x^3 y^2$ differentiated with respect to x gives $3x^2 y^2 dx$, and this differentiated with respect to y gives $6x^2 y\, dx\, dy$. But if we first differentiate with respect to y, we get $2x^3 y\, dy$, and this, with respect to x, gives, as before, $6x^2 y\, dx\, dy$.

IMPLICIT FUNCTIONS.

56. When the function of x and y is an implicit one, it is frequently impossible to solve the equation for either of these variables, or to find $y = fx$ or $x = \phi y$. In cases of this kind we may suppose $f(x, y) = 0$ to be a function of two variables which have a *dependence* on each other, and deduce the method of differentiation from Art. 54.

Thus, let $u = f(x, y) = 0$, then, when the variables are *independent*,

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy \quad . \quad . \quad . \quad . \quad (1).$$

But since $u = 0$, and y is a function of x, we have $du = 0$, and, dividing by dx,

$$\frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which gives the value of $\dfrac{dy}{dx} = -\dfrac{du}{dx} \div \dfrac{du}{dy} \quad . \quad . \quad . \quad . \quad (3).$

If we wish to obtain $\dfrac{dx}{dy}$, we have only to divide the total differential of two variables by dy; then

$$\frac{du}{dy} = \frac{du}{dx} \cdot \frac{dx}{dy} + \frac{du}{dy} = 0, \therefore \frac{dx}{dy} = -\frac{du}{dy} \div \frac{du}{dx} \quad . \quad . \quad . \quad . \quad (4).$$

EXAMPLES.

1. Given $u = y^3 - 3axy + x^3 = 0$, to find $\dfrac{dy}{dx}$, and also $\dfrac{d^2y}{dx^2}$.

Here $\quad \dfrac{du}{dx} = -3ay + 3x^2$, and $\dfrac{du}{dy} = 3y^2 - 3ax$;

$\therefore$ by (3) $\quad \dfrac{dy}{dx} = -\dfrac{du}{dx} \div \dfrac{du}{dy} = \dfrac{ay - x^2}{y^2 - ax}.$

$$\text{Again, } \frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$$

or substituting for $\dfrac{dy}{dx}$ its value $\dfrac{ay - x^2}{y^2 - ax}$, we get $\dfrac{d^2y}{dx^2}$

$$= \frac{(y^2-ax)(a^2y-ax^2-2xy^2+2ax^2)-(ay-x^2)(2ay^2-2x^2y-ay^2+a^2x)}{(y^2-ax)^3}$$

$$= -\frac{2xy\,(y^3 - 3axy + x^3) + 2a^3xy}{(y^2 - ax)^3} = -\frac{2a^3xy}{(y^2 - ax)^3}.$$

2. Given $u = y^2 - 2m\,xy + x^2 - a = 0$, to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$Ans.\ \frac{dy}{dx} = \frac{my - x}{y - mx},\ \frac{d^2y}{dx^2} = \frac{a\,(m^2-1)}{(y-mx)^3}.$$

57. We can now determine the successive differentials of functions of two or more variables. Since (54), if $u = f(x, y)$,

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy;$$

and du must vary in reference to both x and y; therefore we have

$$d^2u = \left(\frac{d^2u}{dx^2}\,.\,dx^2 + \frac{d^2u}{dx\,dy}dx\,dy\right) + \left(\frac{d^2u}{dx\,dy}dx\,dy + \frac{d^2u}{dy^2}dy^2\right),$$

or $d^2u = \frac{d^2u}{dx^2}\,.\,dx^2 + 2\,\frac{d^2u}{dx\,dy}dx\,dy + \frac{d^2u}{dy^2}dy^2;$

where $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dx\,dy}$, and $\frac{d^2u}{dy^2}$ are the partial second differential coefficients of u. The first is found by taking x alone variable, the third by taking y alone variable, and the second by differentiating u, first with respect to one of the quantities x and y as the variable, and the result with respect to the other as the variable. The higher differentials may be found in a similar manner.

Find the second differential of $u = e^x y^4$.

Here $\frac{du}{dx} = e^x y^4$, $\frac{d^2u}{dx^2} = e^x y^4$, $\frac{d^2u}{dx\,dy} = 4e^x y^3$, $\frac{du}{dy} = 4e^x y^3$,

and $\frac{d^2u}{dy^2} = 12e^x y^2$; hence we have

$$d^2u = e^x y^4 dx^2 + 8e^x y^3 dx\,dy + 12e^x y^2 dy^2.$$

MAXIMA AND MINIMA FUNCTIONS OF TWO VARIABLES.

58. Let $u = f(x, y)$, where x and y are two independent variables; but we are at liberty to assume that $y\ (= \phi x)$ is a function of x, provided that ϕx is any *arbitrary* function of x. Now since y is a function of x, we cannot regard y and x as varying uniformly, as in No. 57; hence if dx is considered as constant, dy must be regarded as variable, and differentiating on this supposition, we get

$$\frac{t\,du}{dx} = \frac{du}{dx} + \frac{du}{dy}\cdot\frac{dy}{dx}\ .\ .\ .\ .\ (1),$$

and $\frac{t\,d^2u}{dx^2} = \frac{d^2u}{dx^2} + 2\frac{d^2u}{dx\,dy}\cdot\frac{dy}{dx} + \frac{d^2u}{dy^2}\cdot\frac{dy^2}{dx^2} + \frac{du}{dy}\cdot\frac{d^2y}{dx^2}\ .\ .\ .\ (2),$

where the last term of (2) is found by considering $\frac{dy}{dx}$ as variable, and $t\,du$, $t\,d^2u$, are the *total differentials* of u. Reasoning as in the case of maximum and minimum functions of one variable, it will be found that in case of either a maximum and minimum value of u, we must have $\frac{t\,du}{dx} = 0$, and $\frac{t\,d^2u}{dx^2} < 0$ in case of a maximum, and $\frac{t\,d^2u}{dx^2} > 0$, in

case of a minimum, the exceptions being precisely the same as those mentioned in the case of a single variable (Art. 48).

When $\frac{t\,du}{dx} = 0$, we have by (1), $\frac{dy}{dx}$ being an arbitrary quantity,

$$\frac{du}{dx} = 0 \quad . \ . \ . \ . \ . \quad (3), \text{ and } \frac{du}{dy} = 0 \quad . \ . \ . \ . \ . \quad (4).$$

Equation (4) reduces (2) to

$$\frac{t\,d^2u}{dx^2} = \frac{d^2u}{dx^2} + 2\,\frac{d^2u}{dx\,dy}\cdot\frac{dy}{dx} + \frac{d^2u}{dy^2}\cdot\frac{dy^2}{dx^2} \quad . \ . \ . \ . \quad (5),$$

and the second member of (5) must, with the exception referred to, be *negative* in the case of a *maximum*, and *positive* in case of a *minimum*, not changing its sign whatever values may be attributed to x and y in the immediate vicinity of those which render u or $f(x, y)$ a maximum or minimum. Put now the second member of (5) equal to zero, and solve the equation for $\frac{dy}{dx}$, then

$$\frac{dy}{dx} = \left(-\frac{d^2u}{dx\,dy} \pm \sqrt{\left\{\left(\frac{d^2u}{dx\,dy}\right)^2 - \frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2}\right\}}\right) \div \frac{d^2u}{dy^2}.$$

Hence the second member of (5) will retain its sign unchanged if

$$\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2} > \left(\frac{d^2u}{dx\,dy}\right)^2,$$

a condition which requires that $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ shall have the same sign. Consequently if $u = f(x, y)$ be a maximum or minimum, we must have the conditions $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, and $\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2} > \left(\frac{d^2u}{dx\,dy}\right)^2$; and if $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ be *negative*, u will be a maximum; but if both are *positive*, u will be a minimum.

Examples.

1. To find the maximum and minimum values of $u = x^3 + y^3 - 3axy$.

Here $\frac{du}{dx} = 3x^2 - 3ay = 0$, and $\frac{du}{dy} = 3y^2 - 3ax = 0$;

and these equations give either $x = 0, y = 0$, or $x = a, y = a$.

Again, $\frac{d^2u}{dx^2} = 6x$, $\frac{d^2u}{dx\,dy} = -3a$, $\frac{d^2u}{dy^2} = 6y$; hence

if $x = 0, y = 0$, then $\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2} - \left(\frac{d^2a}{dx\,dy}\right)^2 = 36xy - 9a^2 = -9a^2$;

if $x = a, y = a$, then $\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2 = 36xy - 9a^2 = 27a^2$;

consequently the former system of values does not correspond to either a maximum or minimum value of u; but the latter system fulfils the conditions, and if a be positive, $u = -a^3$ is a minimum, while if a be negative, $u = +a^3$ is a maximum.

2. The perimeter of a triangle is given equal to $2s$; find the sides so that the area may be a maximum.

Let x, y, and $2s - x - y$ denote the three sides of the triangle; then the area is $= \sqrt{\{s(s-x)(s-y)(x+y-s)\}}$, which is to be a maximum. Its square will therefore be a maximum, and the logarithm of the square will also be a maximum; therefore

$$u = \log s + \log(s-x) + \log(s-y) + \log(x+y-s) = \text{a maximum};$$

$$\therefore \frac{du}{dx} = -\frac{1}{s-x} + \frac{1}{x+y+s} = 0 \quad . \quad . \quad . \quad . \quad (1),$$

$$\frac{du}{dy} = -\frac{1}{s-y} + \frac{1}{x+y+s} = 0 \quad . \quad . \quad . \quad . \quad (2).$$

Whence we get $x = y = \frac{2}{3}s$, and $2s - x - y = \frac{2}{3}s$; therefore the triangle is equilateral, and the test shows the area is then a maximum.

4

3. Find the dimensions of a rectangular reservoir, open at top, having the least internal surface, its content being given equal to a^3.

Let x = length of the base, and y = its breadth; then xy = the area of the base, and the depth $= \dfrac{a^3}{xy}$; hence we have

$$u = xy + 2(x+y)\frac{a^3}{xy} = xy + \frac{2a^3}{x} + \frac{2a^3}{y} = \text{a minimum}.$$

Hence $$\frac{du}{dx} = y - \frac{2a^3}{x^2} = 0, \text{ and } \frac{du}{dy} = x - \frac{2a^3}{y^2} = 0;$$

therefore $x^2 y = 2a^3 = xy^2$, and consequently $x = y$, which shows that the base is a square whose side is $x = \sqrt[3]{2a^3} = a\sqrt[3]{2}$. Hence the depth $= \dfrac{a^3}{xy} = \dfrac{a^3}{x^2} = \dfrac{a^3}{a^2\sqrt[3]{4}} = \dfrac{a}{2}\sqrt[3]{2}$ = one-half the side of the base.

Again, $$\frac{d^2u}{dx^2} = \frac{4a^3}{x^3}, \ \frac{d^2u}{dx\,dy} = 1, \text{ and } \frac{d^2u}{dy^2} = \frac{4a^3}{y^3};$$

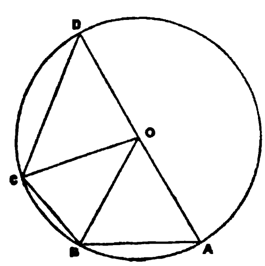
hence $$\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2 = \frac{16a^6}{x^3y^3} - 1 = \frac{16a^6}{4a^6} - 1 = 3;$$

therefore the values $x = y = a\sqrt[3]{2}$ give u a minimum, and $u = \dfrac{3a^2}{2}\sqrt[3]{4}$.

4

4. Inscribe in a given circle a polygon of n sides, having its area a maximum.

Draw from the centre, O, of the given circle to the extremities of the successive sides of the polygon, the radii OA, OB, OC, etc., and denote the successive angles AOB, BOC, COD, etc., by $\theta_1, \theta_2, \theta_3$, etc., and the radius of the circle by r. Then the area of the isosceles triangle AOB $= \frac{1}{2}r^2 \sin\theta_1$, that of BOC $= \frac{1}{2}r^2\sin\theta_2$, and so on; hence the entire area of the polygon will be $=$ $\frac{1}{2}r^2(\sin\theta_1 + \sin\theta_2 + \sin\theta_3 + \ . \ . \sin\theta_n)$.



But $\sin\theta_n = \sin\{2\pi - (\theta_1 + \theta_2 - \theta_3 + \ . \ . \ . \ \theta_{n-1})\} = \sin(2\pi - s)$, if we put

$$s = \theta_1 + \theta_2 + \theta_3 + \ . \ . \ . \ . \ \theta_{n-1};$$

hence, rejecting the constant multiplier $\frac{1}{2} r^2$, we get

$$u = \sin \theta_1 + \sin \theta_2 + \sin \theta_3 + \ . \ . \ . \ + \sin (2\pi - s) = \text{a maximum}.$$

Hence $\dfrac{du}{d\theta_1} = \cos \theta_1 - \cos (2\pi - s)$; $\dfrac{du}{d\theta_2} = \cos \theta_2 - \cos (2\pi - s)$, etc.;

but these partial differential coefficients are severally $= 0$; hence

$$\theta_1 = 2\pi - s, \ \theta_2 = 2\pi - s, \ \theta_3 = 2\pi - s, \text{ etc.};$$

and the polygon has, therefore, all its sides and angles equal, and it is, therefore, a regular polygon.

4 5. Among all rectangular prisms, to determine that which, having a given volume a^3, shall have the least surface.

Let x, y, z denote the three contiguous edges of the prism; then by the conditions

$$xyz = a^3 \ . \ . \ . \ . \ (1),$$

$$2xy + 2xz + 2yz = \text{a minimum} \ . \ . \ . \ (2).$$

From (1) we get $z = \dfrac{a^3}{xy}$, which, substituted for z in (2), and rejecting the constant factor 2, gives $u = xy + \dfrac{a^3(x+y)}{xy} = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y} = \text{a min.}$

Hence $\dfrac{du}{dx} = y - \dfrac{a^3}{x^2} = 0 \ . \ . \ . \ (3)$, and $\dfrac{du}{dy} = x - \dfrac{a^3}{y^2} = 0 \ . \ . \ . \ (4)$;

which give $x = y = a$, and thence $z = a$; therefore the prism is a cube.

6. Divide 24 into three parts, x, y, z, such that $u = xy^2z^3$ may be a maximum. *Ans.* $x = 4, y = 8$, and $z = 12$.

7. The surface of a rectangular parallelopiped is $2a^2$; find when its volume is a maximum. *Ans.* $x = y = z = a$, or a cube whose side is a.

8. Inscribe the greatest parallelopiped in a given ellipsoid.

Ans. If $2a, 2b, 2c$, be the principal diameters of the ellipsoid, then $\dfrac{2a}{3}\sqrt{3}$, $\dfrac{2b}{3}\sqrt{3}$, $\dfrac{2c}{3}\sqrt{3}$ are the edges of the greatest parallelopiped.

9. When is $u = x^2 + xy + y^2 - 9x - 6y$ a maximum or minimum? *Ans.* $x = 4, y = 1$, give $u = -21$, a minimum.

10. Show that the least polygon of a given number of sides which can be described about a given circle is a regular one.

SINGULAR VALUES OF FUNCTIONS.

59. The value of an explicit function can generally be determined by performing the operations indicated, but it sometimes happens that such a value may be given to the variable as shall render the determination of the value of the function impossible in the ordinary manner. Thus, if $x = 0$, the value of the function $u = \dfrac{x - \sin x}{x^3} = \dfrac{0}{0}$, cannot be found in the usual way. Such values are termed ***singular values***, and they may be easily determined by the method of limits. For since (39)

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{etc.};$$

$$\therefore u = \frac{x - \sin x}{x^3} = \frac{1}{1.2.3} - \frac{x^2}{1.2.3.4.5} + \frac{x^4}{1.2\ldots 7} - \text{etc.}$$

Now as x approaches to zero, the second member of the last equation approaches to $\frac{1}{1.2.3}$; hence when $x = 0$, we get $u = \frac{1}{6}$.

These singular values of functions may frequently be detected by the differential calculus in the following manner. Let v and z be the functions of x, and let $u = \frac{v}{z}$ be the function which takes the form $\frac{0}{0}$ when the value of $x = a$. Now since $u z = v$, we get by differentiation

$$u\, dz + z\, du = dv;$$

but when $x = a$, $z = 0$, therefore $u = \frac{dv}{dz}$......(1).

If the value $x = a$ makes also $dv = 0$ and $dz = 0$, then repeating the above process, gives $u = \frac{d(dv)}{d(dz)} = \frac{d^2v}{d^2z}$......(2).

Continue this process till one of the differential coefficients becomes finite, when $x = a$, and the value of the function will be determined.

EXAMPLES.

1. Find the value of the function $u = \frac{x - \sqrt{(2x^2 - a^2)}}{2x - \sqrt{(5x^2 - a^2)}}$ when $x = a$.

Differentiating the numerator and denominator of the function,

$$dv = dx - \frac{2x\,dx}{\sqrt{(2x^2 - a^2)}} = \frac{\sqrt{(2x^2 - a^2)} - 2x}{\sqrt{(2x^2 - a^2)}}\, dx,$$

$$dz = 2\,dx - \frac{5x\,dx}{\sqrt{(5x^2 - a^2)}} = \frac{2\sqrt{(5x^2 - a^2)} - 5x}{\sqrt{(5x^2 - a^2)}}\, dx;$$

$$\therefore u = \frac{dv}{dz} = \frac{\sqrt{(2x^2 - a^2)} - 2x}{\sqrt{(2x^2 - a\,)}} \times \frac{\sqrt{(5x^2 - a^2)}}{2\sqrt{(5x^2 - a^2)} - 5x};$$

and when $x = a$, we get $u = \frac{-a}{a} \cdot \frac{2a}{-a} = 2$, the value required.

2. Find the value of $u = \frac{\sin nx - nx\cos nx}{2x^2 \sin nx}$, when $x = 0$.

$$dv = n\cos nx\; dx - n\,dx\,\cos nx + n^2 x\,\sin nx\,dx = n^2 x \sin nx\,dx;$$

$$dz = 4x \sin nx\,dx + 2nx^2 \cos nx\,dx$$
$$= 2x(2\sin nx + nx\cos nx)\,dx;$$

$$\therefore u = \frac{dv}{dz} = \frac{n^2 \sin nx}{2(2\sin nx + nx\cos nx)} = \frac{0}{0}, \text{ when } x = 0.$$

Again, $d^2v = n^3 \cos nx\,dx^2$,

$$d^2z = (4n\cos nx + 2n\cos nx - 2n^2 x\sin nx)\,dx^2,$$

$$\therefore u = \frac{d^2v}{d^2z} = \frac{n^2 \cos nx}{6\cos nx - 2nx\sin nx} = \frac{n^2}{6}, \text{ when } x = 0.$$

The singular value of u, when $x = 0$, is therefore $\frac{n^2}{6}$.

Find the values of the following functions :—

3. $u = \dfrac{x^3 - 1}{x^3 + 2x^2 - x - 2}$, when $x = 1$. *Ans.* $u = \frac{1}{2}$.

4. $u = \dfrac{x^4 - 1}{x^2 - 1}$, when $x = 1$. *Ans.* $u = 2$.

5. $u = \dfrac{a^x - b^x}{x}$, when $x = 0$. *Ans.* $u = \log \dfrac{a}{b}$.

6. $u = \dfrac{6x^2 - 17x + 12}{8x^2 - 18x + 9}$, when $x = 1\frac{1}{2}$ *Ans.* $u = \frac{1}{6}$.

7. $u = \dfrac{x^2 \tan x}{1 + \tan x}$, when $x = \dfrac{\pi}{2}$. *Ans.* $u = \dfrac{\pi^2}{4}$

8. $u = \dfrac{\sec x}{x} - \cot x$, or $\dfrac{\tan x - x \cos x}{x \sin x}$ when $x = 0$. *Ans.* $u = 0$.

9. $u = \dfrac{\cos x - \cos 2x}{\cos x - \cos 3x}$, when $x = 0$. *Ans.* $u = \frac{3}{8}$.

10. $u = \dfrac{e^{2x} - 1}{e^x \log (1 + x)}$, when $x = 0$. *Ans.* $u = 2$.

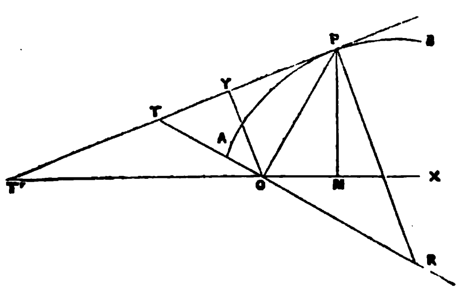
11. $u = \dfrac{a^n - x^n}{\log a - \log x}$, when $x = a$. *Ans.* $u = n a^n$.

12. $u = \dfrac{\pi}{4x} - \dfrac{\pi}{2x(e^{\pi x} + 1)}$, when $x = 0$. *Ans.* $u = \dfrac{\pi}{8}$.

CURVES REFERRED TO POLAR COORDINATES.

60. It is frequently advantageous to refer curves to polar coordinates instead of rectangular ones, especially in the investigations of physical astronomy, and we shall now advert to the method of drawing tangents to spiral and other curves by means of their polar equations.

Let O be the pole, and O P the radius vector of a curve APB. Draw the line O T perpendicular to O P, meeting the tangent P T′ in T, and let O be the origin of rectangular coordinates O M and M P. Let O P $= r$, and angle P O T′ $= \theta$; then



$$x = -r \cos \theta,\ y = r \sin \theta,\ x^2 + y^2 = r^2;$$

and differentiating these, we get

$$dx = -\cos \theta dr + r \sin \theta d\theta;\ dy = \sin \theta dr + r \cos \theta d\theta;\ x dx + y dy = r dr.$$

$$\text{Now } \tan \mathrm{TPO} = \tan (\mathrm{POM} - \mathrm{PT'M}) = \frac{\tan \mathrm{POM} - \tan \mathrm{PT'M}}{1 + \tan \mathrm{POM} \tan \mathrm{PT'M}}$$

$$= \frac{\dfrac{y}{x} - \dfrac{dy}{dx}}{1 + \dfrac{y\,dy}{x\,dx}} = \frac{y\,dx - x\,dy}{x\,dx + y\,dy} = \frac{r^2\,d\theta}{r\,d r} = \frac{r\,d\theta}{d r};$$

therefore $\quad \mathrm{O\,T} = \mathrm{O\,P} \tan \mathrm{T\,P\,O} = \dfrac{r^2\,d\theta}{d r} =$ polar subtangent...(1).

Draw the normal P R perpendicular to the tangent, P T meeting T O produced in R; then we have

$$\mathrm{O\,R} = \frac{\mathrm{O\,P}^2}{\mathrm{O\,T}} = r^2 \div \frac{r^2\,d\theta}{d r} = \frac{d r}{d\theta} = \text{the polar subnormal} \;.\;.\;.\;.\; (2).$$

If O Y be drawn perpendicular to the tangent P T; then if O Y $= p$, we shall have, by the similar triangles P O T and T O Y,

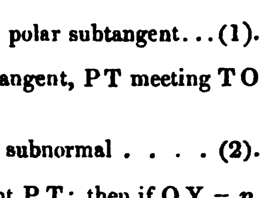
$$\mathrm{O\,Y} = \mathrm{O\,T} \cos \mathrm{T\,O\,Y} = \mathrm{O\,T} \cos \mathrm{T\,P\,O} = \frac{\mathrm{O\,T}}{\sec \mathrm{T\,P\,O}},$$

or $\mathrm{O\,Y}^2 = \dfrac{\mathrm{O\,T}^2}{1 + \tan^2 \mathrm{T\,P\,O}}$; that is $p^2 = \dfrac{r^4\,d\theta^2}{d r^2 + r^2\,d\theta^2} = \dfrac{r^4}{r^2 + \dfrac{d r^2}{d\theta^2}}$....(3).

EXAMPLES.

1. If the straight line O A revolve uniformly round O as a centre, the point P which moves uniformly from O along O A will trace out a curve called the *spiral of Archimedes.*

Let the radius vector O P $= r$, and angle A O P $= \theta$, and let a' be the value of r when the revolving line has made a complete revolution; then $a' : r :: 2\pi : \theta$; which gives $2\pi r = a'\theta$; but if we put $a' = 2\pi a$, then the equation of the curve will be simply $r = a\theta$.



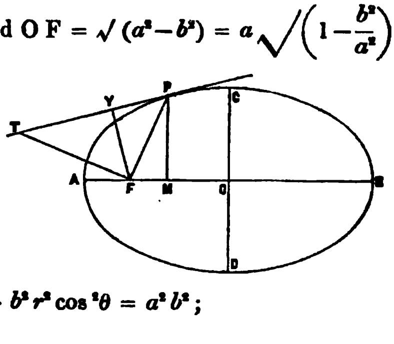
To draw a tangent to this spiral, we have from the equation to the curve,

$$d r = a\,d\theta;\ \text{hence}\ \frac{d\theta}{d r} = 1 \div \frac{d r}{d\theta} = \frac{1}{a};\ \text{consequently}$$

$$\mathrm{O\,T} = r^2 . \frac{d\theta}{d r} = r^2 . \frac{1}{a} = r^2 . \frac{2\pi}{a'} = \frac{2\pi r^2}{a'} = \text{the polar subtangent.}$$

2. Find the polar equation to the ellipse, the focus being the pole, and draw a tangent to the curve.

Let F P $= r$, angle A F P $= \theta$, and O F $= \sqrt{(a^2 - b^2)} = a\sqrt{\left(1 - \frac{b^2}{a^2}\right)}$ $= a e$, where $e^2 = 1 - \dfrac{b^2}{a^2}$; then O M $= a e + r \cos\theta$, and M P $= r \sin\theta$. Substituting these values of O M and M P for x and y in the usual equation to the ellipse, we have $a^2 y^2 + b^2 x^2$, or $a^2 r^2 \sin^2\theta + b^2 (a e + r \cos\theta)^2 = a^2 b^2$, or since $1 - \cos^2\theta = \sin^2\theta$, $a^2 r^2 - a^2 r^2 \cos^2\theta + b^2 (a^2 - b^2) + 2 a e\, b^2 r \cos\theta + b^2 r^2 \cos^2\theta = a^2 b^2$;



$\therefore a^2 r^2 = b^4 - 2ae\,b^2 r\cos\theta + a^2 e^2 r^2 \cos^2\theta = (b^2 - aer\cos\theta)^2$;

hence $ar = b^2 - aer\cos\theta$, and $r = \dfrac{b^2}{a(1+e\cos\theta)} = \dfrac{a(1-e^2)}{1+e\cos\theta}$, the polar equation to the ellipse, the focus being the pole. This equation is much used in physical astronomy, since the orbits of the planetary bodies are ellipses, and the sun is situated in one of their foci.

Differentiating this equation, we get

$$dr = \frac{a(1-e^2)\times e\sin\theta\,d\theta}{(1+e\cos\theta)^2}, \text{ hence } \frac{d\theta}{dr} = \frac{(1+e\cos\theta)^2}{ae(1-e^2)\sin\theta}, \quad \cdots (1)$$

$$\mathrm{FT} = \frac{r^2 d\theta}{dr} = \frac{a^2(1-e^2)^2}{(1+e\cos\theta)^2}\times\frac{(1+e\cos\theta)^2}{ae(1-e^2)\sin\theta} = \frac{b^2}{ae\sin\theta} = \text{subtangent.}$$

Also, by formula (3) we have FY^2 or $p^2 = \dfrac{r^4 d\theta^2}{dr^2 + r^2 d\theta^2}$; but

$$dr^2 + r^2 d\theta^2 = \frac{a^2e^2(1-e^2)^2\sin^2\theta\,d\theta^2}{(1+e\cos\theta)^4} + r^2 d\theta^2 = \frac{e^2r^2\sin^2\theta\,d\theta^2}{(1+e\cos\theta)^2} + r^2 d\theta^2$$

$$= \frac{r^2 d\theta^2}{(1+e\cos\theta)^2}(e^2\sin^2\theta + 1 + 2e\cos\theta + e^2\cos^2\theta)$$

$$= \frac{r^2 d\theta^2\{2(1+e\cos\theta) - (1-e^2)\}}{(1+e\cos\theta)^2}.$$

Now substituting for $1 + e\cos\theta$ its value $\dfrac{a(1-e^2)}{r}$, we get

$$p^2 = \frac{r^2(1+e\cos\theta)^2}{2(1+e\cos\theta)-(1-e^2)} = \frac{a^2(1-e^2)^2}{\dfrac{2a(1-e^2)}{r} - (1-e^2)} = \frac{a^2r(1-e^2)}{2a-r} = \frac{b^2 r}{2a-r}.$$

In a similar manner, the square of the perpendicular from the other focus upon the tangent is found to be ($2a - r$ being the radius vector)

$$p'^2 = \frac{b^2(2a-r)}{r}\text{; consequently } pp' = b^2\text{; that is,}$$

the product of the perpendiculars from the foci to the tangent at any point of an ellipse is equal to the square of the semi-minor axis.

3. Draw a tangent to the *logarithmic* or *equiangular spiral*, its equation being $r = a^\theta$. *Ans.* Subtangent $= \dfrac{r}{\log_e a}$.

4. Find the polar subtangent in the curve whose equation is $r = \dfrac{a}{\theta^{\frac{1}{2}}}$.

Ans. Subtangent $= \dfrac{2a^2}{r}$.

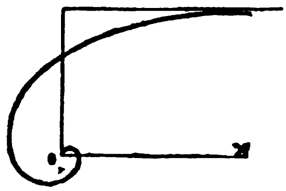
5. Find the polar subtangent in the hyperbolic spiral whose equation is $r = \dfrac{a}{\theta}$. *Ans.* Subtangent $= a$.

ASYMPTOTES TO POLAR CURVES.

61. Since an asymptote is a tangent to a curve, at a point infinitely distant, which passes at a finite distance from the origin, we have only

to assume $r = \infty$, and find the value of θ from the equation to the curve $r = f\theta$; then if the polar subtangent $r^2 \dfrac{d\theta}{dr}$ be finite, there will be an asymptote corresponding to the value assigned to θ.

1. To find whether the curve $r = \dfrac{a}{\theta}$ has an asymptote. Assume $r = \infty$; then $\theta = 0$, and the polar subtangent is $-r^2 \dfrac{d\theta}{dr} = \dfrac{a^2}{\theta^2} \times \dfrac{\theta^2}{a} = a$, since θ diminishes as r increases. Hence there is an asymptote parallel to the fixed line O X.

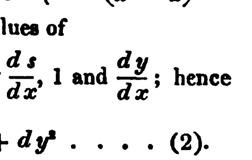


2. Let the equation of the spiral be $r = \dfrac{a\theta^2}{\theta^2 - 1}$.

Let $r = \infty$, then $\theta^2 - 1 = 0$ or $\theta = \pm 1$; hence the polar subtangent is $-r^2 \dfrac{d\theta}{dr} = \dfrac{a^2\theta^4}{(\theta^2 - 1)^2} \times \dfrac{(\theta^2 - 1)^2}{2a\theta} = \dfrac{a\theta^3}{2} = \pm \dfrac{a}{2}$, since $\theta = \pm 1$ when $r = \infty$. Hence the curve has two asymptotes.

DIFFERENTIAL OF AN ARC OF A CURVE.

62. Let A P B be a curve, and let the arc A P $= s$, arc A Q $= s'$ O M $= x$, O N $= x'$, M P $= y$, N Q $= y'$, and the chord P Q $= c$; then the arc P Q $= s' - s$, and by common algebra, we have



$$\frac{s' - s}{x' - x} = \frac{s' - s}{c} \cdot \frac{c}{x' - x} \quad . \; . \; . \; . \; . \; (1).$$

But since $c^2 = (x' - x)^2 + (y' - y)^2$, the equation (1) becomes

$$\frac{s' - s}{x' - x} = \frac{s' - s}{c} \cdot \left\{\frac{(x'-x)^2 + (y'-y)^2}{(x'-x)^2}\right\}^{\frac{1}{2}} = \frac{s' - s}{c}\left\{1 + \left(\frac{y' - y}{x' - x}\right)^2\right\}^{\frac{1}{2}}.$$

Now when Q approaches P, the limiting values of

$\dfrac{s' - s}{x' - x}$, $\dfrac{s' - s}{c}$, and $\dfrac{y' - y}{x' - x}$ are respectively $\dfrac{ds}{dx}$, 1 and $\dfrac{dy}{dx}$; hence

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}, \text{ or } ds^2 = dx^2 + dy^2 \quad . \; . \; . \; . \; (2).$$

Hence *the square of the differential of an arc is equal to the sum of the squares of the differentials of its coordinates.*

63. If $x = r \cos\theta$ and $y = r \sin\theta$; then will $dx = dr \cos\theta - r \sin\theta\, d\theta$, $dy = dr \sin\theta + r \cos\theta\, d\theta$, and hence

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \quad . \; . \; . \; . \; (3),$$

which is the formula for the differential of an arc of a curve referred to polar coordinates.

THE DIRECTION OF CURVATURE OF A CURVE.

64. From the theory of tangents to curves, the direction of their curvature may be easily determined. For since (50)

$$\tan\theta = \frac{dy}{dx}, \text{ or } \theta = \tan^{-1}\frac{dy}{dx},$$

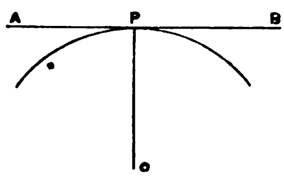
$$\therefore \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}, \text{ or } \left(1 + \frac{dy^2}{dx^2}\right) \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}.$$

Now $1 + \left(\frac{dy}{dx}\right)^2$ is essentially positive; therefore the sign of $\frac{d\theta}{dx}$ is always the same as that of $\frac{d^2y}{dx^2}$, and if $\frac{d\theta}{dx}$ be *positive*, θ increases when x increases, and therefore the concavity of the curve must be turned upwards as in Art. 50 (fig. 2); but if $\frac{d\theta}{dx}$ be negative, θ diminishes when x increases, and therefore the concavity is turned downwards, as in (fig. 1); consequently the curve will be convex towards the axis of x if the second differential coefficient of y be positive, and concave if it be negative. We have supposed that y is positive in the preceding remarks, and if y be negative, the conclusion in respect of the signs will be reversed. The following enunciation of the principle will apply in all cases :—

A curve at any point $(x\ y)$ *is convex to the axis of* x, *if* y *and* d^2y *have the same sign, but concave if the signs be different.*

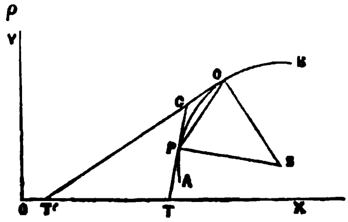
RADIUS OF CURVATURE.

65. If a circle touch a straight line A B in any point P, it is evident that in the immediate vicinity of P the circle will tend to coincide with A B, as the radius O P = ρ, of the circle increases; therefore the greater the radius is, the curvature of the circle is the less, and the less the radius is, the greater is the curvature; consequently the reciprocal of the radius $\frac{1}{\rho}$ is naturally taken as the measure of curvature in different circles. Let P, Q be two points in the curve A P Q B, and let P S, Q S be normals, and P T, Q T′ tangents at these points. Let the arc A P = s, the arc A Q = s', angle P T X = ϕ, angle Q T′ X = ϕ', and the chord of the arc P Q = c; then arc P Q = $s' - s$, and angle T C T′ = $\phi - \phi'$. Now in the triangle P Q S we have



$$\frac{\sin S}{c} = \frac{\sin SQP}{\rho}.$$

But since a circle passes through the four points S, P, C, Q, the angle T C T′ = P S Q = $\phi - \phi'$; hence the last equation becomes



$$\frac{\sin(\phi - \phi')}{c} = \frac{\sin SQP}{\rho}.$$

Now the limit to which this will

approach, as Q approaches P, is $\frac{d\phi}{ds} = \frac{1}{\rho}$ . . . . (1),

since then the chord P Q and the arc P Q approach to equality, and the angle S Q P tends to become the same as S Q T', or a right angle.

Again, the mean curvature of the arc P Q depends on the relative magnitudes of the angle T C T' and the arc P Q; for the greater the change of direction of the curve, or the angle T C T', is for a given distance P Q, the greater must be the curvature of P Q, and the less this change is, the less does P Q deviate from the tangent T C, or the less is its curvature. Now the limit to which the ratio $\frac{\phi - \phi'}{s' - s}$ approaches, when Q approaches P, is $\frac{d\phi}{ds}$, and consequently (1) $\frac{d\phi}{ds}$ or $\frac{1}{\rho}$ is the index of curvature at the point P; and this is the curvature of the circle whose radius is ρ. Hence that circle which has the same curvature as the curve at P, and which coincides with it more nearly than any other circle touching it in the same point, is called the ***circle of curvature***, or the ***osculatory circle*** at the point P. Its radius ρ is called the ***radius of curvature***, and the centre S, situated in the normal P S, is called the ***centre of curvature***.

Since $\tan\phi = \frac{dy}{dx}$ (50), or $\phi = \tan^{-1}\frac{dy}{dx}$; therefore (30)

$$d\phi = d\,.\,\frac{dy}{dx} \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \frac{d^2y\,dx}{dx^2 + dy^2} = \frac{d^2y\,dx}{ds^2} \quad (62);$$

$$\therefore \frac{1}{\rho} = \frac{d\phi}{ds} = \pm\frac{d^2y\,dx}{ds^3}, \text{ and } \rho = \pm\frac{ds^2}{d^2y\,dx} \quad . \; . \; . \; . \quad (2).$$

If dx be considered a variable, then we get from $\phi = \tan^{-1}\frac{dy}{dx}$,

$$\phi = \frac{d^2y\,dx - d^2x\,dy}{dx^2\left(1 + \frac{dy^2}{dx^2}\right)} = \frac{d^2y\,dx - d^2x\,dy}{dx^2 + dy^2} = \frac{d^2y\,dx - d^2x\,dy}{ds^2};$$

$$\therefore \rho = \pm\frac{ds^3}{d^2y\,dx - d^2x\,dy} \quad . \; . \; . \; . \quad (3).$$

Since the magnitude of ρ, without regard to sign, is the object of investigation, the upper or lower sign must be employed, according as the $+$ or the $-$ renders the expression for ρ positive. The direction in which ρ is to be drawn will be determined in Art. 72.

66. For *polar coordinates*, we have $x = r\cos\theta$, and $y = r\sin\theta$;

$\therefore dx = dr\cos\theta - r\sin\theta\,d\theta$; $dy = dr\sin\theta + r\cos\theta\,d\theta$;

$d^2x = d^2r\cos\theta - 2\,dr\,d\theta\sin\theta - r\cos\theta\,d\theta^2$;

$d^2y = d^2r\sin\theta + 2\,dr\,d\theta\cos\theta - r\sin\theta\,d\theta^2$;

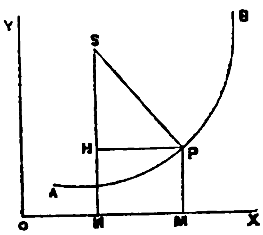
$\therefore d^2y\,dx - d^2x\,dy = (r^2d\theta^2 - r\,d^2r + 2\,dr^2)\,d\theta$;

and $ds^2 = dx^2 + dy^2 = dr^2 + r^2d\theta^2$; consequently (2).

$$= \pm\frac{(dr^2 + r^2d\theta^2)^{\frac{3}{2}}}{(r^2d\theta^2 - r\,d^2r + 2\,dr^2)\,d\theta} \quad . \; . \; . \; . \quad (3).$$

COORDINATES OF THE CENTRE OF CURVATURE.

67. Let S be the limiting position of the point of intersection of two normals to the curve; then $PS = \rho$, and if α and β denote the coordinates of S, and PH be drawn parallel to OX, we shall have



$$\left.\begin{array}{l}\alpha = ON = OM - MN = x - \rho\sin\phi \\ \beta = NS = PM + SH = y + \rho\cos\phi\end{array}\right\}\ldots(1),$$

since the normal PS makes an angle $\frac{\pi}{2} - \phi$ with the axis of x.

But since $\tan\phi = \frac{dy}{dx}$ (50); therefore $\sec\phi$ $= \sqrt{(1 + \tan^2\phi)} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \frac{ds}{dx}$; hence $\cos\phi = \frac{1}{\sec\phi} = \frac{dx}{ds}$; and $\sin\phi = \sqrt{(1 - \cos^2\phi)} = \sqrt{\left(1 - \frac{dx^2}{ds^2}\right)} = \frac{dy}{ds}$. Substituting these values for $\sin\theta$ and $\cos\theta$ in (1), and recollecting the value of ρ in Art. 65, equation (2), we get

$$\left.\begin{array}{l}\alpha = x - \frac{ds^3}{d^2y\,dx}\cdot\frac{dy}{ds} = x - \frac{ds^2}{d^2y}\cdot\frac{dy}{dx} \\ \beta = y + \frac{ds^3}{d^2y\,dx}\cdot\frac{dx}{ds} = y + \frac{ds^2}{d^2y}\end{array}\right\}\ldots.(2).$$

LOCUS OF THE CENTRE OF CURVATURE.

68. If the point P be made to move along the curve A B, the point S will vary in position, and trace out some curve. The locus of S, the centre of curvature, is called the *evolute* of the curve A P B, and the proposed curve, in relation to the evolute, is called its *involute*. The equation of the evolute of a curve is found by eliminating x and y, and their differentials by means of the equation of the curve $y = fx$, and equations (2) in No. 67.

EXAMPLES.

1. To find the radius of curvature of the cycloid, the coordinates of its centre of curvature, and the equation of its evolute.

The equation to the cycloid is (Art. 50, Ex. 4)

$$x = r\,\text{vers}^{-1}\frac{y}{r} - \sqrt{(2ry - y^2)} \text{ or } \frac{y}{r} = \text{vers}\,\frac{x + \sqrt{(2ry - y^2)}}{r}\ldots.(1).$$

Differentiating this equation we get

$$\frac{dy}{dx} = \sqrt{\left(\frac{2r - y}{y}\right)} = \frac{\sqrt{(2ry - y^2)}}{y}, \therefore \frac{d^2y}{dx} = -\frac{r\,dy}{y\sqrt{(2ry - y^2)}}.$$

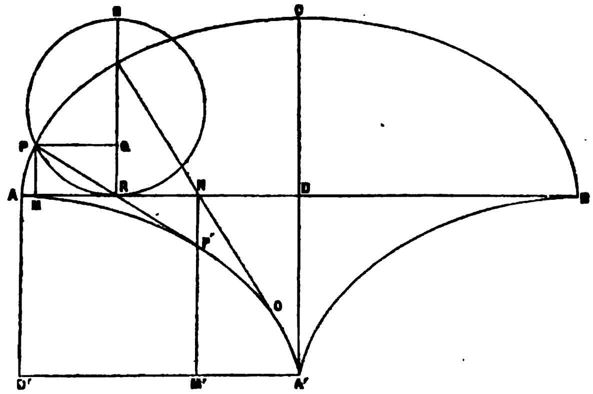
$$\therefore \frac{d^2y}{dx}\cdot\frac{dy}{dx} = -\frac{r\,dy}{y^2}, \text{ or } \frac{d^2y}{dx^2} = -\frac{r}{y}, \therefore d^2y\,dx = -\frac{r\,dx^3}{y^2}.$$

Again $1 + \frac{dy^2}{dx^2} = \frac{dx^2 + dy^2}{dx^2} = \frac{ds}{dx^2} = \frac{2r}{y} \therefore ds^2 = \frac{2r\,dx^2}{y}$;

But by Art. 65, eq. (2), we have

$$\rho^2 = \frac{(ds^2)^2}{(d^2y\,dx)^2} = \frac{8r^3(dx^2)^3}{y^3} \div \left(-\frac{r\,dx^2}{y^2}\right)^2 = 8ry;$$

hence $\rho = 2\sqrt{2ry} = 2\sqrt{\mathrm{SR}\,.\,\mathrm{RQ}} = 2\,\mathrm{PR}$,



and therefore if the intercept of the normal PR be produced till RP′ = RP, then P′ will be the centre of curvature, and PP′ = 2PR = the radius of curvature.

Again, by Art. 67 equations (2), we have

$$\alpha = x - \frac{ds^2}{d^2y}\cdot\frac{dy}{dx} = x + 2y\,.\,\frac{\sqrt{(2ry - y^2)}}{y} = x + 2\sqrt{(2ry - y^2)},$$

$$\beta = y + \frac{ds^2}{d^2y} \qquad = y - 2y \qquad\qquad = -y;$$

which are the coordinates of P′, the centre of curvature.

Lastly, from these values of α and β, we have

$$y = -\beta,\ x = \alpha - 2\sqrt{(2ry - y^2)} = \alpha - 2\sqrt{(-2r\beta - \beta^2)}.$$

Substituting these values of y and x in (1) we get

$$-\frac{\beta}{r} = \text{vers}\,\frac{\alpha - \sqrt{(-2r\beta - \beta^2)}}{r}, \quad . \ . \ . \ . \ (2).$$

which is possible only when β is negative. Produce CD till DA′ = CD and transfer the origin from A to A′. To effect this we must write in the last equation $\beta' - 2r$ for β and $\pi r - \alpha'$ for α (AD being $= \pi r$); hence (2) becomes

$$2 - \frac{\beta'}{r} = \text{vers}\left\{\pi - \frac{\alpha' + \sqrt{(2r\beta' - \beta'^2)}}{r}\right\} = 2 - \text{vers}\,\frac{\alpha' + \sqrt{(2r\beta' - \beta'^2)}}{r};$$

$$\therefore \frac{\beta'}{r} = \text{vers}\frac{\alpha' + \sqrt{(2r\beta' - \beta'^2)}}{r}, \text{ or } \alpha' = r\,\text{vers}^{-1}\frac{\beta'}{r} - \sqrt{(2r\beta' - \beta'^2)} \ldots\ldots (2')$$

which is the equation of the evolute, and is consequently a cycloid having A′ for its origin, and described by a generating circle *equal* to the original one, and moving in the direction A′ D′.

2. Find the radius of curvature at a point (xy) of the parabola whose equation is $y^2 = 4\,m\,x$, and also at the vertex.

$$Ans.\ \rho = \frac{2\,(m+x)^{\frac{3}{2}}}{m^{\frac{1}{2}}},\ \text{and at the vertex}\ \rho = 2\,m.$$

3. Find the coordinates of the centre of curvature of the parabola, and the equation of its evolute.

$Ans.$ $\alpha = 3\,x + 2\,m,\ \beta = -\dfrac{y^3}{4\,m^2}$, and the equation to the evolute is $\beta^2 = \dfrac{4}{27\,m}(\alpha - 2\,m)^3$, denoting a curve called the semi-cubical parabola.

4. Find the radius of curvature at any point in the ellipse whose equation is $a^2\,y^2 + b^2\,x^2 = a^2\,b^2$; and determine the radii of curvature at the extremities of the axes.

$Ans.$ $\rho = \dfrac{(a^4 - e^2\,x^2)^{\frac{3}{2}}}{a\,b} = \dfrac{(a^4\,y^2 + b^4\,x^2)^{\frac{3}{2}}}{a^4\,b^4}$: at the extremity of the minor axis $\rho = \dfrac{a^2}{b}$, and at the extremity of the major axis, $\rho = \dfrac{b^2}{a}$.

5. Find the equation to the evolute of the ellipse.

$$Ans.\ (a\,\alpha)^{\frac{2}{3}} + (b\,\beta)^{\frac{2}{3}} = (a\,e)^{\frac{4}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

6. Find the radius of curvature at any point in the ellipse, in terms of λ, the angle between the normal and the transverse axis.

$$Ans.\ \rho = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2\lambda)^{\frac{3}{2}}}.$$

7. Find the radius of curvature at any point in the logarithmic curve, its equation being $y = a^x$.

$$Ans.\ \rho = \frac{(M_a^2 + y^2)^{\frac{3}{2}}}{M_a\,y}.$$

8. Find the equation of the evolute of the hyperbola, its equation being $a^2\,y^2 - b^2\,x^2 = -\,a^2\,b^2$. $\quad Ans.\ (a\,\alpha)^{\frac{2}{3}} - (b\,\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$

9. Determine the radius of curvature of the rectangular hyperbola, its equation between the asymptotes being $x\,y = a^2$; and find also the equation of the evolute.

$$Ans.\ \rho = \frac{(a^4 + x^4)^{\frac{3}{2}}}{2\,a^2\,x^3},\ \text{and the equation is}\ (\alpha+\beta)^{\frac{2}{3}} - (\alpha-\beta)^{\frac{2}{3}} = (4\alpha)^{\frac{2}{3}}.$$

TANGENTS AND ARCS OF EVOLUTES.

69. The equation of the normal at the point $(x\,y)$ in a curve, and which passes through a point $(\alpha\,\beta)$ in the evolute, is

$$y - \beta = -\frac{d\,x}{d\,y}(x - \alpha) \quad . \quad . \quad . \quad (1).$$

Differentiating (1), and regarding $d\,x$ as constant, we get

$$d\,y - d\,\beta = \frac{d\,x\,d^2 y}{d\,y^2}(x - \alpha) - \frac{d\,x^2}{d\,y} + \frac{d\,\alpha\,d\,x}{d\,y};$$

but Art. 67, equations (2), gives $x - \alpha = \frac{d s^2}{d^2 y} \cdot \frac{d y}{d x}$;

hence $\quad d y - d \beta = \frac{d s^2}{d y} - \frac{d x^2}{d y} + \frac{d \alpha \, d x}{d y} = d y + \frac{\alpha \, d x}{d y}$;

and transposing, $d \alpha \; d x + d \beta \; d y = 0$, or $\frac{d x}{d y} = - \frac{d \beta}{d \alpha}$;

hence (1) becomes $\beta - y = \frac{d \beta}{d \alpha} (\alpha - x) \quad . \quad . \quad . \quad . \quad (2)$,

which is the equation of a line touching the evolute at the point $(\alpha \beta)$, and passing through the point $(x y)$, in the original curve; hence it follows that *the radius of the osculatory circle touches the evolute, and that the centre of that circle is the point of contact;* and *the radius of curvature is at the same time a normal to the involute and a tangent to the evolute.*

70. From equations (1) Art. 67 we have

$$(\alpha - x)^2 + (\beta - y)^2 = \rho^2 \quad . \quad . \quad . \quad . \quad (1),$$

and if σ denote the arc of the evolute corresponding to the coordinates α and β; then by Art. 62, $d \alpha^2 + d \beta^2 = d \sigma^2 \quad . \quad . \quad . \quad . \quad (2)$.

Differentiate (1), then we have

$$(\alpha - x)(d \alpha - d x) + (\beta - y)(d \beta - d y) = \rho \, d \rho,$$

or $(\alpha - x) d \alpha + (\beta - y) d \beta + (x - \alpha) d x + (y - \beta) d y = \rho \, d \rho$,

but $(x - \alpha) d x + (y - \beta) d y = 0$ by eq. (1) Art. 69; therefore

$$(\alpha - x) d \alpha + (\beta - y) \, d \beta = \rho \, d \rho \quad . \quad . \quad . \quad . \quad (3).$$

Substitute the value of $\beta - y$ from eq. (2) Art. 69 in equations (1) and (3); then

$$(\alpha - x)^2 d \sigma^2 = \rho^2 d \alpha^2, \text{ and } (\alpha - x) d \sigma^2 = \rho \, d \rho \, d \alpha;$$

and dividing the square of the latter by the former, gives

$$d \sigma^2 = d \rho^2 \text{ or } d \sigma = \pm d \rho \; \therefore \sigma = C \pm \rho,$$

where C is a constant quantity, for otherwise we should not have $d \sigma = \pm d \rho$. Now if ρ and ρ' be the radii of curvature at any two points, and σ, σ' the corresponding arcs of the evolute, then

$$\sigma' = C \pm \rho' \text{ and } \sigma = C \pm \rho;$$

$\therefore \rho' - \rho = \sigma' - \sigma$, that is, the difference between the two radii is equal to the arc of the evolute comprehended between them. From this property, and the one established in the preceding number, we deduce the following very interesting conclusion :—

If an inextensible thread or cord be applied to the evolute, and being kept tight, be gradually unwound, a fixed point in it will describe the original curve or involute.

Thus if a thread, applied to the evolute A A′ of the figure in Art. 68, be unwound by moving the point P from A towards C, the fixed point P will describe the involute A C; hence the origin of the names *involute* and *evolute*, which in French phraseology are termed the *développante* and the *développée*. The conclusions arrived at in these two last articles afford the means of making a pendulum move in an arc of a cycloid.

SINGULAR POINTS OF CURVES.

71. A *singular point* of a curve is a point at which the curve has some property inherently different from what it has in the immediate

vicinity of that point. Thus points of contrary flexure, cusps, multiple points, conjugate points, and so on, are singular points.

I.—*Points of Contrary Flexure.*

72. If a curve change the direction of its curvature, the point at which the change takes place is called a *point of contrary flexure*, or a *point of inflexion*. To determine such points we have only to refer to Art. 64, where it will be seen that a curve is convex to the axis of x, if the second differential coefficient of y be positive, and concave if it be negative. Hence at a point of contrary flexure $\frac{d^2 y}{d x^2}$ must change its sign, and must therefore be either zero or infinite. If $\frac{d^2 y}{d x^2} = 0$, and $\frac{d^3 y}{d x^3}$ be neither 0 nor ∞, there will be a point of inflexion.

EXAMPLES.

1. Find whether the curve whose equation is $x y^2 = 4 r^2 (2 r - x)$ has a point of contrary flexure.

Here $\quad y = 2 r \sqrt{\left(\frac{2 r - x}{x}\right)}; \; \frac{d y}{d x} = -\frac{2 r^2}{x \sqrt{(2 r x - x^2)}},$

and $$\frac{d^2 y}{d x^2} = \frac{2 r^2 (3 r x - 2 x^2)}{x^2 (2 r x - x^2)^{\frac{3}{2}}} = \frac{2 r^2 (3 r - 2 x)}{x (2 r x - x^2)^{\frac{3}{2}}};$$

$$\therefore \; 3 r - 2 x = 0, \text{ and } x = \frac{3 r}{2},$$

hence the curve has two points of contrary flexure, and their coordinates are $$x = \frac{3 r}{2}, \text{ and } y = \pm \frac{2 r}{3} \sqrt{3}.$$

2. Let the equation of the curve be $y = a x + b x^2 + c x^3$, to determine the point or points of contrary flexure.

Ans. $x = -\frac{b}{3 c}, \; y = -\frac{b (9 a c - b^2)}{27 c^2}.$

3. Find whether the curve, whose equation is $y = \frac{x^2 (a^2 - x^2)}{a^3}$, has a point of inflexion, and if so, determine its coordinates.

Ans. $x = \pm \frac{a}{6} \sqrt{6}$, and $y = \frac{5 a}{36}.$

II.—*Multiple Points.*

73. The most remarkable singular points of a curve, next to points of contrary flexure, are those called multiple points.

A *multiple point* is one in which two or more branches of a curve *intersect* or *touch* each other, and it is called a *double point*, or *triple point*, and so on, according to the number of branches passing through the point. Now, when several branches of a curve intersect one another, it is obvious that there will be as many tangents at this point as there are branches, and consequently as many *different* values of $\frac{d y}{d x}$, which

expresses the tangent of the inclination of any tangent at the point $(x\, y)$ to the axis of x. But when several branches of a curve touch one another, there will be as many *equal* values of the same differential coefficient as there are branches which touch one another. On these principles the theory of multiple points rests.

Let the equation of a curve, divested of radicals and variable denominators, be $\quad u = f(x, y) = 0 \quad . \quad . \quad . \quad . \quad (1);$
then, y being a function of x, we have (Art. 56)

$$\frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad (2);$$

hence $\dfrac{dy}{dx} = -\dfrac{du}{dx} \div \dfrac{du}{dy}$, which, being divested of radicals, can furnish only *one* value of $\dfrac{dy}{dx}$, unless $\dfrac{du}{dx} = 0$, and $\dfrac{du}{dy} = 0$, and then the values of $\dfrac{dy}{dx}$ may be determined from the equation $\dfrac{dy}{dx} = \dfrac{0}{0}$, as in Art. 59. The number of *unequal* or *equal* values of $\dfrac{dy}{dx}$ indicates the number of different intersecting or osculating branches.

Multiple points may also be found thus:—Differentiate eq. (2), then

$$\frac{d^2u}{dx^2} + 2\frac{d^2u}{dx\,dy} \cdot \frac{dy}{dx} + \frac{d^2u}{dy^2}\frac{dy^2}{dx^2} + \frac{du}{dy} \cdot \frac{d^2y}{dx^2} = 0,$$

which since $\dfrac{du}{dy} = 0$, reduces to

$$\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2 + 2\frac{d^2u}{dx\,dy} \cdot \frac{dy}{dx} + \frac{d^2u}{dx^2} = 0, \quad . \quad . \quad . \quad . \quad (3),$$

the solution of which for $\dfrac{dy}{dx}$ contains the radical

$$\left\{\left(\frac{d^2u}{dx\,dy}\right)^2 - \frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2}\right\}^{\frac{1}{2}};$$

hence, if the quantity within the vinculum be positive, there will be a double point by intersection; and if it be equal to zero, there will be a double point by osculation. By proceeding to the higher differentials of eq. (2), we might determine the existence of triple points.

If the multiple point be at the origin, then $x = 0$, and $y = 0$, and the multiplicity of the point may be determined by simple inspection, as will be seen in the following examples.

Ex. 1. To determine the multiple points of the curve

$$u = x^4 + 2ax^2y - ay^3 = 0.$$

Here $\quad \dfrac{du}{dx} = 4x^3 + 4axy = 0; \; \dfrac{du}{dy} = 2ax^2 - 3ay^2 = 0;$

hence there is only one system of values of x and y that can satisfy these three equations, viz., $x = 0$ and $y = 0$, and hence the origin may be a multiple point. To proceed we have

$\dfrac{dy}{dx} = \dfrac{4x^3 + 4axy}{3ay^2 - 2ax^2} = \dfrac{0}{0}$, when $x = 0$, and $y = 0$. Now the terms of

the numerator and denominator of the *lowest* degree are of two dimensions, therefore we must differentiate these terms twice; hence

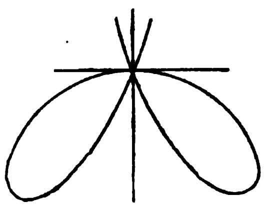
$$\frac{dy}{dx} = \frac{6x^2 + 2ay + 2ax\frac{dy}{dx}}{3ay\frac{dy}{dx} - 2ax} = \frac{0}{0},$$

$$\frac{dy}{dx} = \frac{4a\frac{dy}{dx}}{3a\left(\frac{dy}{dx}\right)^2 - 2a}, \text{ rejecting terms involving } x \text{ and } y;$$

$$\therefore \quad 3a\left(\frac{dy}{dx}\right)^2 - 6a\frac{dy}{dx} = 0,$$

$$\text{or } \frac{dy}{dx}\left\{\left(\frac{dy}{dx}\right)^2 - 2\right\} = 0;$$

$$\therefore \frac{dy}{dx} = 0, \frac{dy}{dx} = +\sqrt{2}, \frac{dy}{dx} = -\sqrt{2}.$$



Hence the origin is a triple point, the axis of x touching one branch, and the tangents of the two other intersecting branches are inclined to the axis of x at angles $\tan^{-1}(+\sqrt{2})$ and $\tan^{-1}(-\sqrt{2})$, as in the annexed figure.

In the final differentiation certain terms involving x and y have been rejected, hence it is evident that the multiplicity of a multiple point at the origin may be determined very simply by rejecting certain terms of the equation to the curve and retaining only those of lowest degree. Thus in the present example we may reject the term x^4 and retain the others; hence $2ax^2y - ay^3 = 0$;

$$\therefore y = 0, \; y = +x\sqrt{2}, \; y = -x\sqrt{2},$$

which are the equations of the tangents to the curve at the origin; hence, as before, the origin is a triple point.

Ex. 2. Let the equation of the curve be $y = c + (x-a)^2\sqrt{(x-b)}$.

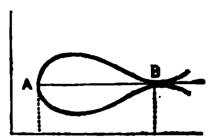
Freeing the equation from radicals, we get

$$u = (y-c)^2 - (x-a)^4(x-b) = 0 \quad . \quad . \quad . \quad . \quad (1).$$

$$\text{Hence} \quad \frac{du}{dy} = 2(y-c); \; \frac{du}{dx} = -(x-a)^3\{5x - (a+4b)\},$$

$$\frac{d^2u}{dy^2} = 2, \; \frac{d^2u}{dx\,dy} = 0,$$

$$\frac{d^2u}{dx^2} = -4(x-a)^2(5x - 2a - 3b);$$



hence, by equation (3), we have

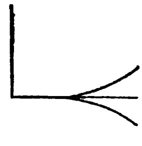
$$2\left(\frac{dy}{dx}\right)^2 - 4(x-a)^2(5x - 2a - 3b) = 0,$$

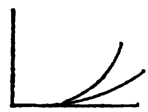
$$\therefore \frac{dy}{dx} = \pm(x-a)\sqrt{(10x - 4a - 6b)};$$

and hence the curve has a double point by osculation, as in the annexed figure. From the equation of the curve we see that when $x = a$, $y = c$, and when $x = b$, $y = c$, which are the coordinates of the points B and A respectively.

III.—*Of Cusps and Isolated Points.*

74. A *cusp*, or *point of regression*, is a double point of the second kind in which the two touching branches terminate, and do not extend beyond this point in one direction. There are two species of cusps, the *ceratoid*, so called from its resemblance to the horns of animals, the curvature of the two branches lying in opposite directions; and the *ramphoid*, so called from its resemblance to the beak of a bird, the curvature of the two branches lying in the same direction. The two annexed figures represent the two different kinds of cusps, the former being an instance of a ceratoid and the latter of a ramphoid. The nature of the cusp will be determined by the direction of the curvature of the two branches, as in Art. 64. If the values of $\dfrac{d^2y}{dx^2}$ have different signs, the cusp is of the first species; and if they have the same signs, the cusp is of the second species.





75. A *conjugate* or *isolated* point is a point the coordinates of which satisfy the equation to the curve, while if to either x or y a value be assigned, differing ever so little from its value at the point, the corresponding value of y or x will be impossible.

Ex. 1. Let the equation of the curve be $(y - x^2)^2 = x^5$.

Resolving the equation for y, we get $y = x^2 \pm x^{\frac{5}{2}}$, and when x is negative y is imaginary, therefore the curve lies on one side of the origin.

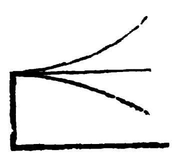
Differentiating, $\dfrac{dy}{dx} = 2x \pm \dfrac{5}{2}x^{\frac{3}{2}}$, and $\dfrac{d^2y}{dx^2} = 2 \pm \dfrac{5}{2}\cdot\dfrac{3}{2}x^{\frac{1}{2}} = 2 \pm \dfrac{15}{4}x^{\frac{1}{2}}$.

Now from the equation of the curve, when $x = 0$, $y = 0$, and when $x = 0$, $\dfrac{dy}{dx} = 0$, therefore the axis of x is a common tangent, and the two branches are both convex towards the axis, since when $x = 0$, the value of $\dfrac{d^2y}{dx^2}$ is *positive*, (Art. 64); therefore the origin of coordinates is a cusp of the second species.

Ex. 2. Let the curve be the semicubical parabola whose equation is

$$a(y - b)^2 = x^3.$$

When $y = b$, $x = 0$, and from the equation of the curve we have $y = b \pm \dfrac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$. Differentiating, we get



$$\frac{dy}{dx} = \pm\frac{3x^{\frac{1}{2}}}{2a^{\frac{1}{2}}} \text{ and } \frac{d^2y}{dx^2} = \pm\frac{3}{4\sqrt{ax}}.$$

Now when $x = 0$, $\dfrac{dy}{dx} = 0$, therefore the common tangent is parallel to the axis of x at the distance b. The values of $\dfrac{d^2y}{dx^2}$ have different

signs, and therefore the cusp at the point $x = 0$, $y = b$ is of the first species, as in the figure.

Ex. 3. Let the equation to the curve be $a(y - b)^2 = x^3 - cx^2$.

When $x = 0$, $y = b$, and from the equation we get

$$y = b \pm \sqrt{\left(\frac{x^3 - cx^2}{a}\right)} = b \pm x\sqrt{\left(\frac{x - c}{a}\right)}.$$

Now by assigning to x a small value less than c, the value of y is imaginary, whether the value of x be positive or negative, and therefore the point $x = 0$, $y = b$, is an isolated point.

Scholium.

76. The singular points of curves may always be determined in the following manner:—The abscissa to which a singular point corresponds will be indicated by considering in what case the differential coefficients of any order whatever become 0, or ∞, or $\frac{0}{0}$. The nature of the point will be assigned

(1). By examining what number of branches pass through the point, and whether they extend on both sides of it.

(2). By determining the position of the tangents to the curve at that point.

(3). By the direction of their curvature.

TRACING OF CURVES.

77. It is sometimes necessary to trace the general form of a curve from its equation without actually calculating its exact dimensions, and the principles we have just established afford considerable assistance in the delineation of the curve. When we proceed to trace a curve from its equation, it is desirable to solve the equation with respect to one or other of the coordinates, when that can be done in a form which enables us to determine readily the values of one coordinate for different values of the other. Let $y = fx$ be the equation when thus resolved, then we may proceed as follows:—

(1). Assign to x all positive values from 0 to ∞, marking those which make $y = 0$, $y = \infty$, or y impossible. The first gives the points where the curve cuts the axis of x, the second gives the infinite branches, and the third gives the limits of the curve in the plane of reference.

(2). Assign to x all negative values from 0 to ∞, and proceed as in the case of the positive values of x, attending to the positive and negative values of y in both cases, so as to obtain the branches on both sides of the axis of x.

(3). Ascertain whether the curve has asymptotes, and draw them if they exist.

(4). Find the value of $\frac{dy}{dx}$, and thence determine the angles at which the curve cuts the axis, as well as the maximum and minimum values of y, if there be such.

(5). Find the value of $\frac{d^2y}{dx^2}$, and thence deduce the nature of the

curvature of the different branches, and the points of contrary flexure, if such exist.

(6). Determine the nature and situation of the singular points, if there be such, by the usual rules.

(7). When the equation to a curve is given by an equation $r = f\theta$, the values of θ, which make $f\theta = 0$ are then to be found, and these give the angles at which the branches of the curve which pass through the origin cut the axis. Find the values of r when the curve cuts the axis, by giving to θ the values 0 and $n\pi$; and by giving to θ the value $\frac{1}{2}(2n+1)\pi$, we determine the values of r when the radius vector is perpendicular to the axis. Lastly, by making $\frac{dr}{d\theta} = 0$, we determine the values of θ, which render r a maximum or a minimum.

Examples.

1. Analyse and trace the curve whose equation is $xy^2+2a^2y-x^3=0$

Solving the equation for y, we have $y = -\frac{a^2}{x} \pm \sqrt{\left(\frac{a^4}{x^2}+x^2\right)}$.

When $x = 0$, $y = 0$, and $y = -\infty$, since the equation takes the form

$$y\left(y+\frac{2a^2}{x}\right) = x^2.$$

Now if we take the upper sign, and expand the radical in ascending powers of x, we get

$$y = -\frac{a^2}{x}+\frac{a^2}{x}\left(1+\frac{1}{2}\cdot\frac{x^4}{a^4}-\frac{1.1}{2.4}\cdot\frac{x^8}{a^8}+\ldots\right)$$

$$=\frac{1}{2}\cdot\frac{x^3}{a^2}-\frac{1.1}{2.4}\cdot\frac{x^7}{a^6}+\frac{1.1.3}{2.4.6}\cdot\frac{x^{11}}{a^{10}}-\ldots$$

When x is small, the first term determines the sign of the whole series, and therefore y is positive. Also since no value of x can make $y = 0$, this branch of the curve lies always in the positive region and extends to infinity, since $y = \infty$, when $x = \infty$.

If we take the lower sign, and expand the radical in descending powers of x, we have $y = -\frac{a^2}{x}-x\left(1+\frac{1}{2}\cdot\frac{a^4}{x^4}-\ldots\right)$, which when $x = \infty$ is negative and infinite, that is $y = -\infty$.

Expanding in ascending powers of x, we get

$$y = -\frac{2a^2}{x}-\frac{1}{2}\cdot\frac{x^3}{a^2}+\frac{1.1}{2.4}\cdot\frac{x^7}{a^6}-\ldots,$$

which, when $x=0$, is negative and infinite; hence this branch lies wholly between the positive axis of x and the negative axis of y.

Since the equation of the curve is unchanged by the substitution of $-x$ and $-y$ for x and y, it follows that the curve is symmetrical in the opposite regions of coordinates.

To determine the asymptotes, we have (Art. 53)

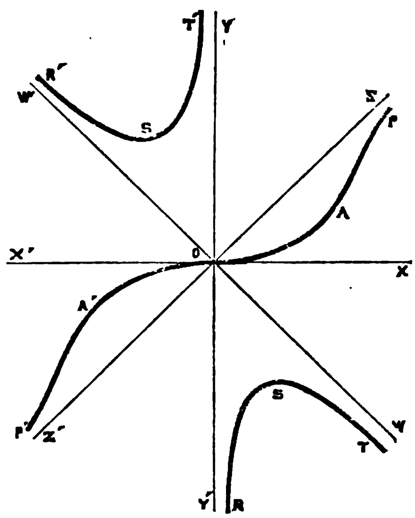
$$\frac{y}{x} = -\frac{a^2}{x^2}\pm\sqrt{\left(\frac{a^4}{x^4}+1\right)};$$

$$\therefore \alpha = \text{limit of } \frac{y}{x} = \pm 1, \text{ when } x \text{ approaches to } \infty.$$

Again, $$y - \alpha x = -\frac{a^2}{x} \pm \sqrt{\left(\frac{a^4}{x^2} + x^2\right)} \mp x,$$

which, when x approaches to ∞, gives $\beta = \pm \infty \mp \infty = 0$.

Hence the equations of the asymptotes are $y = \pm x$, and we have already seen that when $y = -\infty$, then $x = 0$; therefore the axis of y is also an asymptote. The asymptotes Z O Z$'$ and W O W$'$ bisect the angles between the axes of coordinates.



Differentiating the equation of the curve we get

$$\frac{dy}{dx} = \frac{3x^2 - y^2}{2(a^2 + xy)} = 0;$$

hence the axis of x touches the curve at the origin of coordinates. From the last equation $3x^2 - y^2 = 0$; hence $y = \pm x\sqrt{3}$, and by substitution in the given equation we have $2x^3 \pm 2a^2 x\sqrt{3} = 0$, which gives $x = \pm a^4\sqrt{3}$; hence the minimum value of y belongs only to the branches in the second and fourth quadrants, the negative value of y having been employed for the fourth quadrant.

Differentiating the equation of the curve twice, we get

$$\frac{d^2 y}{dx^2} = \frac{6a^2 x + 3x^2 y + y^3 - 4x^3 \frac{dy}{dx}}{2(a^2 + xy)^2};$$

which, when $x = 0$, $y = 0$, gives $\frac{d^2 y}{dx^2} = 0$, since then $\frac{dy}{dx} = 0$; hence the origin is a point of contrary flexure, which is otherwise evident, since the curve at the origin both touches and cuts the axis of x. If the curve has any other points of inflexion, they will be determined by putting the numerator of the last equation equal to zero, and solving the resulting equation.

2. Analyse and trace the curve whose equation is

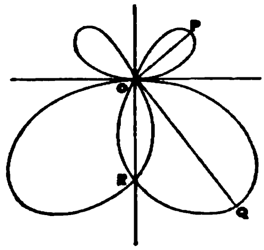
$$r = a(\sin 2\theta - \sin\theta) = a \sin\theta(2\cos\theta - 1).$$

Here r is equal to 0, when $\sin\theta = 0$, and $\cos\theta = \frac{1}{2}$, or when

$$\theta = 0, \theta = \pi, \theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}.$$

It is obvious that the values of r recur when $\theta = 2\pi$, and as r never becomes infinite, the curve consists of four loops arranged round the pole O.

From $\theta = 0$ to $\theta = \frac{\pi}{3}$ the radius vector r is positive, and from $\theta = \frac{\pi}{3}$ to $\theta = \pi$, r is negative, because $2 \cos \theta - 1$ is negative and $\sin \theta$ is positive within those limits.



Again, from $\theta = \pi$ to $\theta = \frac{5\pi}{3}$, r is positive, since both $\sin \theta$ and $2 \cos \theta - 1$ are negative, and from $\theta = \frac{5\pi}{3}$ to $\theta = 2\pi$, r is negative.

Differentiating the equation of the curve, we have

$$\frac{dr}{d\theta} = a \cos \theta (2 \cos \theta - 1) - 2a \sin^2\theta = a(4\cos^2\theta - \cos\theta - 2) = 0;$$

$$\therefore\ 4\cos^2\theta - \cos\theta = 2, \text{ or } \cos^2\theta - \frac{1}{4}\cos\theta = \frac{1}{2}; \text{ whence}$$

$$\cos\theta = \frac{1}{8} \pm \sqrt{\left(\frac{1}{64} + \frac{1}{2}\right)} = \frac{1 \pm \sqrt{33}}{8} = .8430703, \text{ or } -.5930703;$$

$$\therefore\ \theta = 32^\circ\, 32'\, 3'', \text{ or } \theta = 126^\circ\, 22'\, 31'',$$

which determine the maximum and minimum values of r; viz.,

$$r = .369008\, a = \mathrm{OP},\ r = -1.760166\, a = \mathrm{OQ}.$$

When $\theta = \frac{\pi}{2}$, then $r = -a$, and if OR be taken $= a$, then the curve will pass through R.

Exercises.

1. Trace the curve whose equation is $y = x^3 + 5x^2 + 6x$.

2. Show that the curve whose equation is $(a^2 - x^2)y^2 = a^4$ has a double point at the origin of coordinates, and find the direction of its branches.

3. Show that the curve defined by the equation

$$a(x + y - 1)^2 = (x - y + 3)^3$$

has a ceratoid cusp whose coordinates are $x = -1$ and $y = 2$.

4. If the equation of a curve be $(2x + y - x^2)^2 = (x - 1)^5$, it is required to show that the curve has a ramphoid cusp at the point $x = 1$, and $y = -1$.

5. Trace the curve whose equation is

$$(x + 2)y + (x + 1)(x + 3) = 0.$$

6. Analyse the curve whose equation is $x^2(x + y) + x - y = 0$.

7. Trace the curve whose equation is $r = a(1 + \cos\theta)$.

CHANGE OF THE INDEPENDENT VARIABLE.

78. It is frequently necessary to change the independent variable, in an expression involving differentials, and to do this we have only to recollect that the independent variable is that variable whose differential is constant, and that

$$\frac{d^2 y}{d x^2} = \frac{d\left(\frac{d y}{d x}\right)}{d x}; \ \frac{d^3 y}{d x^3} = \frac{d\left\{\frac{d\left(\frac{d y}{d x}\right)}{d x}\right\}}{d x}, \text{ etc. } \ldots \ldots (1).$$

In the first members of these equations, the independent variable is evidently x, since $d x$ is constant; but in the second members the independent variable may be any quantity we please, since the differentiation is only indicated, not performed. If the expression proposed contain the differentials $d x$, $d y$, $d^2 y$, $d^3 y$, etc.; then from equations (1) we have

$$d^2 y = d x\, d\left(\frac{d y}{d x}\right), \ d^3 y = d x^2 d\left\{\frac{d\left(\frac{d y}{d x}\right)}{d x}\right\}, \text{ etc. } \ldots \ldots (2),$$

and by substituting the values of the differential coefficients in (1), or the values of the differentials in (2), and performing the operations indicated in accordance with the hypothesis, we may readily change the independent variable in any expression.

Examples.

1. Let the proposed expression be

$$\frac{d^2 y}{d x^2} - 2\frac{d y}{d x} + 2 y = 0, \ \ldots \ldots (\alpha),$$

where x is the independent variable, it is required to change (α) into another expression where y is the independent variable.

Differentiating the second member of the first of equations (1), considering y as the independent variable, and $d y$ constant, we have

$$\frac{d^2 y}{d x^2} = \frac{-d y \frac{d^2 x}{d x^2}}{d x} = -\frac{d^2 x\, d y}{d x^3};$$

hence equation (α) becomes $-\dfrac{d^2 x\ d y}{d x^3} - 2\dfrac{d y}{d x} + 2 y = 0,$

which, by multiplying by $d x^3$, changing signs, and dividing by $d y^3$ gives

$$\frac{d^2 x}{d y^2} + 2\left(\frac{d x}{d y}\right)^2 - 2 y\left(\frac{d x}{d y}\right)^3 = 0 \ \ldots \ldots (\beta).$$

2. Change the independent variable x to θ, where $\theta = \cos^{-1} x$, in the expression $u = \dfrac{d^2 y}{d x^2} - \dfrac{x}{1 - x^2} \cdot \dfrac{d y}{d x} + \dfrac{y}{1 - x^2}$.

Here $\quad x = \cos\theta,\ d x = -\sin\theta\ d\theta, \dfrac{d y}{d x} = -\dfrac{d y}{\sin\theta\ d\theta};$

whence $d\left(\dfrac{d y}{d x}\right) = -\dfrac{d^2 y \sin\theta - d y \cos\theta\ d\theta}{\sin^2\theta\ d\theta}, \ \dfrac{x}{1 - x^2} = \dfrac{\cos\theta}{\sin^2\theta};$

$$\text{therefore } u = \frac{d\left(\frac{d y}{d x}\right)}{d x} - \frac{x}{1 - x^2} \cdot \frac{d y}{d x} + \frac{y}{1 - x^2}$$

$$= \frac{d^2y \sin\theta - dy\cos\theta\, d\theta}{\sin^3\theta\, d\theta^2} + \frac{\cos\theta\, dy}{\sin^2\theta\, d\theta} + \frac{y}{\sin^2\theta}$$

$$= \left\{\frac{d^2y}{d\theta^2} + y\right\}\frac{1}{\sin^2\theta},$$ the required expression in which θ is the independent variable.

3. Change the variable from x to y in the equation

$$\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 + y\frac{d^2y}{dx^2} = 0. \quad \textit{Ans. } y\frac{d^2x}{dy^2} - \left(\frac{dx}{dy}\right)^2 - 1 = 0.$$

4. Change the equation $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + ay = 0,$

into another having θ as the independent variable, θ being equal to $\log x$.

Ans. $\dfrac{d^3y}{d\theta^3} + ay = 0.$

5. The expression for the radius of curvature, when x is the independent variable, is $\rho = -\dfrac{ds^3}{d^2y\,dx}$; what does it become when y is the independent variable?

Ans. $\rho = \dfrac{ds^3}{d^2x\,dy}.$

6. Change the variable in $y^2\dfrac{d^2u}{dy^2} + Ay\dfrac{du}{dy} + Bu = 0,$

from y to x when $y = e^x$. *Ans.* $\dfrac{d^2u}{dx^2} + (A-1)\dfrac{du}{dx} + Bu = 0.$

II. INTEGRAL CALCULUS.

79. *The integral calculus* is the inverse of the differential, its object being to determine the primitive function from which a given differential expression has been derived. In this great branch of analysis, the primitive function with respect to its differential, is called the *integral* of that differential, and the process of finding the integral is termed *integration*, and when to be performed is signified by prefixing the symbol $\int$, the initial letter of the word *summa*. Thus $\int dx = x$, the symbol $\int$ signifying an operation which is the inverse of that denoted by the symbol d.

80. In assigning the integral of a differential expression, a *constant* quantity should be annexed, because, in the process of differentiation, it was shown that the differential of $x^n \pm C$ is the same as the differential of x^n, viz., $nx^{n-1}dx$; therefore, conversely, the integral of $nx^{n-1}dx$ must be $x^n + C$, where C may be *positive*, *negative*, or *zero*. The value of C is discoverable from the peculiar nature of the problem under consideration, as will be seen when we come to the determination of the areas of plane surfaces.

81. In the integral calculus, the results are not obtained by direct processes as in the differential calculus. They can only be determined by reversing the processes of the differential calculus, and in this way the elementary formulas of differentiation will furnish an equal number of elementary ones for the integral calculus, and to one or other of these elementary forms all other differentials must be transformed, and then their integrals may be found. Various artifices are employed to reduce

differential expressions to known forms, and when this cannot be effected, the expression may be expanded in a series of which the several terms can be integrated by elementary forms.

82. Since a constant multiplier or divisor in a function is retained in the differential (11), it follows that a constant multiplier or divisor may be removed and placed before the sign of integration: thus $\int a x^n dx = a\int x^n dx$, and $\int \frac{x^n dx}{a} = \frac{1}{a}\int x^n dx$. In the differential calculus it was shown (11) that the differential of the sum of any number of functions is equal to the sum of the differentials of those functions; hence to integrate an expression consisting of any number of differentials connected by the signs plus or minus, we must take the integral of each separately, and connect them by their proper signs: thus

$$\int (a x^n dx - b x^p dx + c x^q dx) = a\int x^n dx - b\int x^p dx + c\int x^q dx + C;$$

where *one* constant quantity is annexed, because the aggregate of three or of any number of constants is only one constant.

83. It will be convenient to collect the various elementary forms of integrals, derived from reversing the fundamental processes in the differential calculus, and place them in a tabular form for reference. Hence if

$$d x^n = n x^{n-1} dx \qquad \therefore \int x^{n-1} dx = \frac{x^n}{n} \quad . \quad . \quad . \quad . \quad (1).$$

$$d a^x = a^x \log_e a \,.\, dx \qquad \int a^x dx = \frac{a^x}{\log_e a} \quad . \quad (2).$$

$$d e^x = e^x dx \qquad \int e^x dx = e^x \quad . \quad . \quad . \quad (3).$$

$$d \log_a x = \frac{1}{\log_e a} \cdot \frac{dx}{x} \qquad \int \frac{dx}{x} = \log_e a \log_a x \quad (4).$$

$$d \log_e x = \frac{dx}{x} \qquad \int \frac{dx}{x} = \log_e x \quad . \quad . \quad (5).$$

$$d \sin x = \cos x\, dx \qquad \int \cos x\, dx = \sin x \quad . \quad (6).$$

$$d \cos x = -\sin x\, dx \qquad \int \sin x\, dx = -\cos x \quad . \quad (7).$$

$$d \tan x = \sec^2 x\, dx \qquad \int \sec^2 x\, dx = \tan x \quad . \quad (8).$$

$$d \cot x = -\operatorname{cosec}^2 x\, dx \qquad \int \operatorname{cosec}^2 x\, dx = -\cot x \quad . \quad (9).$$

$$d \sec x = \tan x \sec x\, dx \qquad \int \tan x \sec x\, dx = \sec x \quad . \quad (10).$$

$$d \operatorname{cosec} x = -\cot x \operatorname{cosec} x\, dx \qquad \int \cot x \operatorname{cosec} x\, dx = -\operatorname{cosec} x \quad (11).$$

$$d \operatorname{vers} x = \sin x\, dx \qquad \int \sin x\, dx = \operatorname{vers} x \quad . \quad (12).$$

$$d \sin m x = m \cos m x\, dx \qquad \int \cos m x\, dx = \frac{1}{m} \sin m x \quad (13).$$

$$d \cos m x = -m \sin m x\, dx \qquad \int \sin m x\, dx = -\frac{1}{m} \cos m x \quad (14).$$

$$d \tan m x = m \sec^2 m x\, dx \qquad \int \sec^2 m x\, dx = \frac{1}{m} \tan m x \quad (15).$$

In a similar manner from the inverse trigonometrical functions we get the following elementary forms:—

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a}, \quad \text{or} \int \frac{-dx}{\sqrt{(a^2 - x^2)}} = \cos^{-1}\frac{x}{a} \quad . \quad . \quad . \quad (16).$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \text{ or} \int \frac{-dx}{a^2 + x^2} = \frac{1}{a}\cot^{-1}\frac{x}{a} \quad . \quad . \quad (17).$$

$$\int \frac{dx}{x\sqrt{(x^2-a^2)}} = \frac{1}{a}\sec^{-1}\frac{x}{a}, \text{ or } \int \frac{-dx}{x\sqrt{(x^2-a^2)}} = \frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} \quad (18).$$

$$\int \frac{dx}{\sqrt{(2ax-x^2)}} = \operatorname{vers}^{-1}\frac{x}{a}, \text{ or } \int \frac{-dx}{\sqrt{(2ax-x^2)}} = \operatorname{covers}^{-1}\frac{x}{a} \quad (19).$$

84. It may be worthy of remark here that the same differential admits of two apparently different integral forms: thus in (16) we have

$$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a} + C, \text{ and also } \int \frac{dx}{\sqrt{(a^2-x^2)}} = -\cos^{-1}\frac{x}{a} + C';$$

but since $\sin^{-1}\frac{x}{a} + \cos^{-1}\frac{x}{a} = \frac{\pi}{2} =$ constant; therefore $\sin^{-1}\frac{x}{a}$ $= -\cos^{-1}\frac{x}{a} +$ constant; which shows that the two expressions include precisely the same system of values, when all possible constant values are given to the arbitrary constants. The two integrals differ only by a constant quantity. In a similar manner the forms in (17), (18), and (19), may be shown to be identical; hence if the radius be unity, or $a = 1$, we have

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x + C = -\cos^{-1}x + C' \quad . . . (16').$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + C = -\cot^{-1}x + C' \quad . . . (17').$$

$$\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec^{-1}x + C = -\operatorname{cosec}^{-1}x + C' \quad . . (18').$$

$$\int \frac{dx}{\sqrt{(2x-x^2)}} = \operatorname{vers}^{-1}x + C = -\operatorname{covers}^{-1}x + C' \quad . . (19').$$

85. The forms (16), (17), (18), (19), may be made more general in the following manner. Writing bx for x, and $b\,dx$ for dx, we have (16)

$$\int \frac{b\,dx}{\sqrt{(a^2-b^2x^2)}} = \sin^{-1}\frac{bx}{a}, \text{ or } \int \frac{dx}{\sqrt{(a^2-b^2x^2)}} = \frac{1}{b}\sin^{-1}\frac{b}{a}x.$$

Again in (19) write $\frac{2b^2}{a}x$ instead of x, and $\frac{2b^2}{a}dx$ instead of dx;

$$\therefore \int \frac{\frac{2b^2}{a}dx}{\left(4b^2x - \frac{4b^4}{a^2}x^2\right)^{\frac{1}{2}}} = \operatorname{vers}^{-1}\frac{2b^2}{a^2}x, \text{ or } \int \frac{dx}{\sqrt{(a^2x-b^2x^2)}} = \frac{1}{b}\operatorname{vers}^{-1}\frac{2b^2}{a^2}x.$$

In this manner we obtain the four following forms:—

$$\int \frac{dx}{\sqrt{(a^2-b^2x^2)}} = \frac{1}{b}\sin^{-1}\frac{b}{a}x + C = -\frac{1}{b}\cos^{-1}\frac{b}{a}x + C' . . (20).$$

$$\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{b}{a}x + C = -\frac{1}{ab}\cot^{-1}\frac{b}{a}x + C' . . (21).$$

$$\int \frac{dx}{x\sqrt{(b^2x^2-a^2)}} = \frac{1}{a}\sec^{-1}\frac{b}{a}x + C = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{b}{a}x + C' \; (22).$$

$$\int \frac{dx}{\sqrt{(a^2x-b^2x^2)}} = \frac{1}{b}\operatorname{vers}^{-1}\frac{2b^2}{a^2}x + C = -\frac{1}{b}\operatorname{covers}^{-1}\frac{2b^2}{a^2}x + C' (23).$$

86. Since $d(uv) = u\,dv + v\,du$, where u and v are functions of x; therefore by integration, $uv = \int u\,dv + \int v\,du$, and transposing

$$\int u\,dv = uv - \int v\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

This is called the formula of *integration by parts*, and enables us to find the integral of $u\,dv$, provided the integral of $v\,du$ can be found. The method of integration by parts is extensively employed in reducing integrals to known forms. If we change u into u^{-1} or $\frac{1}{u}$, then we have du changed into $-\frac{du}{u^2}$, and (24) becomes

$$\int \frac{dv}{u} = \frac{v}{u} + \int \frac{v}{u} \cdot \frac{du}{u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

a formula which is sometimes advantageously employed.

87. We may now proceed to explain the methods of reducing integrals of different functions to one or other of the preceding fundamental forms; but before entering upon the various artifices to be employed in these transformations, we shall advert to one or two very useful *theorems* which are of frequent occurrence in the integral calculus. Thus since $dx^n = n\,x^{n-1}\,dx$; therefore $\int x^{n-1}\,dx = \frac{1}{n}\,x^n + C$; hence to integrate a differential in which the differential of a variable is multiplied by a power of that variable, the index of the power being constant, we have the following rule:

Divide by the differential of the variable, add unity to the index; divide by the index thus increased, and annex the constant quantity.

$$\text{Thus } \int x^4\,dx = \tfrac{1}{5}x^5 + C,\ \int x^{\frac{1}{2}}\,dx = \tfrac{2}{3}x^{\frac{3}{2}} + C,$$

$$\int x^{-\frac{1}{2}}\,dx = 2\,x^{\frac{1}{2}} + C,\ \int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

This principle may be extended to the integration of the general expression

$$(a\,x^n + b)^m \; x^{n-1}\,dx.$$

For if we put $a\,x^n + b = z$, then differentiating we have

$$n\,a\,x^{n-1}\,dx = dz, \text{ or } x^{n-1}\,dx = \frac{1}{n\,a}\,dz;$$

hence the proposed differential becomes $\frac{1}{n\,a}\,z^m\,dz$; therefore

$$\int (ax^n + b)^m x^{n-1}\,dx = \frac{1}{n\,a}\int z^m\,dz = \frac{z^{m+1}}{n\,a\,(m+1)} = \frac{(a\,x^n+b)^{m+1}}{n\,a\,(m+1)} + C.$$

The principle here employed is applicable in all cases when the index $(n - 1)$ of the power of x without the vinculum is less by unity than the index (n) of the power within it, except in the cases when $m = -1$, or $n = 0$, which reduce the differential expression to

$$\frac{x^{n-1}\,dx}{a\,x^n + b}, \text{ or } (a + b)^m . \frac{dx}{x},$$

the integrals of which will be considered in the next Article.

The substitution of z for $a\,x^n + b$ was employed to reduce the differential to the form (1), but in practice the principle should be applied directly to the differential whose integral is required. Thus, con-

sidering $(ax^n + b)$ as a monomial, its differential is $nax^{n-1}dx$, and dividing the proposed differential by this, gives $\frac{(ax^n+b)^m}{na}$. Now increase the index m by unity, divide by $m+1$, the index thus increased, and we obtain

$$\int (ax^n + b)^m x^{n-1} dx = \frac{(ax^n+b)^{m+1}}{na(m+1)} + \mathrm{C}.$$

88. Since $d \log (x+a) = \frac{dx}{x+a}$; therefore, conversely $\int \frac{dx}{x+a} = \log (x+a) + \mathrm{C}'$; whence it follows that, *when the numerator of a differential is the differential of the denominator, the integral is the napierian logarithm of the denominator.*

The constant may be incorporated with $\log (x+a)$, for the constant C′ is the logarithm of another constant C, that is, $\mathrm{C}' = \log \mathrm{C}$; hence

$$\int \frac{dx}{x+a} = \log (x+a) + \log \mathrm{C} = \log \mathrm{C}\,(x+a).$$

Reverting to the differential in Article 87, we have, when $m = -1$,

$$\frac{x^{n-1}dx}{ax^n+b} = \frac{1}{na}\cdot\frac{nax^{n-1}dx}{ax^n+b} = \frac{1}{na}\cdot\frac{d\,(ax^n+b)}{ax^n+b};$$

$$\therefore \int \frac{x^{n-1}dx}{ax^n+b} = \frac{1}{na}\log (ax^n+b) + \mathrm{C}' = \frac{1}{na}\log \mathrm{C}\,(ax^n+b).$$

And when $n = 0$, in the same differential, we have

$$\int (a+b)^m \frac{dx}{x} = (a+b)^m \int \frac{dx}{x} = (a+b)^m \log \mathrm{C}x.$$

89. We frequently meet with integrals which may be brought to elementary forms by certain easy algebraic processes, as well as by the substitution of certain differential forms for others equivalent to them, and to assist the student in the transformation and reduction of such integrals, we shall here give the following table.

TABLE OF ALGEBRAIC AND DIFFERENTIAL EQUIVALENTS.

(1.) $x = a + x - a = a - (a - x).$

(2.) $x = \frac{1}{b}(a + bx - a) = \frac{1}{b}(a + bx) - \frac{a}{b}.$

(3.) $x^2 = (a + x - a)^2 = (a+x)^2 - 2a(a+x) + a^2.$

(4.) $2ax + x^2 = (a+x)^2 - a^2.$

(5.) $2ax - x^2 = a^2 - (a-x)^2.$

(6.) $b^2 - x^2 = b^2 - a^2 + (a^2 - x^2).$

(7.) $a + bx + cx^2 = \frac{1}{4c}(4ac + 4bcx + 4c^2x^2)$

$= \frac{1}{4c}\{(2cx+b)^2 + 4ac - b^2\}.$

(8.) $\frac{a+bx}{(x-c)^2} = \frac{a + bc + b(x-c)}{(x-c)^2} = \frac{a+bc}{(x-c)^2} + \frac{b}{x-c}.$

(9.) $(a^2 - x^2)^{\frac{1}{2}} = \frac{a^2 - x^2}{(a^2-x^2)^{\frac{1}{2}}} = \frac{a^2}{(a^2-x^2)^{\frac{1}{2}}} - \frac{x^2}{(a^2-x^2)^{\frac{1}{2}}}.$

(10.) $$\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} = \frac{a+x}{(a^2-x^2)^{\frac{1}{2}}} = \frac{a}{(a^2-x^2)^{\frac{1}{2}}} + \frac{x}{(a^2-x^2)^{\frac{1}{2}}}.$$

(11.) $$(1-x^2)^{\frac{3}{2}} = (1-x^2)(1-x^2)^{\frac{1}{2}} = (1-x^2)^{\frac{1}{2}} - x^2(1-x^2)^{\frac{1}{2}}.$$

(12.) $$(1-x^2)^{\frac{3}{2}} = (x^2)^{\frac{3}{2}}(x^{-2}-1)^{\frac{3}{2}} = x^3(x^{-2}-1)^{\frac{3}{2}}.$$

(13.) $$(a+bx^2)^{\frac{1}{2}} = b^{\frac{1}{2}}\left(\frac{a}{b}+x^2\right)^{\frac{1}{2}} = b^{\frac{1}{2}}(ab^{-1}+x^2)^{\frac{1}{2}}.$$

(14.) $$dx = d(x \pm a) = d(x \pm b), \text{ etc.}$$

(15.) $$dx = \frac{1}{2a}d(2ax \pm b) = \frac{1}{2c}d(2cx \pm h), \text{ etc.}$$

(16.) $$x\,dx = \frac{1}{2}d(x^2 \pm a^2) = \frac{1}{2}d(x^2 \pm b^2), \text{ etc.}$$

90. Many other equivalent expressions will suggest themselves in practice, especially in trigonometrical functions, and several of those now given will be employed in the examples immediately following, as well as in the integration of some differentials whose integrals involve logarithms, and in the process of integration by parts, both of which will presently come under consideration.

Examples in Integration.

1. To integrate $\frac{5}{6}ax^{\frac{3}{4}}dx$.

By the theorem in Art. 87, we have

$$\frac{5}{6}a\int x^{\frac{3}{4}}dx = \frac{5}{6}a \cdot \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} = \frac{5a}{6} \cdot \frac{4}{7}x^{\frac{7}{4}} = \frac{10a}{21}x^{\frac{7}{4}} + C.$$

2. To integrate $(3a^2x^2 + x^{-3})\,dx$.

$$\int(3a^2x^2+x^{-3})\,dx = 3a^2\int x^2dx + \int\frac{dx}{x^3} = a^2x^3 - \frac{1}{2x^2}.$$

3. To integrate $\sqrt{(1+x^2)}\,x\,dx$.

$$\int(1+x^2)^{\frac{1}{2}}x\,dx = \frac{(1+x^2)^{\frac{1}{2}+1}x\,dx}{\frac{3}{2}\cdot 2x\,dx} = \frac{(1+x^2)^{\frac{3}{2}}}{3}.$$

4. To integrate $\frac{dx}{(1-x^2)^{\frac{3}{2}}}$.

$$\int\frac{dx}{(1-x^2)^{\frac{3}{2}}} = \int\frac{dx}{x^3(x^{-2}-1)^{\frac{3}{2}}} \text{ by (12)} = \int x^{-3}(x^{-2}-1)^{-\frac{3}{2}}dx$$

$$= \frac{(x^{-2}-1)^{-\frac{1}{2}}x^{-3}dx}{-\frac{1}{2}(-2x^{-3}dx)} = \frac{1}{(x^{-2}-1)^{\frac{1}{2}}} = \frac{x}{(1-x^2)^{\frac{1}{2}}}.$$

5. To integrate $\frac{x\,dx}{\sqrt{(a^2+x^2)}}$.

$$\int\frac{x\,dx}{\sqrt{(a^2+x^2)}} = \int(a^2+x^2)^{-\frac{1}{2}}x\,dx = \frac{(a^2+x^2)^{\frac{1}{2}}x\,dx}{\frac{1}{2}\cdot 2x\,dx} = (a^2+x^2)^{\frac{1}{2}}.$$

6. To integrate $(a x + b)^n\, d x$.

$$\int (a x + b)^n\, d x = \frac{(a x + b)^{n+1} d x}{(n + 1)\, a\, d x} = \frac{(a x + b)^{n+1}}{(n + 1)\, a}.$$

7. To integrate $\dfrac{x^m\, d x}{(a + b x)^n}$.

Let $a + b x = z$; then $(a + b x)^n = z^n$, $x = \dfrac{z - a}{b}$, $x^{m+1} = \dfrac{(z - a)^{m+1}}{b^{m+1}}$;

therefore $(m + 1)\, x^m\, d x = (m + 1)\, \dfrac{(z - a)^m\, d z}{b^{m+1}}$, and consequently

$$x^m\, d x = \frac{1}{b^{m+1}} (z - a)^m\, d z.$$

Substituting in the proposed expression, we get

$$\int \frac{x^m\, d x}{(a + b x)^n} = \frac{1}{b^{m+1}} \int \frac{(z - a)^m\, d z}{z^n} \quad \ldots\ldots \quad (1).$$

Now $(z - a)^m = z^m - m\, a\, z^{m-1} + \dfrac{m\,(m - 1)}{1\,.\,2}\, a^2 z^{m-2} - \text{etc.}$,

and substituting this in (1), we shall have, after multiplying by $d z$ and dividing by z^n,

$$\int \frac{x^m\, d x}{(a + b x)^n} = \frac{1}{b^{m+1}} \Big\{ \int z^{m-n} d z - m\, a \int z^{m-n-1} d z$$
$$+ \frac{m\,(m - 1)}{1\,.\,2}\, a^2 \int z^{m-n-2} d z - \text{etc.} \Big\}.$$

When m and n are given, the integrals of the terms within the brackets can be obtained by (1) or (5) of the elementary forms.

Let $m = 3$ and $n = 2$; then

$$\int \frac{x^3\, d x}{(a + b x)^2} = \frac{1}{b^4} \left\{ \int z\, d z - 3 a \int d z + 3 a^2 \int \frac{d z}{z} - a^3 \int \frac{d z}{z^2} \right\}$$
$$= \frac{1}{b^4} \left(\frac{z^2}{2} - 3 a z + 3 a^2 \log z + \frac{a^3}{z} \right)$$
$$= \frac{1}{b^4} \left\{ \frac{(a + b x)^2}{2} - 3 a (a + b x) + 3 a^2 \log (a + b x) + \frac{a^3}{a + b x} \right\}.$$

8. To integrate $\dfrac{2 x^4\, d x}{x^5 \pm a}$.

$$\int \frac{2 x^4\, d x}{x^5 \pm a} = \frac{2}{5} \int \frac{5 x^4\, d x}{x^5 \pm a} = \frac{2}{5} \log (x^5 \pm a), \text{ by Art. 88} = \log (x^5 \pm a)^{\frac{2}{5}}.$$

9. To integrate $d u = \dfrac{x^3\, d x}{x + a}$.

Dividing the numerator by the denominator, gives

$$d u = x^2 d x - a x\, d x + a^2\, d x - \frac{a^3\, d x}{x + a};$$

$$\therefore\ u = \int x^2\, d x - a \int x\, d x + a^2 \int d x - a^3 \int \frac{d x}{x + a}.$$
$$= \frac{x^3}{3} - \frac{a x^2}{2} + a^2 x - a^3 \log (x + a).$$

10. To integrate $du = \dfrac{dx}{x\sqrt{(2ax - x^2)}}$.

Here $x\sqrt{(2ax - x^2)} = x^2\sqrt{(2ax^{-1} - 1)}$, and consequently

$$u = \int \frac{dx}{x^2\sqrt{(2ax^{-1} - 1)}} = \int (2ax^{-1} - 1)^{-\frac{1}{2}} x^{-2}\, dx$$

$$= \frac{(2ax^{-1} - 1)^{\frac{1}{2}} x^{-2} dx}{\frac{1}{2}(-2ax^{-2}dx)} = -\frac{1}{a}(2ax^{-1} - 1)^{\frac{1}{2}} = -\frac{(2ax - x^2)^{\frac{1}{2}}}{ax}.$$

11. To integrate $du = \dfrac{x\,dx}{(2ax - x^2)^{\frac{3}{2}}}$.

Here $(2ax - x^2)^{\frac{3}{2}} = (x^2)^{\frac{3}{2}}(2ax^{-1} - 1)^{\frac{3}{2}} = x^3(2ax^{-1} - 1)^{\frac{3}{2}}$; hence

$$u = \int \frac{x\,dx}{x^3(2ax^{-1} - 1)^{\frac{3}{2}}} = \int (2ax^{-1} - 1)^{-\frac{3}{2}} x^{-2} dx$$

$$= \frac{(2ax^{-1} - 1)^{-\frac{1}{2}} x^{-2} dx}{-\frac{1}{2}(-2ax^{-2}dx)} = \frac{1}{a}(2ax^{-1} - 1)^{-\frac{1}{2}} = \frac{x}{a\sqrt{(2ax - x^2)}}.$$

12. To integrate $du = \dfrac{dx}{x^2\sqrt{(1 - x^2)}}$.

Here $x^2\sqrt{(1 - x^2)} = x^3\sqrt{(x^{-2} - 1)}$, and therefore

$$u = \int \frac{dx}{x^3\sqrt{(x^{-2} - 1)}} = \int (x^{-2} - 1)^{-\frac{1}{2}} x^{-3} dx = \frac{(x^{-2} - 1)^{\frac{1}{2}} x^{-3} dx}{\frac{1}{2}(-2x^{-3}dx)}$$

$$= -(x^{-2} - 1)^{\frac{1}{2}} = -\frac{(1 - x^2)^{\frac{1}{2}}}{x}.$$

13. To integrate $du = \dfrac{3\,dx}{4 + 5x^2}$.

Here $u = \displaystyle\int \frac{3\,dx}{5(\frac{4}{5} + x^2)} = \frac{3}{5}\int \frac{dx}{\frac{4}{5} + x^2} = \frac{3}{5}\sqrt{\frac{5}{4}} \cdot \tan^{-1} x\sqrt{\frac{5}{4}}$

$$= \frac{3}{10}\sqrt{5}\tan^{-1}\frac{x\sqrt{5}}{2}.$$

14. To integrate $du = \dfrac{3\,dx}{\sqrt{(2 - 3x^2)}}$.

Since $\sqrt{(2 - 3x^2)} = \sqrt{3} \cdot \sqrt{(\frac{2}{3} - x^2)}$; therefore, by (16), we have

$$u = 3\int \frac{dx}{\sqrt{(2 - 3x^2)}} = \sqrt{3}\int \frac{dx}{\sqrt{(\frac{2}{3} - x^2)}} = \sqrt{3}\sin^{-1} x\sqrt{\frac{3}{2}}.$$

15. To integrate $du = \dfrac{x^2\,dx}{1 + x^2}$.

$$u = \int \frac{x^2\,dx}{1 + x^2} = \int \frac{(1 + x^2 - 1)\,dx}{1 + x^2} = \int dx - \int \frac{dx}{1 + x^2} = x - \tan^{-1} x.$$

16. To integrate $du = \dfrac{x\,dx}{(2ax - x^2)^{\frac{1}{2}}}$.

Here $u = \displaystyle\int \frac{a\,dx}{(2ax - x^2)^{\frac{1}{2}}} - \int \frac{(a - x)\,dx}{(2ax - x^2)^{\frac{1}{2}}}$ by Art. 89 (1)

$$= a\,\text{vers}^{-1}\frac{x}{a} - (2ax - x^2)^{\frac{1}{2}}.$$

17. To integrate $du = (a+x)^{\frac{1}{2}} x\, dx$.

$$u = \int (a+x)^{\frac{1}{2}} x\, dx = \int (a+x-a)(a+x)^{\frac{1}{2}} dx, \text{ Art. 89 (1)}$$
$$= \int (a+x)^{\frac{3}{2}} dx - a \int (a+x)^{\frac{1}{2}} dx$$
$$= \frac{2}{5}\overline{(a+x)}^{\frac{5}{2}} - \frac{2a}{3}(a+x)^{\frac{3}{2}} = \frac{2}{15}(3x-2a)(a+x)^{\frac{3}{2}}.$$

18. To integrate $du = \dfrac{x^2\, dx}{(a+x)^{\frac{3}{2}}}$.

$$u = \int \frac{x^2\, dx}{(a+x)^{\frac{3}{2}}} = \int \left\{ \frac{(a+x)^2 - 2a(a+x) + a^2}{(a+x)^{\frac{3}{2}}} \right\} dx, \text{ Art. 89 (3)}$$
$$= \int (a+x)^{\frac{1}{2}} dx - 2a \int (a+x)^{-\frac{1}{2}} dx + a^2 \int (a+x)^{-\frac{3}{2}} dx$$
$$= \frac{2}{3}(a+x)^{\frac{3}{2}} - 4a(a+x)^{\frac{1}{2}} - 2a^2(a+x)^{-\frac{1}{2}}$$
$$= \frac{2(a+x)^2 - 12a(a+x) - 6a^2}{3(a+x)^{\frac{1}{2}}} = \frac{2(x^2 - 4ax - 8a^2)}{3(a+x)^{\frac{1}{2}}}.$$

19. To integrate $du = \dfrac{(x^2-a^2)^{\frac{1}{2}} dx}{x}$.

$$\text{Here } u = \int \frac{(x^2-a^2)\, dx}{x(x^2-a^2)^{\frac{1}{2}}} = \int \frac{x\, dx}{(x^2-a^2)^{\frac{1}{2}}} - a^2 \int \frac{dx}{x(x^2-a^2)^{\frac{1}{2}}}$$
$$= (x^2-a^2)^{\frac{1}{2}} - a \sec^{-1}\frac{x}{a}.$$

20. To integrate $du = (a^3 - x^3)^{\frac{1}{3}} x^5\, dx$.

$$\text{Here } u = \int (a^3-x^3)^{\frac{1}{3}} \{a^3 - (a^3-x^3)\} x^2\, dx$$
$$= a^3 \int (a^3-x^3)^{\frac{1}{3}} x^2\, dx - \int (a^3-x^3)^{\frac{4}{3}} x^2\, dx$$
$$= -\frac{a^3(a^3-x^3)^{\frac{4}{3}}}{4} + \frac{(a^3-x^3)^{\frac{7}{3}}}{7} = -\frac{1}{28}(3a^3+4x^3)(a^3-x^3)^{\frac{4}{3}}.$$

21. To integrate $du = \dfrac{dx}{x\sqrt{(ax^n - b)}}$.

Let $x^n = z^2$; then $x = z^{\frac{2}{n}}$, $dx = \dfrac{2}{n} z^{\frac{2}{n}-1} dz$, $ax^n - b = az^2 - b$;

$$\therefore\ u = \int \frac{dx}{x\sqrt{(ax^n-b)}} = \frac{2}{n}\int \frac{z^{-1}\, dz}{\sqrt{(az^2-b)}} = \frac{2}{n}\int \frac{dz}{z(az^2-b)^{\frac{1}{2}}}$$
$$= \frac{2}{nb^{\frac{1}{2}}} \sec^{-1} \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} z = \frac{2}{nb^{\frac{1}{2}}} \sec^{-1} \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} x^{\frac{n}{2}}.$$

Additional Exercises.

1. $du = ax^{\frac{3}{2}}\, dx$. *Ans.* $u = \frac{2}{5} a x^{\frac{5}{2}}$.
2. $du = ax\sqrt[3]{x}\, dx$. *Ans.* $u = \frac{3}{7} a x^{\frac{7}{3}}$.

3. $du = \dfrac{3\,dx}{\sqrt{x}}.$ $\qquad$ *Ans.* $u = 6\,x^{\frac{1}{2}}.$

4. $du = \frac{2}{3}\,\dfrac{dx}{x^4}.$ $\qquad$ *Ans.* $u = -\dfrac{2}{9\,x^3}.$

5. $du = a\,(b + c\,x)\,dx.$ $\qquad$ *Ans.* $u = \dfrac{a\,x}{2}\,(2\,b + c\,x).$

6. $du = 6\,x^3\,(2 + 3x^4)\,dx.$ $\qquad$ *Ans.* $u = \dfrac{x^4}{4}\,(12 + 9\,x^4).$

7. $du = a\,dx - b\,x^{-3}\,dx + x^{\frac{3}{2}}dx.$ $\qquad$ *Ans.* $u = a\,x + \dfrac{b}{2\,x^2} + \frac{2}{5}\,x^{\frac{5}{2}}.$

8. $du = (1 - x)^2\,x^2\,dx.$ $\qquad$ *Ans.* $u = x^3\left(\dfrac{1}{3} - \dfrac{x}{2} + \dfrac{x^2}{5}\right).$

9. $du = 5\,x^3\,dx\,\sqrt{(7 + 3\,x^4)}.$ $\qquad$ *Ans.* $u = \frac{5}{18}\,(7 + 3\,x^4)^{\frac{3}{2}}.$

10. $du = \dfrac{x\,dx}{a + b\,x}.$ $\qquad$ *Ans.* $u = \dfrac{x}{b} - \dfrac{a}{b^2}\log\,(a + b\,x).$

11. $du = \dfrac{x^2\,dx}{a + b\,x}.$ $\qquad$ *Ans.* $u = \dfrac{x^2}{2\,b} - \dfrac{a\,x}{b^2} + \dfrac{a^2}{b^3}\log\,(a + bx).$

12. $du = \dfrac{x^2\,dx}{(a + b\,x)^2}.$ $\qquad$ *Ans.* $u = \dfrac{x\,(2\,a + b\,x)}{b^2\,(a + b\,x)} - \dfrac{2\,a}{b^3}\log\,(a + b\,x).$

13. $du = \dfrac{x\,dx}{(a + b\,x)^3}.$ $\qquad$ *Ans.* $u = -\dfrac{a + 2\,b\,x}{2\,b^2\,(a + b\,x)^2}.$

14. $du = (a - x)(2\,ax - x^2)^{\frac{1}{2}}\,dx.$ $\qquad$ *Ans.* $u = \frac{1}{3}\,(2\,a\,x - x^2)^{\frac{3}{2}}.$

15. $du = \dfrac{x^2\,dx}{a^3 - b^3 - x^3}.$ $\qquad$ *Ans.* $u = -\log\,(a^3 - b^3 - x^3)^{\frac{1}{3}}.$

16. $du = \dfrac{2\,dx}{4 + x^2}.$ $\qquad$ *Ans.* $u = \tan^{-1}\dfrac{x}{2}.$

17. $du = \dfrac{3\,dx}{1 + 4\,x^2}.$ $\qquad$ *Ans.* $u = \frac{3}{2}\tan^{-1} 2\,x.$

18. $du = \cos 3\,x\,dx - \sin 7\,x\,dx.$ $\qquad$ *Ans.* $u = \frac{1}{3}\sin 3\,x + \frac{1}{7}\cos 7\,x.$

19. $du = \dfrac{dx}{\sqrt{(2\,x - 5\,x^2)}}.$ $\qquad$ *Ans.* $u = \dfrac{\sqrt{5}}{5}\,\text{vers}^{-1}\,5\,x.$

20. $du = \dfrac{dx}{x\sqrt{(3\,x^2 - 1)}}.$ $\qquad$ *Ans.* $u = \sec^{-1} x\sqrt{3}.$

21. $du = (\tan^5 x + \tan^7 x)\,dx.$ $\qquad$ *Ans.* $u = \frac{1}{6}\tan^6 x.$

22. $du = \dfrac{dx}{a + x} - \dfrac{dx}{b + x}.$ $\qquad$ *Ans.* $u = \log\dfrac{a + x}{b + x}.$

23. $du = \sin 2\,x\,dx.$ $\qquad$ *Ans.* $u = \sin^2 x.$

24. $du = \dfrac{x\,dx}{\sqrt{(1 - x^4)}}.$ $\qquad$ *Ans.* $u = \frac{1}{2}\sin^{-1} x^2.$

25. $du = \dfrac{x\,dx}{1 + x^4}.$ $\qquad$ *Ans.* $u = \frac{1}{2}\tan^{-1} x^2.$

26. $du = \dfrac{x^{\frac{1}{2}}\,dx}{\sqrt{\{2\,(1 - 2\,x^3)\}}}.$ $\qquad$ *Ans.* $u = \frac{1}{3}\sin^{-1} x\sqrt{2\,x}.$

91. We have seen, in Art. 88, that when the numerator of a differential is the differential of the denominator, the integral is the logarithm of the denominator, and as there are some differentials which frequently occur in physical inquiries whose integrals can be expressed in a logarithmic form, we shall now advert to these, especially as the integration of the several forms offers no difficulty.

(1.) To integrate $du = \dfrac{dx}{\sqrt{(x^2 \pm a^2)}}$.

Let $x^2 \pm a^2 = z^2$; then differentiating, we get $x\,dx = z\,dz$, and adding $z\,dx$ to both sides, gives, $x\,dx + z\,dx = z\,dz + z\,dx$:

$$\therefore (x+z)\,dx = z(dz + dx), \text{ or } \frac{dx + dz}{x+z} = \frac{dx}{z};$$

hence $u = \displaystyle\int \frac{dx}{z} = \int \frac{dx + dz}{x+z} = \int \frac{d(x+z)}{x+z} = \log(x+z)$;

$$\therefore \int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log\{x + \sqrt{(x^2 \pm a^2)}\}.$$

(2). To integrate $du = \dfrac{dx}{x\sqrt{(a^2+x^2)}}$.

Let $x = \dfrac{a}{z}$, then $\log x = \log a - \log z$, and $\dfrac{dx}{x} = -\dfrac{dz}{z}$; hence

$$du = \frac{dx}{x\sqrt{(a^2+x^2)}} = \frac{dx}{ax\sqrt{\left(1+\frac{x^2}{a^2}\right)}} = -\frac{dz}{az\sqrt{(1+z^{-2})}} = -\frac{dz}{a\sqrt{(z^2+1)}}:$$

$$\therefore u = -\frac{1}{a}\int \frac{dz}{\sqrt{(z^2+1)}} = -\frac{1}{a}\log\{z + \sqrt{(z^2+1)}\}, \text{ by (1)}$$

$$= -\frac{1}{a}\log\frac{a+\sqrt{(a^2+x^2)}}{x} = -\frac{1}{a}\log\frac{x}{\sqrt{(a^2+x^2)}-a}.$$

Similarly $\displaystyle\int \frac{dx}{x\sqrt{(a^2-x^2)}} = -\frac{1}{a}\log\frac{a+\sqrt{(a^2-x^2)}}{x} = \frac{1}{a}\log\frac{x}{a+\sqrt{(a^2-x^2)}}$.

(3). To integrate $du = \dfrac{dx}{a^2 - x^2}$.

Here $\dfrac{1}{a^2-x^2} = \dfrac{1}{2a}\left\{\dfrac{1}{a+x} + \dfrac{1}{a-x}\right\} \therefore du = \dfrac{1}{2a}\left\{\dfrac{dx}{a+x} + \dfrac{dx}{a-x}\right\}$;

$$\therefore u = \frac{1}{2a}\int \frac{dx}{a+x} + \frac{1}{2a}\int \frac{dx}{a-x}$$

$$= \frac{1}{2a}\{\log(a+x) - \log(a-x)\} = \frac{1}{2a}\log\frac{a+x}{a-x}.$$

Similarly, $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x-a}\right\} = \frac{1}{2a}\log\frac{x-a}{x+a}$.

(4). To integrate $du = \dfrac{dx}{\sqrt{(2ax + x^2)}}$.

Since $2ax + x^2 = (a+x)^2 - a^2$, and $dx = d(a+x)$;

therefore $du = \dfrac{d(a+x)}{\sqrt{\{(a+x)^2 - a^2\}}}$, which is of the same form as the

first of these differentials; hence by (1) we have

$$\int \frac{dx}{\sqrt{(2ax+x^2)}} = \int \frac{d(a+x)}{\sqrt{\{(a+x)^2-a^2\}}}$$
$$= \log\{a+x+\sqrt{(a+x)^2-a^2}\}$$
$$= \log\{a+x+\sqrt{(2ax+x^2)}\}.$$

(5). To integrate $du = \dfrac{dx}{a+bx+cx^2}$, when b^2 is *greater* than $4ac$.

$$\int \frac{dx}{a+bx+cx^2} = 2\int \frac{2c\,dx}{4ac+4bcx+4c^2x^2}$$
$$= 2\int \frac{d(2cx+b)}{(2cx+b)^2-(b^2-4ac)}$$
$$= \frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}, \text{ by (5)}.$$

(6). To integrate $du = \dfrac{dx}{(a+bx+cx^2)^{\frac{1}{2}}}$.

$$\int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}} = \frac{1}{c^{\frac{1}{2}}}\int \frac{2c\,dx}{(4ac+4bcx+4c^2x^2)^{\frac{1}{2}}}$$
$$= \frac{1}{c^{\frac{1}{2}}}\int \frac{d(2cx+b)}{\{(2cx+b)^2+4ac-b^2\}^{\frac{1}{2}}}$$
$$= \frac{1}{c^{\frac{1}{2}}}\log\{2cx+b+2c^{\frac{1}{2}}(a+bx+cx^2)^{\frac{1}{2}}\}, \text{ by (1)}.$$

(7). To integrate $du = \dfrac{dx}{x(a+bx+cx^2)^{\frac{1}{2}}}$.

Let $x = \dfrac{1}{y}$; then $\log x = -\log y$, and $\dfrac{dx}{x} = -\dfrac{dy}{y}$;

hence $du = -\dfrac{dy}{y(a+by^{-1}+cy^{-2})^{\frac{1}{2}}} = -\dfrac{dy}{(ay^2+by+c)^{\frac{1}{2}}}$;

$$\therefore \int \frac{dx}{x(a+bx+cx^2)^{\frac{1}{2}}} = -\int \frac{dy}{(ay^2+by+c)^{\frac{1}{2}}}$$
$$= -\frac{1}{a^{\frac{1}{2}}}\log\{2ay+b+2a^{\frac{1}{2}}(ay^2+by+c)^{\frac{1}{2}}\} \text{ by (6)}$$
$$= -\frac{1}{a^{\frac{1}{2}}}\log\left\{\frac{2a+bx+2a^{\frac{1}{2}}(a+bx+cx^2)^{\frac{1}{2}}}{x}\right\}$$
$$= \frac{1}{a^{\frac{1}{2}}}\log\left\{\frac{x}{2a+bx+2a^{\frac{1}{2}}(a+bx+cx^2)^{\frac{1}{2}}}\right\}.$$

EXAMPLES.

1. $$\int \frac{dx}{x^2+x-1} = \frac{1}{5^{\frac{1}{2}}}\log\frac{2x+1-5^{\frac{1}{2}}}{2x+1+5^{\frac{1}{2}}}.$$

2. $\displaystyle\int\frac{dx}{2x^2+3x+1} \qquad \frac{2x+1}{2x+2}.$

3. $\displaystyle\int\frac{dx}{(x^2+x+1)^{\frac{1}{2}}} = \log\{2x+1+2(x^2+x+1)^{\frac{1}{2}}\}.$

4. $\displaystyle\int\frac{dx}{(x^2-x-1)^{\frac{1}{2}}} = \log\{2x-1+2(x^2-x-1)^{\frac{1}{2}}\}.$

5. $\displaystyle\int\frac{dx}{x(x^2+x+1)^{\frac{1}{2}}} = \log\left\{\frac{x}{2+x+2(x^2+x+1)^{\frac{1}{2}}}\right\}.$

6. $\displaystyle\int\frac{dx}{x(4+5x+7x^2)^{\frac{1}{2}}} = \log\left\{\frac{x}{8+5x+4(4+5x+7x^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$

INTEGRATION BY PARTS.

92. The formula to be employed in integrating by parts is, Art. 86 (24)

$$\int u\,dv = uv - \int v\,du,$$

where it is evident that $\int u\,dv$ will be known, provided $\int v\,du$ can be found. As this method of integration is extensively used in the Integral Calculus, the student should make himself familiar with its application, and endeavour to acquire the power of writing down results without repeating the formula.

Examples for Practice.

1. Integrate $x \log x\,dx$.

Let $\log x = u$, and $x\,dx = dv$; then $du = \dfrac{dx}{x}$, and $v = \int x\,dx = \dfrac{x^2}{2}$,

$$\therefore \int x\log x\,dx = \int u\,dv = uv - \int v\,du = \frac{x^2}{2}\log x - \int\frac{x^2}{2}\cdot\frac{dx}{x}$$

$$= \frac{x^2}{2}\log x - \tfrac{1}{2}\int x\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}.$$

2. Let it be required to integrate $(1+x^2)^3 x^5\,dx$*.

Let $x^4 = u$, and $(1+x^2)^3 x\,dx = dv$; then $du = 4\,x^3\,dx$ and $v = \frac{1}{8}(1+x^2)^4$;

$$\therefore \int(1+x^2)^3 x^5\,dx = \int u\,dv = uv - \int v\,du$$

$$= \tfrac{1}{8}x^4(1+x^2)^4 - \tfrac{1}{2}\int(1+x^2)^4 x^3\,dx \;\ldots\ldots (1).$$

The integral in the last term of (1) which we have still to obtain,

* This example has been chosen for illustrating the principle of integration by parts, but its integral may be more readily obtained by expanding $(1+x^2)^3$. Thus

$$\int(1+x^2)^3 x^5\,dx = \int x^5\,dx + 3\int x^7\,dx + 3\int x^9\,dx + \int x^{11}\,dx$$

$$= \frac{x^6}{6} + \frac{3x^8}{8} + \frac{3x^{10}}{10} + \frac{x^{12}}{12} = \frac{x^6}{120}(10x^6 + 36x^4 + 45x^2 + 20),$$

differing only from the former result by the constant $\frac{1}{120}$.

must be submitted to the same process of integration. Thus let

$$x^2 = u, \text{ and } (1+x^2)^4\, x\, dx = dv;$$

then $\int (1+x^2)^4 x^3\, dx = \int x^2 \,.\, (1+x^2)^4\, x\, dx$

$$= x^2 \,.\, \frac{(1+x^2)^5}{10} - \tfrac{1}{5}\int (1+x^2)^5\, x\, dx = \frac{x^2(1+x^2)^5}{10} - \frac{(1+x^2)^6}{60}.$$

Substituting this result in (1), we get finally

$$\int (1+x^2)^3 x^5\, dx = \frac{x^4(1+x^2)^4}{8} - \frac{x^2(1+x^2)^5}{10} + \frac{(1+x^2)^6}{60}$$

$$= \frac{(10\,x^4 - 4\,x^2 + 1)\,(1+x^2)^4}{120}.$$

3. Let it be required to integrate $\int (x^2+a^2)^{\frac{1}{2}}\, dx$.

Since $(x^2+a^2)^{\frac{1}{2}} = \dfrac{x^2+a^2}{(x^2+a^2)^{\frac{1}{2}}} = \dfrac{a^2}{(x^2+a^2)^{\frac{1}{2}}} + \dfrac{x^2}{(x^2+a^2)^{\frac{1}{2}}}$;

$$\therefore \int (x^2+a^2)^{\frac{1}{2}}\, dx = a^2 \int \frac{dx}{(x^2+a^2)^{\frac{1}{2}}} + \int \frac{x^2\, dx}{(x^2+a^2)^{\frac{1}{2}}}$$

$$= a^2 \log\{\, x + (x^2+a^2)^{\frac{1}{2}}\} + \int x \,.\, \frac{x\, dx}{(x^2+a^2)^{\frac{1}{2}}}$$

$$= a^2 \log\{\, x + (x^2+a^2)^{\frac{1}{2}}\} + x\,(x^2+a^2)^{\frac{1}{2}} - \int (x^2+a^2)^{\frac{1}{2}}\, dx.$$

Transposing the last term, and dividing by 2, gives

$$\int (x^2+a^2)^{\frac{1}{2}}\, dx = \frac{a^2}{2}\log\{\, x + (x^2+a^2)^{\frac{1}{2}}\} + \frac{x\,(x^2+a^2)^{\frac{1}{2}}}{2}.$$

4. Find the integral of $(a^2+x^2)^{\frac{1}{2}}\, x^2\, dx$.

Here $\int (a^2+x^2)^{\frac{1}{2}}\, x^2\, dx = \int x \,.\, (a^2+x^2)^{\frac{1}{2}}\, x\, dx$

$$= \frac{x}{3}(a^2+x^2)^{\frac{3}{2}} - \tfrac{1}{3}\int (a^2+x^2)^{\frac{3}{2}}\, dx$$

$$= \frac{x}{3}(a^2+x^2)^{\frac{3}{2}} - \tfrac{1}{3}\int (a^2+x^2)\,(a^2+x^2)^{\frac{1}{2}}\, dx$$

$$= \frac{x}{3}(a^2+x^2)^{\frac{3}{2}} - \frac{a^2}{3}\int (a^2+x^2)^{\frac{1}{2}}\, dx - \tfrac{1}{3}\int (a^2+x^2)^{\frac{1}{2}}\, x^2\, dx;$$

$$\therefore \int (a^2+x^2)^{\frac{1}{2}}\, x^2\, dx = \frac{x}{4}(a^2+x^2)^{\frac{3}{2}} - \frac{a^2}{4}\int (a^2-x^2)^{\frac{1}{2}}\, dx.$$

But by Example 3, we have

$$\int (a^2+x^2)^{\frac{1}{2}}\, dx = \frac{x}{2}(a^2+x^2)^{\frac{1}{2}} + \frac{a^2}{2}\log\{x + (a^2+x^2)^{\frac{1}{2}}\};$$

$$\therefore \int (a^2+x^2)^{\frac{1}{2}}\, x^2\, dx = \frac{x}{4}(a^2+x^2)^{\frac{3}{2}} - \frac{a^2 x}{8}(a^2+x^2)^{\frac{1}{2}} - \frac{a^4}{8}\log\{x + (a^2+x^2)^{\frac{1}{2}}\}$$

$$= \frac{x\,(a^2+2\,x^2)\,(a^2+x^2)^{\frac{1}{2}}}{8} - \frac{a^4}{8}\log\{x+(a^2+x^2)^{\frac{1}{2}}\}.$$

5. $\int x^n \log x\, dx = \dfrac{x^{n+1}\log x}{n+1} - \dfrac{x^{n+1}}{(n+1)^2}.$

6. $\int (a^2-x^2)^{\frac{1}{2}}\, dx = \dfrac{x(a^2-x^2)^{\frac{1}{2}}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}.$

7. $\displaystyle\int \frac{x^2\, dx}{(a^2-x^2)^{\frac{1}{2}}} = -\frac{x(a^2-x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

8. $\displaystyle\int \frac{x^3\, dx}{(a^2+x^2)^{\frac{1}{2}}} = \frac{(x^2-2a^2)(a^2+x^2)^{\frac{1}{2}}}{3}.$

9. $\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x$
$= (x^2-2)\sin x + 2x\cos x.$

10. $\int x^3 \sin x\, dx = 3(x^2-2)\sin x - x(x^2-6)\cos x.$

11. $\displaystyle\int \frac{x^3\, dx}{(1+x^2)^3} = -\frac{x^2}{6(1+x^2)^2} - \frac{1}{6(1+x^2)} = -\frac{1+2x^2}{6(1+x^2)^2}.$

12. $\displaystyle\int \frac{x^5\, dx}{(1+x^2)^5} = -\frac{4+6x^2+5x^4}{24(1+x^2)^4}.$

INTEGRATION OF RATIONAL FRACTIONS.

93. A rational fraction is a fraction of the form

$$\frac{(x^m + a\,x^{m-1} + b\,x^{m-2} + \ldots)\,dx}{x^n + a'\,x^{n-1} + b'\,x^{n-2} + \ldots},$$

where the indices of x are all positive integers.

If the numerator contains powers of x as high or higher than the highest in the denominator, the fraction can always be reduced by common algebraic division to a mixed quantity, composed of a rational integral function, and a rational fraction in which the index of the highest power of x in the numerator of the differential is less by one or more units than that of the highest power in its denominator. We have only then to consider rational fractions in this case, and as they are not often met with in practice containing high powers of x, we shall confine our investigations to such only as are likely to occur in the most useful inquiries.

Rational fractions are integrated by resolving them into the sum of a number of fractions with simpler denominators, called partial fractions, as in the following cases:

I. *When the simple factors of the denominator are real and unequal.*

In this case, either of the following methods may be employed:

First method.—Let the proposed fraction be

$$du = \frac{U\,dx}{V} = \frac{x\,dx}{(x-a)(x-b)(x-c)};$$

then assume $\dfrac{x}{(x-a)(x-b)(x-c)} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$;

therefore $x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b).$

Now this equation must hold for all values of x; hence if

$$x = a, \text{ then } a = A\,(a-b)\,(a-c), \text{ and } A = \frac{a}{(a-b)\,(a-c)},$$

$$x = b, \quad ,, \quad b = B\,(b-a)\,(b-c), \text{ and } B = \frac{b}{(b-a)\,(b-c)},$$

$$x = c, \quad ,, \quad c = C\,(c-a)\,(c-b), \text{ and } C = \frac{c}{(c-a)\,(c-b)}.$$

Consequently A, B, C, are all known, and thence

$$u = \int \frac{x\,dx}{(x-a)\,(x-b)\,(x-c)} = \int \frac{A\,dx}{x-a} + \int \frac{B\,dx}{x-b} + \int \frac{C\,dx}{x-c}$$
$$= A \log\,(x-a) + B \log\,(x-b) + C \log\,(x-c)$$
$$= \log\,\{\,(x-a)^A\,(x-b)^B\,(x-c)^C\,\}\,.\cdot$$

Second method.—Assume $\dfrac{U}{V} = \dfrac{fx}{Fx} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$,

then $\qquad fx = A\,.\,\dfrac{Fx}{x-a} + B\,\dfrac{Fx}{x-b} + C\,\dfrac{Fx}{x-c}.$

Now since $Fx = (x-a)\,(x-b)\,(x-c)$, it is evident that when $x = a$, the fraction $\dfrac{Fx}{x-a} = \dfrac{0}{0} = \dfrac{dFx}{dx}$, by the theory of the singular values of functions Art. 59, and $\dfrac{Fx}{x-b}$ and $\dfrac{Fx}{x-c}$ are each equal to zero; hence to determine A we have only to divide fx by $\dfrac{dFx}{dx}$, and write a for x in the result. Similarly, it is shown that B and C are determined by the same formula, writing in it b and c for x respectively.

Let it be required to integrate

$$du = \frac{(2x-5)\,dx}{x^3 - 6x^2 + 11x - 6} = \frac{(2x-5)\,dx}{(x-1)\,(x-2)\,(x-3)}.$$

Assume $\dfrac{2x-5}{x^3 - 6x^2 + 11x - 6} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$; then by the second method $fx \div \dfrac{dFx}{dx} = \dfrac{2x-5}{3x^2 - 12x + 11}$.

In this formula, let $x = 1$, then $A = -\frac{3}{2}$; if $x = 2$, $B = 1$ and if $x = 3$, $C = \frac{1}{2}$;

hence $du = -\dfrac{3}{2}\,\dfrac{dx}{x-1} + \dfrac{dx}{x-2} + \dfrac{1}{2}\,\dfrac{dx}{x-3}$, and integrating,

$$u = -\tfrac{3}{2}\log(x-1) + \log(x-2) + \tfrac{1}{2}\log(x-3) = \log C\,\frac{(x-2)\,(x-3)^{\frac{1}{2}}}{(x-1)^{\frac{3}{2}}}.$$

By the first method we get from the assumed equation

$$2x - 5 = A\,(x-2)\,(x-3) + B\,(x-1)\,(x-3) + C\,(x-1)\,(x-2).$$

$$\begin{aligned} \text{If } x = 1, \text{ then } & -3 = A\,(1-2)\,(1-3) = 2A \;.\cdot.\, A = -\tfrac{3}{2}, \\ x = 2, \quad & -1 = B\,(2-1)\,(2-3) = -B \;.\cdot.\, B = 1, \\ x = 3, \quad & \;\;\, 1 = C\,(3-1)\,(3-2) = 2C \;.\cdot.\, C = \tfrac{1}{2}; \end{aligned}$$

the same values of A, B, C, as were obtained by the former method.

II.— *When the simple factors of the denominator are real, but not unequal.*

Let the proposed fraction be

$$d\,u = \frac{\mathrm{U}\,d\,x}{\mathrm{V}} = \frac{(m\,x^2 + n\,x + p)\,d\,x}{(x-a)^2(x-b)};$$

then if we make the assumption

$$\frac{m\,x^2 + n\,x + p}{(x-a)^2(x-b)} = \frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{x-a} + \frac{\mathrm{C}}{x-b},$$

it is evident that the second member cannot be made identical with the first, since $\frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{x-a} = \frac{\mathrm{A}+\mathrm{B}}{x-a}$, is of the form $\frac{\mathrm{D}}{x-a}$, and the common denominator is $(x-a)\,(x-b)$ instead of $(x-a)^2\,(x-b)$.

To obviate this, we may assume

$$\frac{m\,x^2 + n\,x + p}{(x-a)^2(x-b)} = \frac{\mathrm{A}\,x + \mathrm{B}}{(x-a)^2} + \frac{\mathrm{C}}{x-b},$$

since by reducing the second member to a common denominator, the numerator will be of the same form as the proposed numerator, and contain *three* undetermined coefficients. But the fraction $\frac{\mathrm{A}\,x+\mathrm{B}}{(x-a)^2}$ may be separated into two others, by assuming

$$\frac{\mathrm{A}\,x + \mathrm{B}}{(x-a)^2} = \frac{\mathrm{P}}{(x-a)^2} + \frac{\mathrm{Q}}{x-a};$$

and therefore the proposed fraction may be separated into three others in the following manner:

Let $\frac{m\,x^2 + n\,x + p}{(x-a)^2(x-b)} = \frac{\mathrm{A}}{(x-a)^2} + \frac{\mathrm{B}}{x-a} + \frac{\mathrm{C}}{x-b}$; then will

$$m\,x^2 + n\,x + p = \mathrm{A}\,(x-b) + \mathrm{B}\,(x-a)\,(x-b) + \mathrm{C}(x-a)^2 \ldots\ldots (1).$$

Let $x = a$, then $m\,a^2 + n\,a + p = \mathrm{A}\,(a-b)$, and $\mathrm{A} = \frac{m\,a^2+n\,a+p}{a-b}$.

Let $x = b$, then $m\,b^2 + n\,b + p = \mathrm{C}\,(b-a)^2$, and $\mathrm{C} = \frac{m\,b^2+n\,b+p}{(b-a)^2}$.

Since eq. (1) holds for all values of x, the coefficients of the same powers of x in both members must be identical; hence taking the highest powers of x, we get $\mathrm{B} + \mathrm{C} = m$, and consequently

$$\mathrm{B} = m - \mathrm{C} = m - \frac{m\,b^2 + n\,b + p}{(b-a)^2} = \frac{m\,(a^2 - 2\,a\,b) - (n\,b + p)}{(a-b)^2}.$$

The usual way, however, of determining B is to subtract the equation

$$m\,a^2 + n\,a + p = \mathrm{A}\,(a-b)$$

from (1); then we get

$$m\,(x^2 - a^2) + n\,(x-a) = \mathrm{A}\,(x-a) + \mathrm{B}\,(x-a)\,(x-b) + \mathrm{C}\,(x-a)^2,$$

which is manifestly divisible by $x - a$; hence, dividing by $x - a$,

$$m\,(x+a) + n = \mathrm{A} + \mathrm{B}\,(x-b) + \mathrm{C}\,(x-a) \;\ldots\ldots\; (2).$$

Now if in (2) we put $x = a$; then $2\,m\,a + n = \mathrm{A} + \mathrm{B}\,(a-b)$;

$$\therefore \text{B} = \frac{2ma+n-\text{A}}{a-b} = \frac{(2ma+n)(a-b)-(ma^2+na+p)}{(a-b)^2}$$
$$= \frac{m(a^2-2ab)-(nb+p)}{(a-b)^2}.$$

III.—*When the denominator contains unequal quadratic factors, having impossible simple factors.*

Let the proposed fraction be of the form

$$du = \frac{\text{U}\,dx}{\text{V}} = \frac{(x+m)\,dx}{(x^2+ax+b)\,(x^2+c^2)}.$$

Assume $\dfrac{x+m}{(x^2+ax+b)\,(x^2+c^2)} = \dfrac{\text{A}x+\text{B}}{x^2+ax+b} + \dfrac{\text{C}x+\text{D}}{x^2+c^2}$; then

$x+m = (\text{A}x+\text{B})\,(x^2+c^2) + (\text{C}x+\text{D})\,(x^2+ax+b), \ldots\ldots (1),$

an equation which, being identical, must hold for all values of x, whether real or imaginary.

Let $x^2 = -c^2$; then equation (1) gives

$$\begin{aligned} x+m &= (\text{C}x+\text{D})\,(ax+b-c^2) \\ &= \text{C}ax^2 + \{\text{C}(b-c^2)+\text{D}a\}\,x + \text{D}(b-c^2) \\ &= \{\text{C}(b-c^2)+\text{D}a\}\,x - \text{C}ac^2 + \text{D}(b-c^2), \end{aligned}$$

which cannot hold unless the coefficients of x in both members be equal;

$$\therefore 1 = \text{C}(b-c^2) + \text{D}a \quad \ldots \quad (2).$$

and hence $\quad m = \text{D}b - c^2(\text{C}a+\text{D}) \quad \ldots \quad (3).$

These equations serve to determine the values of C and D.

Again, let $x^2 = -(ax+b)$; then by equation (1), we obtain

$$\begin{aligned} x+m &= (\text{A}x+\text{B})\,(c^2-ax-b) \\ &= -\text{A}ax^2 + \{\text{A}(c^2-b)-\text{B}a\}\,x + \text{B}(c^2-b) \\ &= \{\text{A}(a^2+c^2-b)-\text{B}a\}\,x + \text{A}ab + \text{B}(c^2-b); \end{aligned}$$

$$\therefore 1 = \text{A}(a^2+c^2-b) - \text{B}a \quad \ldots\ldots \quad (4).$$
$$m = \text{A}ab + \text{B}(c^2-b) \quad \ldots\ldots \quad (5).$$

These equations serve to determine the values of A and B.

94. It may be proper to remind the student that the values of A, B, C, D, can always be obtained from equation (1), by the method of indeterminate coefficients. Thus multiplying out and arranging the result, we have

$$x+m = (\text{A}+\text{C})x^3 + (\text{B}+\text{C}a+\text{D})x^2 + (\text{A}c^2+\text{C}b+\text{D}a)x + \text{B}c^2 + \text{D}b;$$

and equating the coefficients of the same powers of x, we get

$$\text{A}+\text{C} = 0; \quad \text{A}c^2 + \text{C}b + \text{D}a = 1;$$
$$\text{B}+\text{C}a+\text{D} = 0; \quad \text{B}c^2 + \text{D}b = m;$$

which furnish the values of A, B, C, D. This method may be applied in every case, and not unfrequently with as much success as any of the methods we have adverted to in this and the preceding cases can afford. The rational fraction is therefore reduced to the integration of differentials of the forms

$$\frac{(\text{A}x+\text{B})\,dx}{x^2+ax+b} \text{ and } \frac{(\text{A}x+\text{B})\,dx}{x^2+a^2}.$$

To integrate the former of these expressions, let $x + \frac{1}{2}a = z$; then

$x^2 + ax + \frac{1}{4}a^2 = z^2$, and $x^2 + ax + b = z^2 - (\frac{1}{4}a^2 - b) = z^2 - \frac{1}{4}(a^2 - 4b)$, also $\quad Ax + B = Az - \frac{1}{2}Aa + B$, and $dx = dz$; consequently

$$\int \frac{(Ax+B)\,dx}{x^2+ax+b} = \int \frac{\{Az - (\frac{1}{2}Aa - B)\}\,dz}{z^2 - \frac{1}{4}(a^2-4b)} = A\int \frac{z\,dz}{z^2 - \frac{1}{4}(a^2-4b)}$$

$$- (\tfrac{1}{2}Aa - B)\int \frac{dz}{z^2 - \frac{1}{4}(a^2-4b)},$$

both of which are known forms. Integrating the latter form, we get

$$\int \frac{(Ax+B)\,dx}{x^2+a^2} = A\int \frac{x\,dx}{x^2+a^2} + B\int \frac{dx}{x^2+a^2}$$

$$= \frac{A}{2}\log(x^2+a^2) + \frac{B}{a}\tan^{-1}\frac{x}{a}. \quad \text{[See (5) and (17)]}$$

When the denominator consists of equal factors of the form $x^2 + ax + b$, the same method may be employed as was followed with respect to equal factors of the form $x - a$ in Case II.

95. It only now remains to integrate the form

$$\frac{dx}{(x^2+a^2)^n}.$$

Let $\displaystyle\int \frac{dx}{(x^2+a^2)^n} = \frac{Ax}{(x^2+a^2)^{n-1}} + B\int \frac{dx}{(x^2+a^2)^{n-1}}$; then, differentiating,

$$\frac{dx}{(x^2+a^2)^n} = \frac{A\,dx}{(x^2+a^2)^{n-1}} - \frac{2(n-1)Ax^2\,dx}{(x^2+a^2)^n} + \frac{B\,dx}{(x^2+a^2)^{n-1}},$$

which, reducing to a common denominator, and dividing by dx, gives

$$1 = A(x^2+a^2) - 2(n-1)Ax^2 + B(x^2+a^2),$$

or

$$1 = \{A - 2(n-1)A + B\}x^2 + Aa^2 + Ba^2.$$

Hence $A - 2(n-1)A + B = 0$, and $Aa^2 + Ba^2 = 1$; and these give

$$A = \frac{1}{2(n-1)a^2} \text{ and } B = \frac{2n-3}{2(n-1)a^2};$$

$$\therefore \int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}},$$

a formula which, by successive operations, diminishes the exponent n, and finally reduces the differential to $\dfrac{dx}{x^2+a^2}$, which is formula (17) Art. 83.

When the factors of the denominator of any proposed rational fraction are not given, we must put the denominator equal to zero, and the resolution of the equation thus obtained will determine these factors. The solution of equations, however, does not extend beyond certain limits, except in the case of numerical equations, or when $x^n = \pm 1$.

EXAMPLES.

1. Let it be required to integrate $du = \dfrac{(x^3 + x^2 + 2)\,dx}{(x-1)(x+2)(x-3)(x+4)}$.

The product of the factors in the denominator is $x^4 + 2x^3 - 13x^2 - 14x + 24$; and if we assume

$$\frac{x^3+x^2+2}{x^4+2x^3-13x^2-14x+24}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}+\frac{D}{x+4};$$

then the formula for determining A, B, C, D, is

$$fx \div \frac{d\,\mathrm{F}x}{dx}=\frac{x^3+x^2+2}{4x^3+6x^2-26x-14}.$$

If in this formula we substitute in succession 1, -2, 3, and -4 for x, we get $A=-\frac{2}{15}$, $B=\frac{1}{15}$, $C=\frac{19}{35}$, and $D=\frac{23}{35}$; therefore

$$u=-\frac{2}{15}\int\frac{dx}{x-1}+\frac{1}{15}\int\frac{dx}{x+2}+\frac{19}{35}\int\frac{dx}{x-3}+\frac{23}{35}\int\frac{dx}{x+4}$$

$$=-\frac{2}{15}\log(x-1)+\frac{1}{15}\log(x+2)+\frac{19}{35}\log(x-3)+\frac{23}{35}\log(x+4).$$

2. Let it be required to integrate $du=\dfrac{x\,dx}{(x-1)(x^3-1)}$.

Since $(x-1)(x^3-1)=(x-1)^2(x^2+x+1)$, we may hence assume

$$\frac{x}{(x-1)(x^3-1)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1};$$

$$\therefore\ x=A(x^2+x+1)+B(x^3-1)+(Cx+D)(x-1)^2,$$

or $x=(B+C)x^3+(A-2C+D)x^2+(A+C-2D)x+A-B+D$; and equating the coefficients of the same powers of x in both members,

$$B+C=0\ .\ .\ .\ .\ (1),\qquad A+C-2D=1\ .\ .\ .\ .\ (3),$$
$$A-2C+D=0\ .\ .\ .\ .\ (2),\qquad A-B+D=0\ .\ .\ .\ .\ (4).$$

From these four equations we derive the following values:—

$$A=\tfrac{1}{3},\ B=0,\ C=0,\ D=-\tfrac{1}{3};$$

hence $$u=\tfrac{1}{3}\int\frac{dx}{(x-1)^2}-\tfrac{1}{3}\int\frac{dx}{x^2+x+1}$$

$$=\tfrac{1}{3}\int(x-1)^{-2}\,dx-\tfrac{1}{3}\int\frac{d(x+\frac{1}{2})}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$=-\frac{1}{3(x-1)}-\tfrac{2}{9}\sqrt{3}\tan^{-1}\frac{2x}{\sqrt{3}},\ \text{by Art. 87, and Art. 83 (17).}$$

3. Integrate $du=\dfrac{x^2\,dx}{1-x^4}$.

Here $1-x^4=(1-x^2)(1+x^2)=(1-x)(1+x)(1+x^2)$; then if we assume $\dfrac{x^2}{1-x^4}=\dfrac{A}{1-x}+\dfrac{B}{1+x}+\dfrac{Cx+D}{1+x^2}$, we get

$$x^2=A(1+x)(1+x^2)+B(1-x)(1+x^2)+(Cx+D)(1-x)(1+x)$$

or $x^2=(A-B-C)x^3+(A+B-D)x^2+(A-B+C)x+A+B+D$

$$\therefore A-B-C=0\ .\ .\ .\ (1)\qquad A-B+C=0\ .\ .\ .\ .\ (3)$$
$$A+B-D=1\ .\ .\ .\ (2)\qquad A+B+D=0\ .\ .\ .\ (4).$$

From these equations we get the following values:—

$$A=\tfrac{1}{4},\ B=\tfrac{1}{4},\ C=0,\ D=-\tfrac{1}{2};$$

$$\therefore\ u=\frac{1}{4}\int\frac{dx}{1-x}+\frac{1}{4}\int\frac{dx}{1+x}-\frac{1}{2}\int\frac{dx}{1+x^2}$$

$$= -\tfrac{1}{4}\log(1-x)+\tfrac{1}{4}\log(1+x)-\tfrac{1}{2}\tan^{-1}x=\log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}-\tfrac{1}{2}\tan^{-1}x.$$

4. Integrate $du = \dfrac{dx}{ax^2+bx+c}$, when b^2 is *less* than $4ac$.

Since $ax^2+bx+c=a\left(x^2+\dfrac{b}{a}x+\dfrac{c}{a}\right)=a\left\{\left(x+\dfrac{b}{2a}\right)^2+\dfrac{4ac-b^2}{4a^2}\right\}$,

let $\quad x+\dfrac{b}{2a}=z$, and $\dfrac{4ac-b^2}{4a^2}=\alpha^2$; then we have $dx=dz$,

and $\displaystyle\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dz}{z^2+\alpha^2}=\frac{1}{a\alpha}\tan^{-1}\frac{z}{\alpha}$, by Art. 83 (17),

or $\displaystyle\int\frac{dx}{ax^2+bx+c}=\frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2ax+b}{\sqrt{(4ac-b^2)}}.$

If b^2 is greater than $4ac$, the integral now obtained would contain imaginary quantities, since $4ac-b^2$ is, in this case, a negative quantity. But considering α^2 as negative, we get,

$$\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dz}{z^2-\alpha^2}=\frac{1}{2a\alpha}\log\frac{z-\alpha}{z+\alpha},\text{ by Art. 91 (3),}$$

or, $\displaystyle\int\frac{dx}{ax^2+bx+c}=\frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2ax+b-\sqrt{(b^2-4ac)}}{2ax+b+\sqrt{(b^2-4ac)}}$,
as in Art. 91, (5).

If $b^2=4ac$, then both these methods fail, since then $\alpha=0$, and we have

$$\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dz}{z^2}=-\frac{1}{az}=\frac{1}{ax+\frac{1}{2}b}.$$

5. Let it be required to integrate $\dfrac{dx}{(ax^2+bx+c)^n}$.

Let, as before, $x+\dfrac{b}{2a}=z$; and $\dfrac{4ac-b^2}{4a^2}=\alpha^2$; then

$$(ax^2+bx+c)^n=a^n\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}^n=a^n(z^2+\alpha^2)^n;$$

hence $\displaystyle\int\frac{dx}{(ax^2+bx+c)^n}=\frac{1}{a^n}\int\frac{dz}{(z^2+\alpha^2)^n}$, a form which has already been discussed in a preceding page, where it has been shown that

$$\int\frac{dz}{(z^2+\alpha^2)^n}=\frac{z}{2(n-1)\alpha^2(z^2+\alpha^2)^{n-1}}+\frac{2n-3}{2(n-1)\alpha^2}\int\frac{dz}{(z^2+\alpha^2)^{n-1}}.$$

Let $\quad n=2$, then $\displaystyle\int\frac{dz}{(z^2+\alpha^2)^2}=\frac{z}{2\alpha^2(z^2+\alpha^2)}+\frac{1}{2\alpha^2}\int\frac{dz}{z^2+\alpha^2}$;

$$\therefore\int\frac{dx}{(ax^2+bx+c)^2}=\frac{1}{a^2}\int\frac{dz}{(z^2+\alpha^2)^2}=\frac{z}{2a^2\alpha^2(z^2+\alpha^2)}+\frac{1}{2a^2\alpha^3}\tan^{-1}\frac{z}{\alpha}$$

$$=\frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)}+\frac{4a}{(4ac-b^2)^{\frac{3}{2}}}\tan^{-1}\frac{2ax+b}{\sqrt{(4ac-b^2)}}.$$

Exercises in Rational Fractions.

1. $\dfrac{du}{dx} = \dfrac{5x+1}{x^2+x-2}$. *Ans.* $u = \log\{(x-1)^2(x+2)^3\}$.

2. $\dfrac{du}{dx} = \dfrac{x-1}{x^2+6x+8}$. *Ans.* $u = \log\dfrac{(x+4)^{\frac{5}{2}}}{(x+2)^{\frac{3}{2}}}$.

3. $\dfrac{du}{dx} = \dfrac{2x+3}{x^3+x^2-2x}$. *Ans.* $u = \log\dfrac{(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}}(x+2)^{\frac{1}{6}}}$.

4. $\dfrac{du}{dx} = \dfrac{x^2-x+2}{x^4-5x^2+4}$. *Ans.* $u = \log\left\{\dfrac{(x+1)^2(x-2)}{(x-1)(x+2)^2}\right\}^{\frac{1}{3}}$.

5. $\dfrac{du}{dx} = \dfrac{x^2}{x^3+5x^2+8x+4}$. *Ans.* $u = \dfrac{4}{x+2} + \log(x+1)$.

6. $\dfrac{du}{dx} = \dfrac{3x-1}{x^2-2x}$. *Ans.* $u = \log\{x(x-2)^5\}^{\frac{1}{2}}$.

7. $\dfrac{du}{dx} = \dfrac{x^2+x}{(x-2)^2(x-1)}$. *Ans.* $u = -\dfrac{6}{x-2} + \log\dfrac{(x-1)^2}{x-2}$.

8. $\dfrac{du}{dx} = \dfrac{x^2-2}{x^3(x-1)}$. *Ans.* $u = -\dfrac{2x+1}{x^2} + \log\dfrac{x}{x-1}$.

9. $\dfrac{du}{dx} = \dfrac{1}{x(a+bx^3)^2}$. *Ans.* $u = \dfrac{1}{3a(a+bx^3)} - \dfrac{1}{3a^2}\log\dfrac{a+bx^3}{x^3}$.

10. $\dfrac{du}{dx} = \dfrac{x}{x^3+x^2+x+1}$. *Ans.* $u = \frac{1}{2}\log\dfrac{(x^2+1)^{\frac{1}{2}}}{x+1} + \frac{1}{2}\tan^{-1}x$.

11. $\dfrac{du}{dx} = \dfrac{x^2}{(x^2-1)(x^2+2)}$. *Ans.* $u = \frac{1}{6}\log\dfrac{1-x}{1+x} + \dfrac{\sqrt{2}}{3}\tan^{-1}\dfrac{x\sqrt{2}}{2}$.

12. $\dfrac{du}{dx} = \dfrac{x^2}{(x-1)^2(x^2+1)}$. *Ans.* $u = \frac{1}{4}\log\dfrac{(x-1)^2}{x^2+1} - \dfrac{1}{2(x-1)}$.

13. $\dfrac{du}{dx} = \dfrac{1}{1+x^3}$. *Ans.* $u = \dfrac{1}{3}\log\dfrac{x+1}{(x^2-x+1)^{\frac{1}{2}}} + \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x-2}{x\sqrt{3}}$.

14. $\dfrac{du}{dx} = \dfrac{1}{1+x^4}$. *Ans.* $u = \dfrac{\sqrt{2}}{8}\log\dfrac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \dfrac{\sqrt{2}}{4}\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$.

15. $\dfrac{du}{dx} = \dfrac{1}{1-x^4}$. *Ans.* $u = \log\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}} + \frac{1}{2}\tan^{-1}x$.

16. $\dfrac{du}{dx} = \dfrac{1}{x^4(a+bx^3)}$. *Ans.* $u = -\dfrac{1}{3ax^3} + \dfrac{b}{3a^2}\log\dfrac{a+bx^3}{x^3}$.

17. $\dfrac{du}{dx} = \dfrac{2x+3}{x^2+4x+13}$. *Ans.* $u = \log(x^2+4x+13) - \dfrac{1}{3}\tan^{-1}\dfrac{x+2}{3}$.

18. $\dfrac{du}{dx} = \dfrac{5x-2}{x^2+6x+13}$. *Ans.* $u = \dfrac{5}{2}\log(x^2+6x+13) - \dfrac{17}{2}\tan^{-1}\dfrac{x+3}{2}$.

19. $\dfrac{du}{dx} = \dfrac{4x^3-17x^2+9x+10}{x^2-5x+6}$. *Ans.* $u = 2x^2+3x+\log\left(\dfrac{x-2}{x-3}\right)^8$.

INTEGRATION OF IRRATIONAL DIFFERENTIALS.

96. Among the elementary integrals (Art. 83) there are several with irrational differential coefficients, and others may sometimes be reduced to such forms as to admit of integration by means of these, or to rational forms which can be integrated by the principles developed for the integration of rational fractions.

1. Let it be required to integrate $du = \dfrac{(x^{\frac{1}{2}} - x^{\frac{2}{3}})\,dx}{1 + x^{\frac{1}{3}}}$.

The least common denominator of the fractional indices is 6; therefore if we assume $x = z^6$, or $x^{\frac{1}{6}} = z$; then $x^{\frac{1}{2}} = z^3$, $x^{\frac{1}{3}} = z^2$, $x^{\frac{2}{3}} = z^4$, $dx = 6z^5\,dz$; $\therefore du = \dfrac{z^3 - z^4}{1 + z^2} \cdot 6z^5\,dz = -6\dfrac{z^9 - z^8}{z^2+1}dz$, or by division,

$$du = -6\left\{z^7 - z^6 - z^5 + z^4 + z^3 - z^2 - z + 1 + \frac{z}{z^2+1} - \frac{1}{z^2+1}\right\}dz;$$

$$\therefore u = -6\left\{\frac{z^8}{8} - \frac{z^7}{7} - \frac{z^6}{6} + \frac{z^5}{5} + \frac{z^4}{4} - \frac{z^3}{3} - \frac{z^2}{2} + z\right\} - 3\log(z^2+1) + 6\tan^{-1} z$$

$$= -\frac{3}{4}x^{\frac{4}{3}} + \frac{6}{7}x^{\frac{7}{6}} + x - \frac{6}{5}x^{\frac{5}{6}} - \frac{3}{2}x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}}$$

$$- 3\log(1 + x^{\frac{1}{3}}) + 6\tan^{-1} x^{\frac{1}{6}}.$$

2. To integrate $du = \dfrac{dx}{(a + bx)^{\frac{1}{2}}}$.

Let $a + bx = z^2$, then $(a + bx)^{\frac{1}{2}} = z$, $x = \dfrac{z^2 - a}{b}$ and $dx = \dfrac{2z\,dz}{b}$;

$$\therefore u = \int \frac{2z\,dz}{bz} = \frac{2}{b}\int dz = \frac{2z}{b} = \frac{2(a + bx)^{\frac{1}{2}}}{b}.$$

3. To integrate $du = \dfrac{dx}{(a + bx + cx^2)^{\frac{1}{2}}}$.

Let $(a+bx+cx^2)^{\frac{1}{2}} = c^{\frac{1}{2}}(z + x)$; then by squaring both sides, we get

$$a + bx + cx^2 = cz^2 + 2czx + cx^2; \;\therefore\; x = \frac{cz^2 - a}{b - 2cz},$$

$$dx = \frac{-2c(a - bz + cz^2)\,dz}{(b - 2cz)^2}, \text{ and } z + x = \frac{-(a - bz + cz^2)}{b - 2cz};$$

hence $$du = \frac{dx}{(a + bx + cx^2)^{\frac{1}{2}}} = \frac{dx}{c^{\frac{1}{2}}(z + x)} = \frac{2c^{\frac{1}{2}}\,dz}{b - 2cz};$$

therefore $$u = \frac{1}{c^{\frac{1}{2}}}\int \frac{2c\,dz}{b - 2cz} = -\frac{1}{c^{\frac{1}{2}}}\int \frac{2c\,dz}{2cz - b} = -\frac{\log(2cz - b)}{c^{\frac{1}{2}}}.$$

But $$z = \frac{(a + bx + cx^2)^{\frac{1}{2}}}{c^{\frac{1}{2}}} - x;$$ therefore we have finally

$$\int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}} = -\frac{1}{c^{\frac{1}{2}}} \log \{2c^{\frac{1}{2}}(a+bx+cx^2)^{\frac{1}{2}} - 2cx - b\}.$$

If $b = 0$, then $\displaystyle\int \frac{dx}{(a+cx^2)^{\frac{1}{2}}} = -\frac{1}{c^{\frac{1}{2}}} \log \{(a+cx^2)^{\frac{1}{2}} - c^{\frac{1}{2}}x\} - \log 2.$

If $c = 1$, then $\displaystyle\int \frac{dx}{(a+bx+x^2)^{\frac{1}{2}}} = -\log \{2(a+bx+x^2)^{\frac{1}{2}} - 2x - b\}.$

4. To integrate $du = \dfrac{dx}{(a+bx-cx^2)^{\frac{1}{2}}}$.

Let α and β denote the two roots of the equation $x^2 - \frac{b}{c}x - \frac{a}{c} = 0$;

then $x^2 - \frac{b}{c}x - \frac{a}{c} = (x-\alpha)(x-\beta)$, or $a + bx - cx^2 = -c(x-\alpha)(x-\beta)$.

Assume $a + bx - cx^2$, or $-c(x-\alpha)(x-\beta) = c^2(x-\alpha)^2 z^2$; then we have $-(x-\beta) = c(x-\alpha)z^2$, and hence $x = \dfrac{c\alpha z^2 + \beta}{cz^2+1}$;

$\therefore dx = \dfrac{2c(\alpha-\beta)z\,dz}{(cz^2+1)^2}$, and $(a+bx-cx^2)^{\frac{1}{2}} = -\dfrac{c(\alpha-\beta)z}{cz^2+1}$.

Hence $u = -\displaystyle\int \frac{2\,dz}{cz^2+1} = -\frac{2}{c^{\frac{1}{2}}}\tan^{-1} c^{\frac{1}{2}} z$, by (21) Art. 85; but from above we have $z^2 = -\dfrac{x-\beta}{c(x-\alpha)}$, and therefore $c^{\frac{1}{2}} z = \left\{-\dfrac{x-\beta}{x-\alpha}\right\}^{\frac{1}{2}}$;

hence, finally, $\displaystyle\int \frac{dx}{(a+bx-cx^2)^{\frac{1}{2}}} = C - \frac{2}{c^{\frac{1}{2}}}\tan^{-1}\left\{\frac{\beta-x}{x-\alpha}\right\}^{\frac{1}{2}}.$

If $a = 1$, $b = 3$, and $c = 4$; then the equation $x^2 - \frac{3}{4}x - \frac{1}{4} = 0$, gives $x = 1$, or $x = -\frac{1}{4}$; therefore $\alpha = 1$, $\beta = -\frac{1}{4}$,

and $$\int \frac{dx}{(1+3x-4x^2)^{\frac{1}{2}}} = C - \tan^{-1}\frac{(1+4x)^{\frac{1}{2}}}{2(1-x)^{\frac{1}{2}}}.$$

5. Integrate $du = \dfrac{X\,dx}{(a+bx+cx^2)^{\frac{1}{2}}}$, where X is a rational function of x.

Let $(a+bx+cx^2)^{\frac{1}{2}} = c^{\frac{1}{2}}(z-x)$; then, by squaring both sides, we get $a + bx + cx^2 = cz^2 - 2czx + cx^2$;

hence $x = \dfrac{cz^2-a}{2cz+b}$, $dx = \dfrac{2c(a+bz+cz^2)\,dz}{(2cz+b)^2}$,

$$(a+bx+cx^2)^{\frac{1}{2}} = c^{\frac{1}{2}}(z-x) = \frac{c^{\frac{1}{2}}(a+bz+cz^2)}{2cz+b}.$$

The substitution of these values in the proposed differential will transform it into another of the form Z dz, where Z is a rational function of

z, and then $\int Z\, dz$ may be found by the principles already developed. If c be negative, then $c^{\frac{1}{2}}$ would be impossible, and the method adopted in the last example must be employed.

If $X = x$, then we have $\quad d\,(a + bx + cx^2)^{\frac{1}{2}}$

$$= \frac{\frac{1}{2} b\, dx + cx\, dx}{(a + bx + cx^2)^{\frac{1}{2}}} = \frac{cx\, dx}{(a + bx + cx^2)^{\frac{1}{2}}} + \frac{\frac{1}{2} b\, dx}{(a + bx + cx^2)^{\frac{1}{2}}};$$

$$\therefore \frac{x\, dx}{(a + bx + cx^2)^{\frac{1}{2}}} = \frac{1}{c}\, d\,(a + bx + cx^2)^{\frac{1}{2}} - \frac{b}{2c} \frac{dx}{(a + bx + cx^2)^{\frac{1}{2}}};$$

$$\therefore \int \frac{x\, dx}{(a + bx + cx^2)^{\frac{1}{2}}} = \frac{(a + bx + cx^2)^{\frac{1}{2}}}{c} - \frac{b}{2c} \int \frac{dx}{(a + bx + cx^2)^{\frac{1}{2}}},$$

and the integral may be found by Examples (3) or (4), according as c is positive or negative.

6. Let $du = \dfrac{dx}{(1 + x)\sqrt{(1 - x^2)}}$.

Assume $1 - x^2 = z^2 (1 - x)^2$; then $1 + x = z^2 (1 - x)$;

and $\quad x = \dfrac{z^2 - 1}{z^2 + 1}$; therefore $1 + x = \dfrac{2z^2}{z^2 + 1}$, $1 - x = \dfrac{2}{z^2 + 1}$;

$$dx = \frac{4z\, dz}{(z^2 + 1)^2}, \text{ and consequently } du = \frac{dz}{z^2};$$

therefore $u = C - \dfrac{1}{z} = C - \left(\dfrac{1 - x}{1 + x}\right)^{\frac{1}{2}} = C - \dfrac{1 - x}{\sqrt{(1 - x^2)}}$.

Examples for Practice.

1. $\displaystyle\int \frac{dx}{x\,(x^2 + 2x - 1)^{\frac{1}{2}}} = \sin^{-1} \frac{x - 1}{x\sqrt{2}}$.

2. $\displaystyle\int \frac{dx}{(1 + 2x + 3x^2)^{\frac{3}{2}}} = \frac{3x + 1}{2\,(1 + 2x + 3x^2)^{\frac{1}{2}}}$.

3. $\displaystyle\int \frac{x\, dx}{(1 + 2x + 3x^2)^{\frac{3}{2}}} = -\frac{x + 1}{2\,(1 + 2x + 3x^2)^{\frac{1}{2}}}$.

4. $\displaystyle\int \frac{dx}{(1 + x^2)\sqrt{(1 - x^2)}} = \frac{1}{2^{\frac{1}{2}}} \tan^{-1} \frac{2^{\frac{1}{2}} x}{\sqrt{(1 - x^2)}}$.

5. $\displaystyle\int \frac{dx}{(1 + x^2)\sqrt{(x^2 - 1)}} = \frac{1}{2^{\frac{1}{2}}} \log \frac{(x^2 - 1)^{\frac{1}{2}} + 2^{\frac{1}{2}} x}{\sqrt{(x^2 + 1)}}$.

6. $\displaystyle\int \frac{dx}{(1 + x)\sqrt{(1 + x + x^2)}} = \log \frac{1 - x - 2\sqrt{(1 + x + x^2)}}{1 + x}$.

7. $\displaystyle\int \frac{dx}{(1 + x)\sqrt{(1 + x - x^2)}} = \tan^{-1} \frac{1 + 3x}{2\sqrt{(1 + x - x^2)}}$.

8. $\displaystyle\int\frac{dx}{(1+x^m)(2x^m-1)^{\frac{1}{2m}}} = \int\frac{z^{2m-2}\,dz}{1-z^{2m}}$, where $z = \dfrac{(2x^m-1)^{\frac{1}{2m}}}{x}$.

9. $\displaystyle\int\frac{x^{m-1}\,dx}{(1-x^m)(2x^m-1)^{\frac{1}{2m}}} = 2\int\frac{z^{2m-2}\,dz}{1-z^{2m}}$, where $z=(2x^m-1)^{\frac{1}{2m}}$.

10. $$\int\frac{dx}{(a+bx)\sqrt{x}} = \frac{2}{\sqrt{(ab)}}\tan^{-1}\left(\frac{bx}{a}\right)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{(-ab)}}\log\frac{a-bx+2\sqrt{x}\,.\,\sqrt{(-ab)}}{a+bx}.$$

INTEGRATION OF BINOMIAL DIFFERENTIALS.

97. Differential expressions of this kind may be included under the form
$$du = (a+bx^n)^{\frac{p}{q}}x^{m-1}\,dx \;.\;.\;.\;.\;.\;. \quad (1),$$
where m, n, p, q are whole numbers, positive or negative, except n, which is only positive.

1. Let $a+bx^n = z^q$, then $(a+bx^n)^{\frac{p}{q}} = z^p$ and $x = \left(\dfrac{z^q-a}{b}\right)^{\frac{1}{n}}$;

$$\therefore\; x^m = \left(\frac{z^q-a}{b}\right)^{\frac{m}{n}}, \text{ and } x^{m-1}\,dx = \frac{q}{nb}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}z^{q-1}\,dz;$$

hence $$du = z^p x^{m-1}\,dx = \frac{q}{nb}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}z^{p+q-1}\,dz \;.\;.\;.\;.\;. \quad (2),$$

which will always be rational and integrable if $\dfrac{m}{n}$ be a whole number.

2. Let $a+bx^n = z^q x^n$, then $(a+bx^n)^{\frac{p}{q}} = z^p x^{\frac{pn}{q}}$, $x = a^{\frac{1}{n}}(z^q-b)^{-\frac{1}{n}}$;

$$\therefore\; x^m = a^{\frac{m}{n}}(z^q-b)^{-\frac{m}{n}}, \text{ and } x^{m-1}\,dx = -\frac{q a^{\frac{m}{n}}}{n}(z^q-b)^{-\frac{m}{n}-1}z^{q-1}\,dz;$$

hence $$du = z^p x^{\frac{pn}{q}} x^{m-1}\,dx = z^p a^{\frac{p}{q}}(z^q-b)^{-\frac{p}{q}}x^{m-1}\,dx$$
$$= -a^{\frac{p}{q}}z^p(z^q-b)^{-\frac{p}{q}} \times \frac{q a^{\frac{m}{n}}}{n}(z^q-b)^{-\frac{m}{n}-1}z^{q-1}\,dz$$
$$= -\frac{q}{n}a^{\frac{m}{n}+\frac{p}{q}}(z^q-b)^{-\left(\frac{m}{n}+\frac{p}{q}+1\right)}z^{p+q-1}\,dz \;.\;.\;.\;.\;.\;. \quad (3),$$

which will be rational and integrable if $\dfrac{m}{n}+\dfrac{p}{q}$ be a whole number.

These are the only two cases in which the differential can be rendered rational; therefore a binomial differential of the form proposed is always integrable, not only when $\dfrac{m}{n}$ is a whole number, but also when $\dfrac{m}{n}+\dfrac{p}{q}$ is a whole number.

First criterion. If $\dfrac{m}{n}$ = a whole number, assume $a+bx^n = z^q$.

Second criterion. If $\frac{m}{n}+\frac{p}{q}$ = a whole number, assume $a+bx^n = z^q x^n$.

Examples.

1. Let $du = \dfrac{x\,dx}{\sqrt{(a+x)}} = (a+x)^{-\frac{1}{2}} x\,dx$.

Here $\frac{m}{n} = \frac{2}{1} = 2$, and the first criterion is satisfied.

Let then $a+x = z^2$; then $x^2 = (z^2-a)^2 = z^4 - 2az^2 + a^2$;

$$\therefore\ x\,dx = 2z^3dz - 2az\,dz = 2(z^3 - az)\,dz;$$

$$\therefore \int \frac{x\,dx}{\sqrt{(a+x)}} = 2\int \frac{(z^3-az)\,dz}{z} = 2\int z^2dz - 2a\int dz$$

$$= \frac{2}{3}z^3 - 2az = (a+x)^{\frac{1}{2}}\left\{\frac{2}{3}(a+x) - 2a\right\}$$

$$= \frac{2}{3}(x-2a)(a+x)^{\frac{1}{2}}.$$

2. Let $du = \dfrac{\sqrt{(a^2-x^2)}\,.\,dx}{x^6} = (a^2-x^2)^{\frac{1}{2}}x^{-6}dx$.

Here $\frac{m}{n} = -\frac{5}{2}$, and $\frac{m}{n}+\frac{p}{q} = -\frac{5}{2}+\frac{1}{2} = -2$, and the second criterion is satisfied; hence we may assume

$a^2 - x^2 = z^2x^2$; then $\sqrt{(a^2-x^2)} = zx$, and $x^2 = a^2(z^2+1)^{-1}$;

hence $x^4 = a^4(z^2+1)^{-2}$, or $x^{-4} = a^{-4}(z^2+1)^2$;

and differentiating, $-x^{-5}dx = a^{-4}(z^2+1)z\,dz$;

hence $\dfrac{dx}{x^5} = -\dfrac{(z^2+1)z\,dz}{a^4}$, and since $\dfrac{\sqrt{(a^2-x^2)}}{x} = z$,

we have $du = \dfrac{\sqrt{(a^2-x^2)}\,dx}{x^6} = -\dfrac{1}{a^4}z^2(z^2+1)\,dz$;

$$\therefore \int \frac{\sqrt{(a^2-x^2)}\,dx}{x^6} = -\frac{1}{a^4}\left\{\int z^4dz + \int z^2dz\right\} = -\left(\frac{z^5}{5a^4}+\frac{z^3}{3a^4}\right)$$

$$= -\frac{(a^2-x^2)^{\frac{3}{2}}}{15a^4x^3}\left\{\frac{3(a^2-x^2)}{x^2}+5\right\} = -\frac{3a^2+2x^2}{15a^4x^5}(a^2-x^2)^{\frac{3}{2}}.$$

3. Let $\dfrac{du}{dx} = \dfrac{1}{x^4(1+x^2)^{\frac{1}{2}}}$. *Ans.* $u = \dfrac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}$.

4. Let $\dfrac{du}{dx} = x^3(1+x^2)^{\frac{1}{2}}$. *Ans.* $u = \dfrac{x^2-2}{15}(1+x^2)^{\frac{3}{2}}$.

5. Let $\dfrac{du}{dx} = x^3(a^2-x^2)^{-\frac{1}{2}}$. *Ans.* $u = -\dfrac{2a^2+x^2}{3}(a^2-x^2)^{\frac{1}{2}}$.

6. Let $\dfrac{du}{dx} = \dfrac{1}{x^4\sqrt{(a^2-x^2)}}$. *Ans.* $u = -\dfrac{a^2+2x^2}{3a^4x^3}(a^2-x^2)^{\frac{1}{2}}$.

INTEGRATION BY SUCCESSIVE REDUCTION.

98. When neither of these criteria is satisfied, binomial differentials may still be modified so as to assume a simpler form, and to admit of integration by other means. If we can make $\int x^m (a + b x^n)^p \, dx$ depend on another integral of the same form, but with smaller values of m or p, it is evident that by successive repetitions of the process we shall finally arrive at an integral which can be determined by an elementary form. This is called the *method of reduction:* it is applicable to a great number of functions, and is very convenient in practice.

I. Let it be required to integrate $du = \dfrac{x^n \, dx}{(a^2 - x^2)^{\frac{1}{2}}}$.

$$\int \frac{x^n \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = \int x^{n-1} \cdot \frac{x \, dx}{(a^2 - x^2)^{\frac{1}{2}}}$$

$$= - x^{n-1} (a^2 - x^2)^{\frac{1}{2}} + (n - 1) \int x^{n-2} (a^2 - x^2)^{\frac{1}{2}} dx$$

$$= - x^{n-1} (a^2 - x^2)^{\frac{1}{2}} + (n - 1) \int \frac{x^{n-2} (a^2 - x^2) \, dx}{(a^2 - x^2)^{\frac{1}{2}}}$$

$$= - x^{n-1} (a^2 - x^2)^{\frac{1}{2}} + (n - 1) a^2 \int \frac{x^{n-2} \, dx}{(a^2 - x^2)^{\frac{1}{2}}} - (n-1) \int \frac{x^n \, dx}{(a^2 - x^2)^{\frac{1}{2}}}.$$

Transposing the last term of the second member, and dividing by n,

$$\int \frac{x^n \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = - \frac{x^{n-1} (a^2 - x^2)^{\frac{1}{2}}}{n} + \frac{(n - 1) a^2}{n} \int \frac{x^{n-2} \, dx}{(a^2 - x^2)^{\frac{1}{2}}} \quad . \; . \; (A).$$

By the successive application of this formula of reduction, the proposed integral will be reduced, according as n is *even* or *odd*, to

$$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \sin^{-1} \frac{x}{a}, \text{ or } \int \frac{x \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = - (a^2 - x^2)^{\frac{1}{2}}.$$

EXAMPLES.

1. $$\int \frac{x^2 \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = - \frac{x (a^2 - x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

2. $$\int \frac{x^3 \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = - \frac{x^2 (a^2 - x^2)^{\frac{1}{2}}}{3} - \frac{2 a^2}{3} (a^2 - x^2)^{\frac{1}{2}}.$$

3. $$\int \frac{x^4 \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = - \frac{x^3 (a^2 - x^2)^{\frac{1}{2}}}{4} + \frac{3 a^2}{4} \int \frac{x^2 \, dx}{(a^2 - x^2)^{\frac{1}{2}}}; \text{ but,}$$

$$\int \frac{x^2 \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = - \frac{x (a^2 - x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}; \text{ therefore}$$

$$\int \frac{x^4 \, dx}{(a^2 - x^2)^{\frac{1}{2}}} = - \frac{x^3 (a^2 - x^2)^{\frac{1}{2}}}{4} - \frac{3 a^2 x}{4 \, . \, 2} (a^2 - x^2)^{\frac{1}{2}} + \frac{3 a^4}{4 \, . \, 2} \sin^{-1} \frac{x}{a}.$$

$$= -\left(\frac{x^3}{4} + \frac{3a^2x}{4 \cdot 2}\right)(a^2 - x^2)^{\frac{1}{2}} + \frac{3 \cdot 1\, a^4}{4 \cdot 2}\sin^{-1}\frac{x}{a}.$$

4. $$\int \frac{x^5\,dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\left(\frac{x^4}{5} + \frac{4a^2x^2}{5 \cdot 3} + \frac{4 \cdot 2a^4}{5 \cdot 3 \cdot 1}\right)(a^2 - x^2)^{\frac{1}{2}}.$$

5. $$\int \frac{x^6\,dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\left(\frac{x^5}{6} + \frac{5a^2x^3}{6 \cdot 4} + \frac{5 \cdot 3a^4x}{6 \cdot 4 \cdot 2}\right)(a^2 - x^2)^{\frac{1}{2}}$$
$$+ \frac{5 \cdot 3 \cdot 1\, a^6}{6 \cdot 4 \cdot 2}\sin^{-1}\frac{x}{a}.$$

These integrals may be continued at pleasure, and the method employed in the investigation of the formula of reduction (A) should be followed by the student in each of the preceding examples. *The method, and not the formula,* should be recollected.

II. Let it be required to integrate $du = \dfrac{x^n\,dx}{(2ax - x^2)^{\frac{1}{2}}}$.

Here $$u = \int x^{n-1} \cdot \frac{x\,dx}{(2ax - x^2)^{\frac{1}{2}}} = \int x^{n-1}\frac{\{a - (a - x)\}\,dx}{(2ax - x^2)^{\frac{1}{2}}}$$
$$= a\int \frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}} - \int x^{n-1} \cdot \frac{(a - x)\,dx}{(2ax - x^2)^{\frac{1}{2}}}$$
$$= a\int \frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}} - x^{n-1}(2ax - x^2)^{\frac{1}{2}} + (n-1)\int x^{n-2}(2ax - x^2)^{\frac{1}{2}}\,dx.$$

But $$\int x^{n-2}(2ax - x^2)^{\frac{1}{2}}dx = \int x^{n-2}\frac{(2ax - x^2)dx}{(2ax - x^2)^{\frac{1}{2}}} = 2a\int \frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}} - u,$$

and substituting in the preceding equality, we get

$$u = a\int \frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}} - x^{n-1}(2ax - x^2)^{\frac{1}{2}} + 2a(n-1)\int \frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}}$$
$$- (n - 1)\,u;$$

therefore by transposition, and collecting like terms, we obtain

$$nu = -x^{n-1}(2ax - x^2)^{\frac{1}{2}} + (2n - 1)\,a\int \frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}};$$

or $$\int \frac{x^n\,dx}{(2ax - x^2)^{\frac{1}{2}}} = -\frac{x^{n-1}(2ax - x^2)^{\frac{1}{2}}}{n} + \frac{(2n - 1)a}{n}\int \frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}} \cdot \text{(B)},$$

which reduces the integral to $\displaystyle\int \frac{dx}{(2ax - x^2)^{\frac{1}{2}}} = \text{vers}^{-1}\frac{x}{a}$.

EXAMPLES.

1. $$\int \frac{x\,dx}{(2ax - x^2)^{\frac{1}{2}}} = -(2ax - x^2)^{\frac{1}{2}} + a\,\text{vers}^{-1}\frac{x}{a}.$$

2. $$\int \frac{x^2\,dx}{(2ax-x^2)^{\frac{1}{2}}} = -\frac{x(2ax-x^2)^{\frac{1}{2}}}{2} - \frac{3a}{2}\int \frac{x\,dx}{(2ax-x^2)^{\frac{1}{2}}}$$
$$= -\left(\frac{x}{2}+\frac{3a}{2}\right)(2ax-x^2)^{\frac{1}{2}} + \frac{3a^2}{2}\text{vers}^{-1}\frac{x}{a}.$$

3. $$\int \frac{x^3\,dx}{(2ax-x^2)^{\frac{1}{2}}} = -\left(\frac{x^2}{3}+\frac{5ax}{3.2}+\frac{5.3a^2}{3.2}\right)(2ax-x^2)^{\frac{1}{2}}$$
$$+\frac{5.3a^3}{3.2}\text{vers}^{-1}\frac{x}{a}.$$

4. $$\int \frac{x^4\,dx}{(2ax-x^2)^{\frac{1}{2}}} = -\left(\frac{x^3}{4}+\frac{7ax^2}{4.3}+\frac{7.5a^2x}{4.3.2}+\frac{7.5.3a^3}{4.3.2}\right)(2ax-x^2)^{\frac{1}{2}}$$
$$+\frac{7.5.3a^4}{4.3.2}\text{vers}^{-1}\frac{x}{a}.$$

III. Let it be required to integrate $du = \dfrac{x^m\,dx}{(a^2+x^2)^n}$.

Here $$\int \frac{x^m\,dx}{(a^2+x^2)^n} = \int x^{m-1}\frac{x\,dx}{(a^2+x^2)^n}$$
$$= -\frac{1}{2n-2}\cdot\frac{x^{m-1}}{(a^2+x^2)^{n-1}} + \frac{m-1}{2n-2}\int\frac{x^{m-2}\,dx}{(a^2+x^2)^{n-1}} \quad \ldots \quad \text{(C)}.$$

Examples.

1. Let $m = 3$, $n = 2$; then
$$\int\frac{x^3\,dx}{(a^2+x^2)^2} = -\frac{1}{2}\cdot\frac{x^2}{(a^2+x^2)} + \int\frac{x\,dx}{a^2+x^2}$$
$$= -\frac{x^2}{2(a^2+x^2)} + \log(a^2+x^2)^{\frac{1}{2}}.$$

2. Let $m = 2$, $n = 3$; then
$$\int\frac{x^2\,dx}{(a^2+x^2)^3} = -\frac{x}{4(a^2+x^2)^2} + \frac{1}{4}\int\frac{dx}{(a^2+x^2)^2}$$
$$= -\frac{x}{4(a^2+x^2)^2} + \frac{x}{4.2a^2(a^2+x^2)} + \frac{1}{4.2}\cdot\frac{1}{a^3}\tan^{-1}\frac{x}{a} \quad \text{(Art. 95).}$$

3. Let $m = 4$, $n = 2$;

then $$\int\frac{x^4\,dx}{(a^2+x^2)^2} = -\frac{x^3}{2(a^2+x^2)} + \frac{3}{2}\int\frac{x^2\,dx}{a^2+x^2}$$
$$= -\frac{x^3}{2(a^2+x^2)} + \frac{3}{2}\int\frac{(a^2+x^2-a^2)\,dx}{a^2+x^2}$$
$$= -\frac{x^3}{2(a^2+x^2)} + \frac{3}{2}x - \frac{3}{2}a\tan^{-1}\frac{x}{a}.$$

IV. Integrate $du = (a^2-x^2)^{\frac{n}{2}}dx$, where n is an odd number.

Here $u = \int(a^2-x^2)^{\frac{n}{2}}\,.\,dx = x(a^2-x^2)^{\frac{n}{2}} + \frac{n}{2}\int 2x^2(a^2-x^2)^{\frac{n}{2}-1}dx$

$$= x(a^2 - x^2)^{\frac{n}{2}} + n\int \{a^2 - (a^2 - x^2)\}\,(a^2 - x^2)^{\frac{n-2}{2}}\,dx$$

$$= x(a^2 - x^2)^{\frac{n}{2}} + a^2 n\int (a^2 - x^2)^{\frac{n-2}{2}}\,dx - n\int (a^2 - x^2)^{\frac{n}{2}}\,dx;$$

$\therefore (n+1)\,u = x(a^2-x^2)^{\frac{n}{2}} + n\,a^2\int (a^2 - x^2)^{\frac{n-2}{2}}\,dx$, and consequently

$$\int (a^2 - x^2)^{\frac{n}{2}}\,dx = \frac{x(a^2 - x^2)^{\frac{n}{2}}}{n+1} + \frac{n\,a^2}{n+1}\int (a^2 - x^2)^{\frac{n-2}{2}}\,dx \;.\;.\;.\; \text{(D)}.$$

EXAMPLES.

1. $\int (a^2 - x^2)^{\frac{1}{2}}dx = \dfrac{1}{2}\,x(a^2 - x^2)^{\frac{1}{2}} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}.$

2. $\int (a^2 - x^2)^{\frac{3}{2}}\,dx = \dfrac{x(a^2 - x^2)^{\frac{3}{2}}}{4} + \dfrac{3\,a^2}{4}\int (a^2 - x^2)^{\frac{1}{2}}\,dx$

$$= \frac{x(a^2 - x^2)^{\frac{3}{2}}}{4} + \frac{3\,a^2\,x}{4\,.\,2}\,(a^2 - x^2)^{\frac{1}{2}} + \frac{3\,a^4}{4\,.\,2}\sin^{-1}\frac{x}{a}.$$

3. $\int (a^2 - x^2)^{\frac{5}{2}}\,dx = \dfrac{x(a^2 - x^2)^{\frac{5}{2}}}{6} + \dfrac{5\,a^2}{6}\int (a^2 - x^2)^{\frac{3}{2}}\,dx$

$$= \frac{x(a^2-x^2)^{\frac{5}{2}}}{6} + \frac{5a^2x}{6.4}(a^2-x^2)^{\frac{3}{2}} + \frac{5\,.3a^4x}{6.4.2}(a^2-x^2)^{\frac{1}{2}} + \frac{5\,.3a^6}{6.4.2}\sin^{-1}\frac{x}{a}.$$

V. Integrate $du = \dfrac{dx}{x^n\,(x^2 - a^2)^{\frac{1}{2}}}$.

Here $u = \displaystyle\int \frac{1}{x^{n+1}}\cdot\frac{x\,dx}{(x^2-a^2)^{\frac{1}{2}}} = \frac{(x^2-a^2)^{\frac{1}{2}}}{x^{n+1}} + (n+1)\int \frac{(x^2-a^2)^{\frac{1}{2}}\,dx}{x^{n+2}}$

$$= \frac{(x^2 - a^2)^{\frac{1}{2}}}{x^{n+1}} + (n+1)\int \frac{(x^2 - a^2)\,dx}{x^{n+2}\,(x^2 - a^2)^{\frac{1}{2}}}$$

$$= \frac{(x^2 - a^2)^{\frac{1}{2}}}{x^{n+1}} + (n+1)\,u - (n+1)\,a^2\int \frac{dx}{x^{n+2}\,(x^2 - a^2)^{\frac{1}{2}}}.$$

Change n into $n - 2$, or $n + 2$ into n, then will

$$\int \frac{dx}{x^{n-2}(x^2-a^2)^{\frac{1}{2}}} = \frac{(x^2-a^2)^{\frac{1}{2}}}{x^{n-1}} + (n-1)\int \frac{dx}{x^{n-2}(x^2 - a^2)^{\frac{1}{2}}} - (n-1)\,a^2\,u;$$

whence, by transposition, collecting, and dividing by $(n - 1)\,a^2$,

$$\int \frac{dx}{x^{n}(x^2 - a^2)^{\frac{1}{2}}} = \frac{(x^2 - a^2)^{\frac{1}{2}}}{(n-1)\,a^2 x^{n-1}} + \frac{n-2}{(n-1)\,a^2}\cdot\int \frac{dx}{x^{n-2}(x^2 - a^2)^{\frac{1}{2}}}\,\text{(E)}.$$

This formula makes the final integral depend on

$$\int \frac{dx}{x\,(x^2 - a^2)^{\frac{1}{2}}} = \frac{1}{a}\sec^{-1}\frac{x}{a},\ \text{or}\int \frac{dx}{x^2\,(x^2 - a^2)^{\frac{1}{2}}} = \frac{(x^2 - a^2)^{\frac{1}{2}}}{a^2\,x}.$$

EXAMPLES.

1. $\int \frac{dx}{x^3(x^2-a^2)^{\frac{1}{2}}} = \frac{(x^2-a^2)^{\frac{1}{2}}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{x}{a}.$

2. $\int \frac{dx}{x^4(x^2-a^2)^{\frac{1}{2}}} = \frac{(x^2-a^2)^{\frac{1}{2}}}{3a^2x^3} + \frac{2}{3a^2}.\frac{(x^2-a^2)^{\frac{1}{2}}}{a^2x}$

3. $\int \frac{dx}{x^5(x^2-a^2)^{\frac{1}{2}}} = \frac{(x^2-a^2)^{\frac{1}{2}}}{4a^2x^4} + \frac{3(x^2-a^2)^{\frac{1}{2}}}{4.2a^4x^2} + \frac{3.1}{4.2a^5}\sec^{-1}\frac{x}{a}$

$$= \left(\frac{1}{4a^2x^4} + \frac{3.1}{4.2a^4x^2}\right)(x^2-a^2)^{\frac{1}{2}} + \frac{3.1}{4.2a^5}\sec^{-1}\frac{x}{a}.$$

VI. To integrate $du = x^m(a+bx^n)^p dx$, by reducing m, where p may be either positive or negative.

$$\int x^m(a+bx^n)^p dx = \int x^{m-n+1}(a+bx^n)^p x^{n-1} dx$$

$$= \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx \quad . \quad . \quad (F).$$

But $\int x^{m-n}(a+bx^n)^{p+1}dx = \int x^{m-n}(a+bx^n)(a+bx^n)^p dx$

$$= a\int x^{m-n}(a+bx^n)^p dx + bu.$$

Substituting in (F), gives

$$u = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)} - \frac{a(m-n+1)}{nb(p+1)}\int x^{m-n}(a+bx^n)^p dx$$

$$- \frac{m-n+1}{n(p+1)}u,$$

which, after transposing, and dividing by the resulting coefficient of u, gives $\int x^m(a+bx^n)^p dx$

$$= \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)} - \frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p dx \quad . \quad (G).$$

VII. To integrate $du = \frac{(a+bx^n)^p dx}{x^m}$, by reducing m, where p may be either positive or negative.

By freeing the formula (G) from fractions, transposing, and dividing,

$$\int x^{m-n}(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{a(m-n+1)}$$

$$- \frac{b(np+m+1)}{a(m-n+1)}\int x^m(a+bx^n)^p dx.$$

In this, write $-m$ instead of $m-n$, or $-m+n$ instead of m; then

$$\int \frac{(a+bx^n)^p dx}{x^m}$$

$$= \frac{x^{-m+1}(a+bx^n)^{p+1}}{a(1-m)} - \frac{b(np-m+n+1)}{a(1-m)}\int \frac{(a+bx^n)^p dx}{x^{m-n}} \quad . \quad . \quad . \quad (H).$$

VIII. To integrate $du = x^m (a + b x^n)^p dx$, by reducing p, where m may be either positive or negative.

Since $(a + b x^n)^p = a (a + b x^n)^{p-1} + b x^n (a + b x^n)^{p-1}$; multiply by $x^m dx$ and integrate; then

$$\int x^m (a+b x^n)^p dx = a \int x^m (a+b x^n)^{p-1} dx + b \int x^{m+n} (a+b x^n)^{p-1} dx.$$

Now if in (G) we change p into $p - 1$, and m into $m + n$; then will

$$\int x^{m+n}(a+bx^n)^{p-1}dx = \frac{x^{m+1}(a+bx^n)^p}{b(np+m+1)} - \frac{a\,(m+1)}{b(np+m+1)}\int x^m(a+bx^n)^{p-1}dx.$$

Substituting in the previous equality, we get

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{a\,n\,p}{np+m+1}\int x^m (a+bx^n)^{p-1} dx \quad \text{(K)}.$$

IX. To integrate $du = \dfrac{x^m\,dx}{(a+b\,x^n)^p}$, by reducing p, where m may be positive or negative.

By freeing the formula (K) from fractions, transposing, and dividing,

$$\int x^m(a+bx^n)^{p-1}dx = -\frac{x^{m+1}(a+bx^n)^p}{a\,n\,p} + \frac{np+m+1}{a\,n\,p}\int x^m (a+b\,x^n)^p\,dx$$

In this write $-p$ instead of $p - 1$, or $-p + 1$ instead of p; then we have

$$\int \frac{x^m\,dx}{(a+b\,x^n)^p} = -\frac{x^{m+1}}{a\,n(1-p)\,(a+b\,x^n)^{p-1}}$$
$$+ \frac{-np+m+n+1}{a\,n\,(1-p)}\int \frac{x^m\,dx}{(a+b\,x^n)^{p-1}} \quad \ldots\ldots\ldots\ldots \text{(L)}.$$

99. Instead of pursuing the investigation of general formulas for the reduction of integrals, we shall advert to some particular cases of those we have already deduced, which are often useful.

In formula (G), let $n = 2$ and $p = -\frac{1}{2}$; then

$$\left.\begin{aligned}\int \frac{x^m\,dx}{\sqrt{(a+b\,x^2)}} &= \frac{x^{m-1}\sqrt{(a+b\,x^2)}}{m\,b} - \frac{(m-1)\,a}{m\,b}\int \frac{x^{m-2}\,dx}{\sqrt{(a+b\,x^2)}}\\ \int \frac{x^m\,dx}{\sqrt{(a-b\,x^2)}} &= -\frac{x^{m-1}\sqrt{(a-b\,x^2)}}{m\,b} + \frac{(m-1)\,a}{m\,b}\int \frac{x^{m-2}\,dx}{\sqrt{(a-b\,x^2)}}\end{aligned}\right\}\ldots(1).$$

In formula (H), let $n = 2$ and $p = -\frac{1}{2}$; then

$$\left.\begin{aligned}\int \frac{dx}{x^m\sqrt{(a+b\,x^2)}} &= -\frac{\sqrt{(a+b\,x^2)}}{(m-1)\,a\,x^{m-1}} - \frac{(m-2)\,b}{(m-1)\,a}\int \frac{dx}{x^{m-2}\sqrt{(a+b\,x^2)}}\\ \int \frac{dx}{x^m\sqrt{(a-b\,x^2)}} &= -\frac{\sqrt{(a-b\,x^2)}}{(m-1)\,a\,x^{m-1}} + \frac{(m-2)\,b}{(m-1)\,a}\int \frac{dx}{x^{m-2}\sqrt{(a-b\,x^2)}}\end{aligned}\right\}(2).$$

In the first of formulas (2), let $b = 1$, $a = -a^2$, and $m = 2n+1$; then

$$\int \frac{dx}{x^{2n+1}\sqrt{(x^2-a^2)}} = \frac{\sqrt{(x^2-a^2)}}{2\,n\,a^2\,x^{2n}} + \frac{2n-1}{2\,n\,a^2}\int \frac{dx}{x^{2n-1}\sqrt{(x^2-a^2)}}\ \ldots(3).$$

This formula is useful in finding the length of the arc of an hyperbola.

In formula (G), let $n = 1$ and $p = -\frac{1}{2}$; then

$$\int \frac{x^m\,dx}{(a+b\,x)^{\frac{1}{2}}} = \frac{2\,x^m\,(a+b\,x)^{\frac{1}{2}}}{(2\,m+1)\,b} - \frac{2\,m}{2\,m+1}\cdot\frac{a}{b}\int \frac{x^{m-1}\,dx}{(a+b\,x)^{\frac{1}{2}}}\ \ldots(4).$$

100. We have already (Art. 98) recommended the student to recollect the modes of investigating the different formulas of reduction which we have obtained, and to apply them to particular cases, instead of substituting in the general formulas the particular values of the indices and coefficients. We shall here give one or two instances of integration as patterns to be followed by the student in other cases.

1. To integrate $du = \dfrac{x^m\,dx}{\sqrt{(x^2 \pm a^2)}}$.

Commencing with the elementary forms, where $m = 0$ and $m = 1$,

$$\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log\{x + \sqrt{(x^2 \pm a^2)}\}, \int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)}} = \sqrt{(x^2 \pm a^2)}.$$

If $m = 2$, then $du = \dfrac{x^2\,dx}{\sqrt{(x^2 \pm a^2)}}$, and, integrating by parts,

$$\begin{aligned} u = \int x \,.\, \frac{x\,dx}{\sqrt{(x^2 \pm a^2)}} &= x\sqrt{(x^2 \pm a^2)} - \int dx\sqrt{(x^2 \pm a^2)} \\ &= x\sqrt{(x^2 \pm a^2)} - \int \frac{(x^2 \pm a^2)\,dx}{\sqrt{(x^2 \pm a^2)}} \\ &= x\sqrt{(x^2 \pm a^2)} - u \mp a^2 \int \frac{dx}{\sqrt{(x^2 \pm a^2)}}. \end{aligned}$$

Transposing $-u$, and dividing by 2, gives

$$u = \frac{x\sqrt{(x^2 \pm a^2)}}{2} \mp \frac{a^2}{2} \,.\, \log\{x + \sqrt{(x^2 \pm a^2)}\}.$$

If $m = 3$, then $du = \dfrac{x^3\,dx}{\sqrt{(x^2 \pm a^2)}}$, and, integrating by parts,

$$\begin{aligned} u = \int x^2 . \frac{x\,dx}{\sqrt{(x^2 \pm a^2)}} &= x^2\sqrt{(x^2 \pm a^2)} - 2\int x\,dx\sqrt{(x^2 \pm a^2)} \\ &= x^2\sqrt{(x^2 \pm a^2)} - 2\int \frac{(x^2 \pm a^2)\,x\,dx}{\sqrt{(x^2 \pm a^2)}} \\ &= x^2\sqrt{(x^2 \pm a^2)} - 2u \mp 2a^2 \int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)}}. \end{aligned}$$

Transposing $-2u$, and dividing by 3, gives

$$u = \frac{x^2\sqrt{(x^2 \pm a^2)}}{3} \mp \frac{2a^2}{3}\sqrt{(x^2 \pm a^2)} = \left(\frac{x^2}{3} \mp \frac{2a^2}{3}\right)\sqrt{(x^2 \pm a^2)}.$$

In this manner we may ascend from one integral to another, until the proposed integral be obtained.

2 To integrate $du = x^m (a^2 + x^2)^{\frac{1}{2}}\,dx$.

Let $m = 2$; then $du = x^2 (a^2 + x^2)^{\frac{1}{2}}\,dx$, and, integrating by parts,

$$\begin{aligned} u = \int x . x\,dx\,(a^2 + x^2)^{\frac{1}{2}} &= \frac{x}{3}\,(a^2 + x^2)^{\frac{3}{2}} - \frac{1}{3}\int (a^2 + x^2)^{\frac{3}{2}}\,dx \\ &= \frac{x}{3}\,(a^2 + x^2)^{\frac{3}{2}} - \frac{1}{3}\sqrt{(a^2 + x^2)}\,(a^2 + x^2)^{\frac{1}{2}}\,dx \\ &= \frac{x}{3}\,(a^2 + x^2)^{\frac{3}{2}} - \frac{a^2}{3}\int (a^2 + x^2)^{\frac{1}{2}}\,dx - \frac{u}{3}; \end{aligned}$$

$$\therefore\ u = \frac{x}{4}\,(a^2+x^2)^{\frac{3}{2}} - \frac{a^2}{4}\int (a^2+x^2)^{\frac{1}{2}}\,dx,$$

$$= \frac{x}{4}\,(a^2+x^2)^{\frac{3}{2}} - \frac{a^4}{4}\int\frac{dx}{(a^2+x^2)^{\frac{1}{2}}} - \frac{a^2}{4}\int\frac{x^2\,dx}{(a^2+x^2)^{\frac{1}{2}}}.$$

$$= \frac{x}{4}\,(a^2+x^2)^{\frac{3}{2}} - \frac{a^4}{4}\log\{\,x+\sqrt{(a^2+x^2)}\,\}$$

$$-\frac{a^2}{4}\left\{\frac{x\sqrt{(a^2+x^2)}}{2} - \frac{a^2}{2}\log\{\,x+\sqrt{(a^2+x^2)}\,\}\right\}$$

$$= \left(\frac{x^3}{4}+\frac{a^2x}{4.2}\right)\sqrt{(a^2+x^2)} - \frac{a^4}{4.2}\log\{\,x+\sqrt{(a^2+x^2)}\,\}.$$

Integrals of this class may be obtained in the following manner: thus

$$\int x^2(a^2+x^2)^{\frac{1}{2}}\,dx = a^2\int\frac{x^2\,dx}{(a^2+x^2)^{\frac{1}{2}}} + \int\frac{x^4\,dx}{(a^2+x^2)^{\frac{1}{2}}},$$

$$\int x^3(a^2+x^2)^{\frac{1}{2}}\,dx = a^2\int\frac{x^3\,dx}{(a^2+x^2)^{\frac{1}{2}}} + \int\frac{x^5\,dx}{(a^2+x^2)^{\frac{1}{2}}},\ \text{and so on.}$$

3. To integrate $du = \dfrac{dx}{x^m\sqrt{(x^2+2ax)}}$.

The elementary integral is

$$\int\frac{dx}{\sqrt{(x^2+2ax)}} = \log\{\,x+a+\sqrt{(x^2+2ax)}\,\}.$$

If $m = 1$; then putting $x = \dfrac{1}{y}$, we have $dx = -\dfrac{dy}{y^2}$, and

$$du = -\frac{dy}{\sqrt{(1+2ay)}} = -(2ay+1)^{-\frac{1}{2}}\,dy,\ \text{and integrating,}$$

$$u = -\frac{\sqrt{(1+2ay)}}{a} = -\frac{\sqrt{(x^2+2ax)}}{ax}.$$

If $m = 2$, then $du = \dfrac{dx}{x^2\sqrt{(x^2+2ax)}}$.

Let $x = \dfrac{1}{y}$; then, $du = -\dfrac{y\,dy}{\sqrt{(2ay+1)}} = -(2ay+1)^{-\frac{1}{2}}\,y\,dy$;

$$\therefore\ u = -\int y\,.\,(2ay+1)^{-\frac{1}{2}}\,dy,$$

$$= -\frac{y}{a}\,(2ay+1)^{\frac{1}{2}} + \frac{1}{a}\int(2ay+1)^{\frac{1}{2}}\,dy,$$

$$= -\frac{y}{a}\,(2ay+1)^{\frac{1}{2}} + \frac{1}{a}\int\frac{dy}{\sqrt{(2ay+1)}} - 2u;$$

$$\therefore\ u = -\frac{y}{3a}\,(2ay+1)^{\frac{1}{2}} + \frac{1}{a^2}\,(2ay+1)^{\frac{1}{2}}$$

$$= -\left(\frac{1}{3ax^2} - \frac{1}{a^2x}\right)\sqrt{(x^2+2ax)}.$$

In a similar manner, the integrals of various other differentials may be found; and if the power of x be a factor of the denominator, the integral may be readily found by assuming $y = \dfrac{1}{x}$.

EXAMPLES.

1. $\displaystyle\int \frac{x\,dx}{(a+bx)^{\frac{1}{2}}} = \frac{2(bx-2a)}{3b^2}(a+bx)^{\frac{1}{2}}.$

2. $\displaystyle\int \frac{x^2\,dx}{(a+x)^{\frac{1}{2}}} = \frac{2}{15}(3x^2-4ax+8a^2)(a+x)^{\frac{1}{2}}.$

3. $\displaystyle\int \frac{x^2\,dx}{(a-x)^{\frac{1}{2}}} = -\frac{2}{15}(3x^2+4ax+8a^2)(a-x)^{\frac{1}{2}}.$

4. $\displaystyle\int \frac{x^2\,dx}{(a+bx)^{\frac{1}{3}}} = \frac{3(a+bx)^{\frac{2}{3}}}{b^3}\left\{\frac{(a+bx)^2}{8} - \frac{2}{5}a(a+bx) + \frac{a^2}{2}\right\}.$

5. $\displaystyle\int x^2(a+bx)^{\frac{1}{2}}\,dx = \left\{\frac{(a+bx)^2}{7} - \frac{2a(a+bx)}{5} + \frac{a^2}{3}\right\}\frac{2(a+bx)^{\frac{3}{2}}}{b^3}.$

6. $\displaystyle\int x^3(1+x^2)^{\frac{3}{2}}\,dx = \frac{5x^2-2}{5.7}(1+x^2)^{\frac{5}{2}}.$

7. $\displaystyle\int x^3(1+x^2)^{\frac{5}{2}}\,dx = \frac{7x^2-2}{7.9}(1+x^2)^{\frac{7}{2}}.$

8. $\displaystyle\int \frac{dx}{(a+bx^2)^{\frac{5}{2}}} = \left\{\frac{1}{3a(a+bx^2)} + \frac{2}{3a^2}\right\}\frac{x}{\sqrt{(a+bx^2)}}.$

9. $\displaystyle\int \frac{x^3\,dx}{(1+x^2)^{\frac{3}{2}}} = \frac{x^2+2}{\sqrt{(1+x^2)}}.$

10. $\displaystyle\int \frac{x^3\,dx}{(1+x^2)^{\frac{5}{2}}} = -\frac{3x^2+2}{3(1+x^2)^{\frac{3}{2}}}.$

11. $\displaystyle\int \frac{x^5\,dx}{(1+x^2)^{\frac{3}{2}}} = \frac{3x^4+12x^2+8}{3(1+x^2)^{\frac{1}{2}}}.$

12. $\displaystyle\int \frac{x^4\,dx}{(1+x^2)^{\frac{3}{2}}} = \frac{x^3+3x}{2\sqrt{(1+x^2)}} + \frac{3}{2}\log\{x+\sqrt{(1+x^2)}\}.$

13. $\displaystyle\int x^5(1+x^2)^{\frac{1}{2}}\,dx = \left\{\frac{x^4}{7} - \frac{4x^2}{7.5} + \frac{8}{7.5.3}\right\}(1+x^2)^{\frac{3}{2}}.$

14. $\displaystyle\int \frac{x^5\,dx}{\sqrt{(1-x^2)}} = -\left\{\frac{x^4}{5} + \frac{4x^2}{5.3} + \frac{8}{5.3}\right\}(1-x^2)^{\frac{1}{2}}.$

15. $\displaystyle\int \frac{x^4\,dx}{(1-x^2)^{\frac{3}{2}}} = -\frac{x^3-3x}{2\sqrt{(1-x^2)}} - \frac{3}{2}\sin^{-1}x.$

16. $\displaystyle\int \frac{dx}{x^4\sqrt{(1-x^2)}} = -\left\{\frac{1}{3x^3} + \frac{2}{3x}\right\}\sqrt{(1-x^2)}.$

INTEGRATION OF TRANSCENDENTAL FUNCTIONS.

101. Transcendental functions are such as have differential coefficients involving logarithmic, exponential, or circular functions, and we shall now advert shortly to these in their order.

I. *Logarithmic Functions.*

Let it be required to integrate $du = x^m (\log x)^n \, dx$.

$$\int (\log x)^n . x^m \, dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} . \int x^m (\log x)^{n-1} \, dx,$$

which is the formula of reduction, the final integral being $\int x^m \, dx = \dfrac{x^{m+1}}{m+1}$.

If $m = 1$, and $n = 3$, then

$$\int x (\log x)^3 \, dx = \frac{x^2}{2} (\log x)^3 - \frac{3}{2} \int x (\log x)^2 dx;$$

$$\int x (\log x)^2 \, dx = \frac{x^2}{2} (\log x)^2 - \int x \log x \, dx, \text{ if } n = 2,$$

$$\int x \log x \, dx = \frac{x^2}{2} \log x - \frac{x^2}{2}, \text{ if } n = 1;$$

$$\therefore \int x (\log x)^3 \, dx = \frac{x^2}{2} (\log x)^3 - \frac{3x^2}{4} (\log x)^2 + \frac{3x^2}{4} \log x - \frac{3x^2}{8}.$$

Let it be required to integrate $du = \dfrac{x^m \, dx}{(\log x)^n}$.

$$\int \frac{x^m \, dx}{(\log x)^n} = \int x^{m+1} . \frac{dx}{x (\log x)^n} = \int x^{m+1} . (\log x)^{-n} d (\log x)$$

$$= - \frac{x^{m+1}}{(n-1) (\log x)^{n-1}} + \frac{m+1}{n-1} . \int \frac{x^m \, dx}{(\log x)^{n-1}},$$

which is the formula of reduction, the final integral being $\displaystyle\int \frac{x^m \, dx}{\log x}$, which must be found by diminishing n successively. To reduce this to a simpler form, let $x^{m+1} = z$, then $(m + 1) \log x = \log z$, and $(m+1) x^m \, dx = dz$; therefore

$\displaystyle\int \frac{x^m \, dx}{\log x} = \int \frac{dz}{\log z}$, a formula which has never been integrated except by series.

If $m = -1$, then $\displaystyle\int \frac{dx}{x \log x} = \int \frac{d \log x}{\log x} = \log (\log x.$

II. *Exponential Functions.*

102. Let it be required to integrate $du = x^m a^x \, dx$, where m is positive.

Integrating by parts, we have

$$\int x^m . a^x \, dx = \frac{x^m a^x}{\log a} - \frac{m}{\log a} \int x^{m-1} a^x \, dx.$$

By the successive application of this formula, the index of x will be continually diminished. In a similar manner we get

$$\int \frac{a^x \, dx}{x^m} = \int a^x . x^{-m} \, dx = - \frac{a^x}{(m-1) \, x^{m-1}} + \frac{\log a}{m-1} . \int \frac{a^x \, dx}{x^{m-1}}.$$

When $m = 1$, the formula evidently fails, since $x^{m-1} = 0$, and there-

fore $\int \frac{a^x\,dx}{x}$ cannot be found except by expanding a^x in a series. Thus $a^x = 1 + \log a . x + (\log a)^2 \frac{x^2}{1.2} + (\log a)^3 \frac{x^3}{1.2.3} + \text{etc.}$;

$$\therefore \int \frac{a^x\,dx}{x} = \log x + \log a . x + \tfrac{1}{2}(\log a)^2 . \frac{x^2}{1.2} + \tfrac{1}{3}(\log a)^3 . \frac{x^3}{1.2.3} + \text{etc.}$$

The formula $\frac{dz}{\log z}$ may be integrated by series in a similar manner; for if $\log z = x$, or $z = e^x$, then we have

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{etc.};$$

$$\therefore \int \frac{dz}{\log z} = \int \frac{e^x\,dx}{x} = \log x + x + \frac{1}{2} . \frac{x^2}{1 \cdot 2} + \frac{1}{3} . \frac{x^3}{1 \cdot 2 \cdot 3} + \text{etc.}$$

$$= \log {}^2 z + \log z + \frac{1}{2} . \frac{(\log z)^2}{1.2} + \frac{1}{3} . \frac{(\log z)^3}{1.2.3} + \text{etc.},$$

where $\log {}^2 z$ denotes the logarithm of $\log z$.

Examples.

1. $\int x (\log x)^2\, dx = \frac{x^2}{2}\left\{ (\log x)^2 - \log x + 1 \right\}.$
2. $\int x^2 \log x\, dx = \frac{x^3}{3} \log x - \frac{x^3}{9}.$
3. $\int x^3 (\log x)^2\, dx = \frac{x^4}{4}\left\{ (\log x)^2 - \tfrac{1}{2}\log x + \tfrac{1}{8} \right\}.$
4. $\int \frac{x\,dx}{(\log x)^3} = - \frac{x^2}{2(\log x)^2} - \frac{x^2}{\log x} + 2\int \frac{x\,dx}{\log x}.$
5. $\int x a^x\, dx = \frac{a^x}{\log a}\left\{ x - \frac{1}{\log a} \right\}.$
6. $\int x^3 a^x\, dx = \frac{a^x}{\log a}\left\{ x^3 - \frac{3x^2}{\log a} + \frac{6x}{(\log a)^2} - \frac{6}{(\log a)^3} \right\}.$
7. $\int \frac{a^x\,dx}{x^2} = - \frac{a^x}{x} + \log a \int \frac{a^x\,dx}{x}.$
8. $\int \frac{e^x\,dx}{x^3} = - \frac{e^x}{2x^2} - \frac{e^x}{2x} + \frac{1}{2}\int \frac{e^x\,dx}{x}.$
9. $\int \frac{\log x\,dx}{(1-x)^2} = \frac{x \log x}{1-x} + \log(1-x).$
10. $\int e^{ax} x^2\, dx = \frac{e^{ax} x^2}{a} - \frac{2}{a}\left\{ \frac{e^{ax} x}{a} - \frac{e^{ax}}{a^2} \right\} = e^{ax}\left\{ \frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right\}.$

III. *Circular Functions.*

103. If the differential contains a variable arc, as $\tan^{-1} x$, it may generally be made to disappear in the differential derived in the method of integration by parts. Thus

$$\int x^3 \tan^{-1}x \,.\, dx = \int \tan^{-1}x \,.\, x^3\, dx = \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\int \frac{x^4\, dx}{1+x^2}$$
$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\int x^2\, dx + \frac{1}{4}\int dx - \frac{1}{4}\int \frac{dx}{1+x^2}$$
$$= \frac{x^4 \tan^{-1}x}{4} - \frac{x^3}{12} + \frac{x}{4} - \frac{1}{4}\tan^{-1}x.$$

104. Circular functions may always be reduced to algebraic functions by assuming $\sin x$ or $\cos x = z$; thus the expression $\sin^m x \cos^n x\, dx$ may be reduced to a binomial differential by assuming $\sin x = z$; since then $\cos x = \sqrt{(1-z^2)}$, and $dx = \dfrac{dz}{\sqrt{(1-z^2)}}$; consequently we have

$$\int \sin^m x \cos^n x\, dx = \int z^m (1-z^2)^{\frac{n-1}{2}}\, dz.$$

But the integrals of circular functions will be obtained more elegantly by the usual process of integration by parts, or by means of the following trigonometrical values of the powers of $\sin x$ and $\cos x$.

$$2 \sin^2 x = 1 - \cos 2x.$$
$$4 \sin^3 x = 3 \sin x - \sin 3x.$$
$$8 \sin^4 x = 3 - 4 \cos 2x + \cos 4x.$$
$$16 \sin^5 x = 10 \sin x - 5 \sin 3x + \sin 5x.$$
$$32 \sin^6 x = 10 - 15 \cos 2x + 6 \cos 4x - \cos 6x.$$
etc. etc.

$$2 \cos^2 x = 1 + \cos 2x.$$
$$4 \cos^3 x = 3 \cos x + \cos 3x.$$
$$8 \cos^4 x = 3 + 4 \cos 2x + \cos 4x.$$
$$16 \cos^5 x = 10 \cos x + 5 \cos 3x + \cos 5x.$$
$$32 \cos^6 x = 10 + 15 \cos 2x + 6 \cos 4x + \cos 6x.$$
etc. etc.

To exemplify the use of these values of the powers of $\sin x$ and $\cos x$, let $\int \cos^5 x\, dx$ be required. We have

$$\cos^5 x = \frac{5}{8}\cos x + \frac{5}{16}\cos 3x + \frac{1}{16}\cos 5x;$$
$$\therefore \int \cos^5 x\, dx = \frac{5}{8}\int \cos x\, dx + \frac{5}{16}\int \cos 3x\, dx + \frac{1}{16}\int \cos 5x\, dx$$
$$= \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x.$$

Similarly, $\int \sin^4 x\, dx = \frac{3}{8}\int dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{8}\int \cos 4x\, dx$
$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x.$$

105. Let it be required to integrate the forms $\dfrac{dx}{\sin x}$ and $\dfrac{dx}{\cos x}$.

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x} = \int \frac{d(\frac{1}{2}x)}{\tan \frac{1}{2}x \cos^2 \frac{1}{2}x} = \int \frac{\sec^2 \frac{1}{2}x\, d(\frac{1}{2}x)}{\tan \frac{1}{2}x}$$

$$= \int \frac{d \tan \frac{1}{2} x}{\tan \frac{1}{2} x} = \log \tan \tfrac{1}{2} x.$$

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(\frac{1}{2}\pi + x)} = \int \frac{d(\frac{1}{2}\pi + x)}{\sin(\frac{1}{2}\pi + x)} = \log \tan \tfrac{1}{2}(\tfrac{1}{2}\pi + x).$$

106. To integrate the forms $\tan x\, dx$ and $\cot x\, dx$.

$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x} = \int \frac{-d \cos x}{\cos x} = -\log \cos x = \log \sec x.$$

$$\int \cot x\, dx = \int \frac{\cos x\, dx}{\sin x} = \int \frac{d \sin x}{\sin x} = \log \sin x.$$

107. To integrate the form $\dfrac{dx}{\sin x \cos x}$.

Here $\displaystyle\int \frac{dx}{\sin x \cos x} = \int \frac{2\, dx}{\sin 2x} = \int \frac{d(2x)}{\sin 2x} = \log \tan x$, by Art. 105.

108. To integrate the forms $du = \sin^n x\, dx$ and $du = \cos^n x\, dx$.

$$\begin{aligned}
u &= \int \sin^{n-1} x \sin x\, dx = -\int \sin^{n-1} x\, d \cos x \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1 - \sin^2 x)\, dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)u;
\end{aligned}$$

$$\therefore u = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx \quad . \; . \; . \; . \; . \quad (A).$$

Writing $n-2$ for n in (A), we get

$$\int \sin^{n-2} x\, dx = -\frac{1}{n-2}\sin^{n-3} x \cos x + \frac{n-3}{n-2}\int \sin^{n-4} x\, dx \ldots (A').$$

In this manner the index may be reduced to 1 or 0, according as n is odd or even, and the final integral will be either $\int \sin x\, dx = -\cos x$, or $\int dx = x$.

Again, $u = \int \cos^{n-1} x \cos x\, dx = \int \cos^{n-1} x\, d \sin x$

$$\begin{aligned}
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)u;
\end{aligned}$$

$$\therefore u = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx \quad . \; . \; . \; . \quad (B).$$

Similarly, $\displaystyle\int \cos^{n-2} x\, dx = \frac{1}{n-2}\cos^{n-3} x \sin x + \frac{n-3}{n-2}\int \cos^{n-4} x\, dx \ldots (B')$,

and the final integral is either $\int \cos x\, dx = \sin x$ or $\int dx = x$.

109. To integrate the forms $du = \dfrac{dx}{\sin^n x}$ and $du = \dfrac{dx}{\cos^n x}$.

$$\begin{aligned}
\int \frac{dx}{\sin^n x} &= \int \frac{(\cos^2 x + \sin^2 x)\, dx}{\sin^n x} = \int \cos x \frac{\cos x\, dx}{\sin^n x} + \int \frac{dx}{\sin^{n-2} x} \\
&= \frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{1}{n-1}\int \frac{dx}{\sin^{n-2} x} + \int \frac{dx}{\sin^{n-2} x} \\
&= \frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n}{n-1}\int \frac{dx}{\sin^{n-2} x} \quad . \; . \; . \; . \; . \quad (C),
\end{aligned}$$

and the integral is reduced to $\displaystyle\int \frac{dx}{\sin x}$ or $\int dx$, both of which are known.

$$\int\frac{dx}{\cos^n x}=\int\frac{(\cos^2 x+\sin^2 x)\,dx}{\cos^n x}=\int\frac{d\,x}{\cos^{n-2}x}+\int\sin x\,.\frac{\sin x\,dx}{\cos^n x}$$

$$=\int\frac{dx}{\cos^{n-2}x}+\frac{\sin x}{(n-1)\cos^{n-1}x}-\frac{1}{n-1}\int\frac{dx}{\cos^{n-2}x}$$

$$=\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2}x}\quad.\quad\ldots\quad(D),$$

which reduces the integral to $\int\frac{dx}{\cos x}$ or $\int dx$, both of which are known.

110. To integrate the form $du=\sin^m x\cos^n x\,dx$.

If m be an odd positive integer of the form $2p+1$, then we have $u=\int\cos^n x\,(1-\cos^2 x)^p\sin x\,dx=-\int\cos^n x\,(1-\cos^2 x)^p\,d\cos x$, which may be expanded by the binomial theorem and integrated, the general term being of the form $\cos^n x\,d\cos x$, whose integral is $\frac{\cos^{n+1}x}{n+1}$.

If n be an odd positive integer of the form $2p+1$, then we have $u=\int\sin^m x\,(1-\sin^2 x)^p\cos x\,dx=\int\sin^m x\,(1-\sin^2 x)^p\,d\sin x$, which by expansion is easily integrated.

If $m+n$ be an even negative integer of the form $-2p$; then

$$u=\int\tan^m x\cos^{m+n}x\,dx=\int\tan^m x\sec^{2p}x\,dx$$
$$=\int\tan^m x\,(1+\tan^2 x)^{p-1}\sec^2 x\,dx$$
$$=\int\tan^m x\,(1+\tan^2 x)^{p-1}\,d\tan x,$$

which by expansion is integrable, each term being of the form $x^m\,dx$. But when neither of these conditions is satisfied, we must proceed by reducing the indices m and n successively.

111. To reduce m or n in $\int\sin^m x\cos^n x\,dx$.

Since $du=\sin^m x\cos^n x\,dx=-\sin^{m-1}x\cos^n x\,d\cos x$;

$$\therefore u=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^{n+2}x\,dx$$

$$=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n x\,(1-\sin^2 x)\,dx$$

$$=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n x\,dx-\frac{m-1}{n+1}u;$$

$$\therefore u=-\frac{\sin^{m-1}x\cos^{n+1}x}{m+n}+\frac{m-1}{m+n}\int\sin^{m-2}x\cos^n x\,dx\;\ldots\ldots\;(E).$$

In a similar manner we have $du=\sin^m x\cos^{n-1}x\cos x\,dx$, and

$$\int\sin^m x\cos^n x\,dx=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}\int\sin^m x\cos^{n-2}x\,dx..(F).$$

By freeing the formulas (E) and (F) from fractions, transposing and dividing the former by $m-1$, and the latter by $n-1$, we get

$$\int\sin^{m-2}x\cos^n x\,dx=\frac{m+n}{m-1}\int\sin^m x\cos^n x\,dx+\frac{\sin^{m-1}x\cos^{n+1}x}{m-1},$$

$$\int\sin^m x\cos^{n-2}x\,dx=\frac{m+n}{n-1}\int\sin^m x\cos^n x\,dx-\frac{\sin^{m+1}x\cos^{n-1}x}{n-1}.$$

Writing $-m+2$ for m in the former of these, and $-n+2$ for n in the latter, we get

$$\int \frac{\cos^n x\, d\, x}{\sin^m x} = -\frac{\cos^{n+1} x}{(m-1)\sin^{m-1} x} + \frac{m-n-2}{m-1}\int \frac{\cos^n x\, d\, x}{\sin^{m-2} x} \ldots (G).$$

$$\int \frac{\sin^m x\, d\, x}{\cos^n x} = \frac{\sin^{m+1} x}{(n-1)\cos^{n-1} x} - \frac{m-n+2}{n-1}\int \frac{\sin^m x\, d\, x}{\cos^{n-2} x} \ldots (H).$$

Changing n into $-n$ in (G), and m into $-m$ in (H), we get

$$\int \frac{d\, x}{\sin^m x \cos^n x} = -\frac{1}{m-1} \cdot \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1}\int \frac{d\, x}{\sin^{m-2} x \cos^n x} \ldots (K).$$

$$\int \frac{d\, x}{\sin^m x \cos^n x} = \frac{1}{n-1} \cdot \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1}\int \frac{d}{\sin^m x \cos^{n-2} x} \ldots (L).$$

112. To integrate $d\, u = \tan^m x\, d\, x$.

$$\begin{aligned} \int \tan^m x\, d\, x &= \int \tan^{m-2} x\,(\sec^2 x - 1)\, d\, x, \\ &= \int \tan^{m-2} x\, d \tan x - \int \tan^{m-2} x\, d\, x \\ &= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x\, d\, x \ldots\ldots\ldots\ldots (M). \end{aligned}$$

113. To integrate $d\, u = \dfrac{d\, x}{a + b\cos x}$.

Since $a = a\cos^2 \frac{1}{2} x + a \sin^2 \frac{1}{2} x$, and $b \cos x = b \cos^2 \frac{1}{2} x - b \sin^2 \frac{1}{2} x$,

$$\therefore \quad \int \frac{d\, x}{a + b\cos x} = \int \frac{d\, x}{(a+b)\cos^2 \frac{1}{2} x + (a-b)\sin^2 \frac{1}{2} x}$$

$$= \frac{1}{a+b}\int \frac{\sec^2 \frac{1}{2} x\, d\, x}{1 + \frac{a-b}{a+b}\tan^2 \frac{1}{2} x} = \frac{2}{a+b}\int \frac{d \tan \frac{1}{2} x}{1 + \frac{a-b}{a+b}\tan^2 \frac{1}{2} x}$$

$$= \frac{2}{(a^2-b^2)^{\frac{1}{2}}}\int \frac{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} d \tan \frac{1}{2} x}{1 + \frac{a-b}{a+b}\tan^2 \frac{1}{2} x} = \frac{2}{(a^2-b^2)^{\frac{1}{2}}} \tan^{-1}\left\{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} \tan \tfrac{1}{2} x\right\}.$$

114. To integrate $d\, u = \dfrac{d\, x}{a + b\tan x}$.

$$\begin{aligned} \int \frac{d\, x}{a + b\tan x} &= \int \frac{\cos x\, d\, x}{a\cos x + b\sin x} \\ &= \frac{1}{a^2+b^2}\int \left\{a + b\,\frac{-a\sin x + b\cos x}{a\cos x + b\sin x}\right\} d\, x \\ &= \frac{a}{a^2+b^2}\int d\, x + \frac{b}{a^2+b^2}\int \frac{-a\sin x + b\cos x}{a\cos x + b\sin x}\, d \\ &= \frac{a x + b\log(a\cos x + b\sin x)}{a^2+b^2}. \end{aligned}$$

115. To integrate $d\, u = \int \sin m x \cos n x\, d\, x$.

Since $\sin m x \cos n x = \frac{1}{2}\{\sin (m+n) x + \sin (m-n) x\}$; (Trig. Art. 16)

$$\therefore \int \sin m x \cos n x\, d\, x = -\tfrac{1}{2}\left\{\frac{\cos (m+n) x}{m+n} + \frac{\cos (m-n) x}{m-n}\right\}.$$

Similarly, $\displaystyle\int \sin m x \sin n x\, d\, x = -\tfrac{1}{2}\left\{\frac{\sin (m+n) x}{m+n} - \frac{\sin (m-n) x}{m-n}\right\}.$

And $\displaystyle\int \cos m x \cos n x\, d\, x = \tfrac{1}{2}\left\{\frac{\sin (m+n) x}{m+n} + \frac{\sin (m-n) x}{m-n}\right\}.$

EXAMPLES.

1. $\int \sin^3 x \, dx = -\frac{\cos x}{3}(\sin^2 x + 2) = \frac{\cos 3x}{12} - \frac{3}{4}\cos x.$

2. $\int \cos^2 x \, dx = \frac{1}{2}(\sin x \cos x + x) = \frac{x}{2} + \frac{1}{4}\sin 2x.$

3. $\int \sin^3 x \cos^2 x \, dx = \frac{1}{5}\sin^3 x\left(\cos^2 x + \frac{2}{3}\right).$

4. $\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\log\tan\frac{1}{2}x.$

5. $\int \sin^3 x \cos^2 x \, dx = \left(\frac{1}{5}\sin^4 x - \frac{1}{15}\sin^2 x - \frac{2}{15}\right)\cos x.$

6. $\int \frac{\sin^3 x \, dx}{\cos^2 x} = \cos x + \sec x.$

7. $\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3}\left\{\frac{1}{\cos^3 x} + \frac{2}{\cos x}\right\}.$

8. $\int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2\cos^2 x} + \log\tan x.$

9. $\int \frac{dx}{\sin^3 x \cos^2 x} = \frac{1}{\sin^2 x \cos x} - \frac{3\cos x}{2\sin^2 x} + \frac{3}{2}\log\tan\frac{1}{2}x.$

10. $\int \tan^4 x \, dx = \frac{1}{3}\tan^3 x - \tan x + x.$

11. $\int \frac{dx}{\tan^5 x} = -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log\sin x.$

12. $\int \frac{dx}{(a + b\cos x)^2} =$

$$\frac{1}{a^2 - b^2}\left[\frac{-b\sin x}{a + b\cos x} + \frac{2a}{\sqrt{(a^2 - b^2)}}\tan^{-1}\left\{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}x\right\}\right].$$

APPLICATION OF THE INTEGRAL CALCULUS TO GEOMETRY.

1. TO FIND THE AREAS OF CURVES.

116. The determination of the quadrature and rectification of curves, as well as that of the volumes and surfaces of solids, will afford both useful and interesting applications of the Integral Calculus; but before proceeding to these, it will be necessary to premise the following lemma.

LEMMA.

If $A + A_1 h + A_2 h^2 + A_3 h^3 + \text{etc.}$ . . . (A_0),

$B + B_1 h + B_2 h^2 + B_3 h^3 + \text{etc.}$ . . . (B_0),

$C + C_1 h + C_2 h^2 + C_3 h^3 + \text{etc.}$ . . . (C_0),

be three series such that, however small h may be, the value of the second is less than the first and greater than the third; then if $A = C$, we shall have $A = B = C$.

For h may be taken so small that, however small δ may be,

$$A_0 < A + \delta \text{ but } > A,$$
$$B_0 < B + \delta \text{ but } > B,$$
$$C_0 < C + \delta \text{ but } > C,$$

$\therefore A_0 - B_0 < A + \delta - B$ or $< A - B + \delta$, since $B_0 > B$,*

$B_0 - C_0 < B + \delta - C$ or $< B - C + \delta$, since $C_0 > C$.

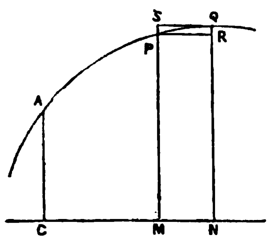
But by hypothesis $A_0 - B_0$ and $B_0 - C_0$ are both positive; consequently

$A - B + \delta$ and $B - C + \delta$ are both positive.

Let now $A = C$; then $A - B + \delta$ and $B - A + \delta$ are both positive; and if $A - B = D$, then $B - A = -D$, and consequently $\delta - D$ is positive, however small δ may be, which can only be the case when $D = 0$; hence $A - B = 0$, or $A = B = C$.

117. To find a general expression for the area of any curve referred to rectangular coordinates.

Let A P Q be a curve, and let it be required to find the area of the space A C M P included by the curve, one of the axes, and two ordinates. Draw N Q so that the successive ordinates between it and P M may form either an increasing series or a decreasing one. This can always be done by taking M N sufficiently small. Let now



$$C M = x, M P = y, M N = h,$$
$$\text{area } A C M P = A, \ N Q = y',$$
$$\text{and area } A C N Q = A',$$

then the area A C M P is a function of x, and A C N Q is a like function of $x + h$; therefore by Taylor's theorem we have

$$A' = A + \frac{dA}{dx}h + \frac{d^2A}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3A}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.};$$

$$\therefore A' - A, \text{or area } P M N Q = \frac{dA}{dx}h + \frac{d^2A}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3A}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.}$$

Again, since $y = fx$, we have

$N Q$ or $y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \text{etc.}$; but it is evident that area P M N Q > area P N < area N S; therefore

$$\frac{dA}{dx}h + \frac{d^2A}{dx^2}\cdot\frac{h^2}{1.2} + \text{etc.} > yh < \left(y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \text{etc.}\right)h,$$

$$\text{or } \frac{dA}{dx} + \frac{d^2A}{dx^2}\cdot\frac{h}{1.2} + \text{etc.} > y < y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \text{etc.}$$

Consequently by the lemma

$$\frac{dA}{dx} = y, \text{or } dA = y\,dx; \text{ therefore } A = \int y\,dx \ldots (A).$$

Hence, *to find the area of a curve referred to rectangular coordinates, multiply one of the coordinates by the differential of the other, and integrate the result.*

---

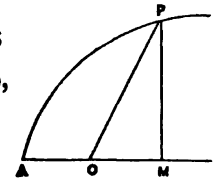
* Since $A_0 < A + \delta$, if B be taken from both, then $A^0 - B < A + \delta - B$; but $B_0 > B$; therefore much more will $A_0 - B_0$ be $< A + \delta - B$.

If the coordinates are oblique, let θ be the angle of ordination; then will $A = \sin\theta \int y\,dx$ . . . . (A').

118. Let the curve be referred to polar coordinates; then since

$AM = x$, $MP = y$, $AO = a$, $OP = r$, angle $AOP = \theta$;

$\therefore x = a - r\cos\theta$, and $y = r\sin\theta$.



Now area AOP = area AMP − triangle OMP;

consequently area $AOP = \int y\,dx - \frac{1}{2}y(x-a)$,

and by differentiating,

$$2d \text{ area } AOP = 2y\,dx - (x-a)\,dy - y\,dx$$
$$= y\,dx - (x-a)\,dy \quad . \; . \; (a).$$

But $dx = -dr\cos\theta + r\sin\theta\,d\theta$, and $dy = dr\sin\theta + r\cos\theta\,d\theta$; therefore by (a),

$$2d.\text{area } AOP = r^2(\sin^2\theta + \cos^2\theta)\,d\theta = r^2\,d\theta:$$

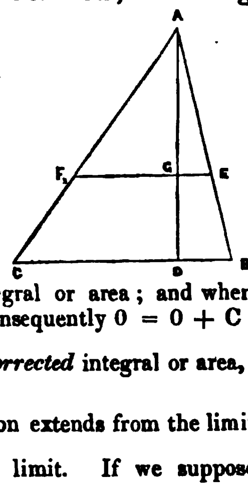
$$\therefore \text{area } AOP = \int \frac{r^2\,d\theta}{2} = \frac{1}{2}\int r^2\,d\theta \quad . \; . \; . \; (B).$$

Examples of finding Areas.

4

1. To find the area of the plane triangle ABC.

Draw AD perpendicular to CB, and FE parallel to CB, intersecting AD in G. Let $BC = a$, $AD = b$, $AG = x$, $FE = y$; then (Euc. vi. 4)



$AG : FE :: AD : CB$, or $x : y :: b : a$;

consequently $by = ax$, or $y = \frac{ax}{b}$;

hence by formula (A), $A = \int y\,dx$

$$= \int \frac{ax\,dx}{b} = \frac{a}{b}\int x\,dx = \frac{ax^2}{2b} + C.$$

This is called the *general* value of the integral or area; and when $x = 0$, the area of the triangle becomes 0; consequently $0 = 0 + C$;

$\therefore C = 0$, and $A = \frac{a}{b}\int_0^x x\,dx = \frac{ax^2}{2b}$, is the *corrected* integral or area,

where the symbol $\int_0^x$ shows that the integration extends from the limit $x = 0$, but does not fix any other particular limit. If we suppose $x = b = AD$, then $A = \frac{a}{b}\int_0^b x\,dx = \frac{ab^2}{2b} = \frac{1}{2}ab = \frac{1}{2}BC \,.\, AD$.

The expression $\frac{a}{b}\int_0^b x\,dx$ is called the *definite* integral, taken between the limits $x = 0$, and $x = b$, and gives the area of the triangle ABC.

Suppose it were required to determine the area of the figure CBEF, where $CB = a$, $FE = c$, and $DG = h$; then we have, as before,

$A = \frac{a}{b}\int x\,dx = \frac{ax^2}{2b} + C$, which is the *indefinite* or *general* integral.

$$\text{Let } x = \text{AD} = b, \text{ then area ABC} = \frac{a\,b^2}{2\,b} + \text{C}.$$

$$\text{let } x = \text{AG} = b - h, \text{ then area AFE} = \frac{a\,(b-h)^2}{2\,b} + \text{C}.$$

Subtracting the latter from the former, gives

$$\text{area BCFE} = \frac{a}{b}\int_{b-h}^{b} x\,dx = \frac{a\,\{b^2 - (b-h)^2\}}{2\,b} = \frac{a\,h\,(2\,b - h)}{2\,b}.$$

But by similar triangles AD : BC :: AG : FE : that is

$$b : a :: b - h : c; \;\therefore\; b = \frac{a\,h}{a - c};$$

and substituting this for b in the above expression, gives

$$\text{area BCFE} = \frac{a\,h + c\,h}{2} = \frac{(a + c)\,h}{2}.$$

2. To find the area of the common parabola.

The equation of the curve is $y^2 = 4\,a\,x$; hence $y = 2\,a^{\frac{1}{2}}\,x^{\frac{1}{2}}$, and

$$\text{A} = \int y\,dx = \int 2\,a^{\frac{1}{2}}\,x^{\frac{1}{2}}\,dx = 2\,a^{\frac{1}{2}} \int x^{\frac{1}{2}}\,dx$$

$$= \frac{4}{3}\,a^{\frac{1}{2}}\,x^{\frac{3}{2}} = \frac{2}{3}.2\,a^{\frac{1}{2}}\,x^{\frac{1}{2}}.x = \frac{2}{3}y\,x = \frac{2}{3}\text{circumscribed rectangle.}$$

There is no correction since A $= 0$, when $x = 0$, therefore C $= 0$.

3. To find the area of a circle.

The equation of the circle is $y^2 = a^2 - x^2$, the centre being the origin; hence $y = (a^2 - x^2)^{\frac{1}{2}}$, and $\text{A} = \int y dx = \int (a^2 - x^2)^{\frac{1}{2}}\,dx$

$$= \int \frac{(a^2 - x^2)\,dx}{(a^2 - x^2)^{\frac{1}{2}}} = a^2 \int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} - \int \frac{x^2\,dx}{(a^2 - x^2)^{\frac{1}{2}}}$$

$$= a^2 \sin^{-1}\frac{x}{a} + \frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} - \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \text{C (Art. 98.)}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \tfrac{1}{2}\,x\,y + \text{C}.$$

When $x = 0$, A $= 0$; therefore $0 = 0 + \text{C}$, and C $= 0$; and when $x = a$, $y = 0$; consequently the area of a quadrant of the circle is

$$\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi\,a^2}{4}, \text{ and the entire circle} = \pi\,a^2.$$

4. To find the area of an ellipse.

The equation of the ellipse is $a^2 y^2 = a^2 b^2 - b^2 x^2$, or $y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$;

$$\therefore\; \text{A} = \int y\,dx = \frac{b}{a}\int (a^2 - x^2)^{\frac{1}{2}}\,dx = \frac{b}{a}.\frac{\pi\,a^2}{4} = \frac{\pi\,a\,b}{4}\;(\textit{Ex.}\ 3):$$

and the entire area of the ellipse $= \pi\,a\,b$.

From these two last examples, it is evident that if the area of any portion of a circle whose radius is a be found, the area of the corresponding portion of the ellipse will be found by multiplying it by b and dividing the product by a.

5. To find the area of an hyperbola.

The centre being the origin, the equation is $y = \frac{b}{a}(x^2 - a^2)^{\frac{1}{2}}$; hence

$$A = \int y\, dx = \frac{b}{a}\int (x^2 - a^2)^{\frac{1}{2}}\, dx = \frac{b}{a}\int \frac{(x^2 - a^2)\, dx}{(x^2 - a^2)^{\frac{1}{2}}}$$

$$= \frac{b}{a}\int \frac{x^2\, dx}{(x^2 - a^2)^{\frac{1}{2}}} - a\, b \int \frac{dx}{(x^2 - a^2)^{\frac{1}{2}}}$$

$$= \frac{b}{a}\cdot\frac{x(x^2 - a^2)^{\frac{1}{2}}}{2} - \frac{a\,b}{2}\log\{x + (x^2 - a^2)^{\frac{1}{2}}\} + C.$$

Now when $x = a$, then $A = 0$, and therefore $0 = -\frac{a\,b}{2}\log a + C$,

$\therefore C = \frac{a\,b}{2}\log a$, and $A = \frac{x\,y}{2} - \frac{a\,b}{2}\log\{x + (x^2 - a^2)^{\frac{1}{2}}\} + \frac{a\,b}{2}\log a$

$$= \frac{x\,y}{2} + \frac{a\,b}{2}\log\frac{a}{x + (x^2 - a^2)^{\frac{1}{2}}} = \frac{x\,y}{2} + \frac{a\,b}{2}\log\{x - (x^2 - a^2)^{\frac{1}{2}}\}.$$

6. To find the area of the cycloid.

The equations of the cycloid given in Art. 51 (*Ex.* 4), are

$$x = r(\theta - \sin\theta) \text{ and } y = r(1 - \cos\theta);$$

$$\therefore A = \int y\, dx = \int r^2(1 - \cos\theta)(1 - \cos\theta)\, d\theta = r^2\int(1 - \cos\theta)^2\, d\theta$$
$$= r^2\int(1 - 2\cos\theta + \cos^2\theta)\, d\theta$$
$$= r^2\int(1 - 2\cos\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta)\, d\theta$$
$$= r^2(\theta - 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta) + C.$$

When $\theta = 0$, then $A = 0$, and therefore $0 = 0 + C$, or $C = 0$; and when $\theta = \pi$, then $A = r^2(\pi + \frac{1}{2}\pi) = \frac{3}{2}\pi r^2 =$ area of semicycloid. Hence the area of the entire cycloid is $= 3\pi r^2 =$ three times the area of the generating circle.

If we take the vertex as origin, we may also find the area by means of the equation,

$$y = r\,\text{vers}^{-1}\frac{x}{r} + (2rx - x^2)^{\frac{1}{2}}.$$

$$A = \int y\, dx = xy - \int x\, dy = xy - \int(2rx - x^2)^{\frac{1}{2}}\, dx$$

$$= xy - 2r\int\frac{x\, dx}{(2rx - x^2)^{\frac{1}{2}}} + \int\frac{x^2\, dx}{(2rx - x^2)^{\frac{1}{2}}}$$

$$= xy + \frac{r - x}{2}(2rx - x^2)^{\frac{1}{2}} - \frac{r^2}{2}\text{vers}^{-1}\frac{x}{r} + C$$

$$= r\,\frac{(2x - r)}{2}\,\text{vers}^{-1}\frac{x}{r} + \frac{x + r}{2}(2rx - x^2)^{\frac{1}{2}} + C.$$

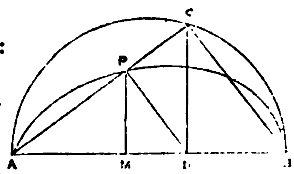
When $x = 0$, then $A = 0$; $\therefore 0 = 0 + C$, and $C = 0$; and when $x = 2r$, the area of the semicycloid is $= \frac{3r^2}{2}\text{vers}^{-1}2 = \frac{3}{2}\pi r^2$, as before.

7. A C B is a given semicircle, and C D any ordinate; join A C, and

draw D P perpendicular to A C; find the equation and area of the curve, which is the locus of P.

Let $AB = 2r$, and take x and y for the coordinates of P, the extremity of the diameter, A, being the origin of coordinates. Draw B C, then we have $AM = x$, $MP = y$, $AP = (x^2 + y^2)^{\frac{1}{2}}$, and from the similar triangles A B C, A D P, and A M P, we have $AB : AD :: AC : AP :: AD : AM$; hence $AD^2 = AB \,.\, AM = 2rx$; consequently $AP^2 = AD . AM = x\sqrt{2rx}$; but $AP^2 = x^2 + y^2$; therefore the equation of the curve is $y = (x\sqrt{2rx} - x^2)^{\frac{1}{2}}$.

Let $2rx = z^2$; then $x = \dfrac{z^2}{2r}$, and $dx = \dfrac{z\,dz}{r}$; consequently



$$y = \left(\frac{z^3}{2r} - \frac{z^4}{4r^2}\right)^{\frac{1}{2}} = \frac{z(2rz - z^2)^{\frac{1}{2}}}{2r}:$$

$$\therefore\ A = \int y\,dx = \int_0 \frac{(2rz - z^2)^{\frac{1}{2}} z^2\,dz}{2r^2}$$

$$= \frac{1}{2r^2}\int \frac{(2rz - z^2)z^2\,dz}{(2rz - z^2)^{\frac{1}{2}}} = \frac{1}{r}\int \frac{z^3\,dz}{(2rz - z^2)^{\frac{1}{2}}} - \frac{1}{2r^2}\int \frac{z^4\,dz}{(2rz - z^2)^{\frac{1}{2}}}$$

$$= (2rz - z^2)^{\frac{1}{2}}\left(\frac{z^3}{8r^2} - \frac{z^2}{24r} - \frac{5z}{48} - \frac{5r}{16}\right) + \frac{5r^2}{16}\operatorname{vers}^{-1}\frac{z}{r} + C.$$

Now when $x = 0$, then $A = 0$, and $z = 0$; therefore $C = 0$, and when $x = 2r$, then $z = 2r$, and the area of the entire curve A P B is

$$\frac{5r^2}{16}\operatorname{vers}^{-1} 2 = \frac{5}{16}\pi r^2 = \frac{5}{8}\text{ semicircle A C B.}$$

To find the area of the same curve by polar coordinates.

Let $AB = 2a$, $AP = r$, and angle $BAP = \theta$; then we have

$AC = 2a\cos\theta$, $AD = 2a\cos^2\theta$, $AP = 2a\cos^3\theta$;

$\therefore\ r = 2a\cos^3\theta$ is the polar equation of the curve.

By formula (B) we have $\frac{1}{2}\int r^2 d\theta = 2a^2\int \cos^6\theta\,d\theta$

$$= 2a^2\int_0 \left(\frac{5}{16} + \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta + \frac{1}{32}\cos 6\theta\right) d\theta$$

$$= 2a^2\left(\frac{5}{16}\theta + \frac{15}{64}\sin 2\theta + \frac{3}{64}\sin 4\theta + \frac{1}{192}\sin 6\theta\right) + C.$$

Now, supposing the point C to move from B towards A, the area would be 0 when $\theta = 0$; hence $C = 0$, and when $\theta = \frac{1}{2}\pi$, the entire area is

$$2a^2 . \frac{5}{16} . \frac{\pi}{2} = \frac{5}{16}\pi a^2 = \frac{5}{8}\text{ semicircle A C B.}$$

The area of the semicircle A C B may readily be found by means of polar coordinates. Since $AC = 2a\cos\theta$; therefore

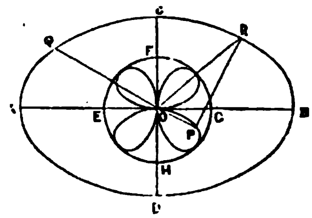
$$\text{area } ABC = \tfrac{1}{2}\int r^2\,d\theta = 2a^2\int \cos^2\theta\,d\theta = 2a^2\int \cos\theta\,d\sin\theta$$
$$= 2a^2\cos\theta\sin\theta + 2a^2\int \sin^2\theta\,d\theta;$$
$$\therefore\ 2\text{ area } ABC = 2a^2\cos\theta\sin\theta + 2a^2\int(\cos^2\theta + \sin^2\theta)\,d\theta$$
$$= a^2\sin 2\theta + 2a^2\theta + C.$$

And since area $= 0$, when $\theta = 0$, therefore $C = 0$, and when $\theta = \frac{1}{2}\pi$,

area of semicircle ABC $= \frac{2a^2}{2} \cdot \frac{\pi}{2} = \frac{1}{2}\pi a^2$, as in *Example* 3.

8. Let OQ and OR be semiconjugate diameters of an ellipse ACBD, whose semiaxes are AO $= a$, CO $= b$; and let RP be perpendicular to QO produced; it is required to find the area of the curve which is the locus of P.

Let O be the origin of coordinates, and OB the axis of x; let x' y' be the coordinates of the point R, then the equation of OR is $yx' - xy' = 0$, and the equation of the semiconjugate diameter OQ is



$$a^2 yy' + b^2 xx' = 0 \ldots . (1.)$$

Also the equations of RP (a perpendicular to OQ) and the curve are

$$y - y' = \frac{a^2 y'}{b^2 x'}(x - x') \ldots . (2.)$$

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2 \ldots . (3.)$$

Eliminate x', y' by means of these equations, and the resulting equation

$$(a^2 y^2 + b^2 x^2)(x^2 + y^2)^2 = (a^2 - b^2)^2 x^2 y^2 \ldots . (4.)$$

is the equation of the curve, which is the locus of P. To transform this to polar coordinates, we have $x = r \cos\theta$, and $y = r\sin\theta$, where $r =$ OP, and $\theta =$ angle BOP; hence, by substitution in (4), we get

$$r^2 = \frac{(a^2 - b^2)^2 \sin^2\theta \cos^2\theta}{a^2 \sin^2\theta + b^2\cos^2\theta}, \text{ the polar equation of the curve.}$$

To determine the maximum value of r, divide the preceding fractional value of r^2 by $\sin^2\theta\cos^2\theta$; then will

$$\frac{a^2}{\cos^2\theta} + \frac{b^2}{\sin^2\theta} = \text{a minimum.}$$

Differentiating, we find $\sin^2\theta = \frac{b}{a+b}$, $\cos^2\theta = \frac{a}{a+b}$, and $r = a - b$, which is the maximum value of r; hence a circle described from centre O, with a radius equal to $a - b$, will circumscribe the four loops of the curve, since there is one loop in each quadrant of the ellipse.

To determine the area of the curve we have, from the polar equation,

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta - \frac{a^2 b^2}{a^2\sin^2\theta + b^2\cos^2\theta};$$

$$\therefore \tfrac{1}{2}\int r^2 d\theta = \tfrac{1}{2}a^2\int\cos^2\theta\, d\theta + \tfrac{1}{2}b^2\int\sin^2\theta\, d\theta - \frac{a^2b^2}{2}\int\frac{\operatorname{cosec}^2\theta\, d\theta}{a^2 + b^2\cot^2\theta}$$

$$= \frac{a^2 - b^2}{8}\sin 2\theta + \frac{a^2 + b^2}{4}\theta - \frac{ab}{2}\cot^{-1}\left(\frac{b}{a}\cot\theta\right) + C,$$

and $\frac{1}{2}\int_0^{\frac{1}{2}\pi} r^2 d\theta = \frac{\pi(a^2+b^2)}{8} - \frac{\pi ab}{4} = \frac{\pi(a-b)^2}{8}$ = area of one loop;

hence the entire area of the four loops $= \frac{\pi(a-b)^2}{2}$ = one-half the area of the circle EFGH, which circumscribes the four loops.

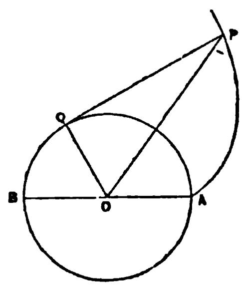
9. To find the area of the involute of a circle.

If a thread, wrapped round the circumference of a given circle, be gradually unwound, any point of it will describe the involute. Let A be the point which describes the involute, and P any point in it; let PQ be a tangent to the evolute at Q, and draw OQ, OP, the point O being the centre of the given circle.

Let $AO = a$, $OP = r$, and angle $AOP = \theta$; then, since the tangent $PQ =$ the arc AQ, we have the arc $AQ = PQ = \sqrt{(r^2 - a^2)}$, and by trigonometry,

$$\cos POQ = \frac{a}{r}, \text{ or } POQ = \cos^{-1}\frac{a}{r};$$

$$\therefore AOQ = AOP + POQ = \theta + \cos^{-1}\frac{a}{r},$$



and semicircumference $AQB = \pi a$;
hence $\pi : AOQ :: \pi a : AQ$; that is,

$$\pi : \theta + \cos^{-1}\frac{a}{r} :: \pi a : \sqrt{(r^2 - a^2)};$$

$$\therefore a\theta = \sqrt{(r^2 - a^2)} - a\cos^{-1}\frac{a}{r}$$

$$= \sqrt{(r^2 - a^2)} - a\sec^{-1}\frac{r}{a},$$

is the polar equation of the involute AP.

Differentiating this equation, we get

$$a\,d\theta = \frac{r\,dr}{\sqrt{(r^2 - a^2)}} - a \cdot \frac{a\,dr}{r\sqrt{(r^2 - a^2)}} = \frac{dr\sqrt{(r^2 - a^2)}}{r};$$

$$\therefore \text{area} = \tfrac{1}{2}\int r^2\,d\theta = \frac{1}{2a}\int r\,dr\sqrt{(r^2 - a^2)} = \frac{(r^2 - a^2)^{\frac{3}{2}}}{6a} + C.$$

When the area of APO is 0, then $r = a$; hence $0 = 0 + C$, and $C = 0$; and when $PQ =$ the entire circumference of the circle, or $2\pi a$, then $r^2 = a^2 + 4\pi^2 a^2$, or $(r^2 - a^2)^{\frac{3}{2}} = (4\pi^2 a^2)^{\frac{3}{2}} = 8\pi^3 a^3$;

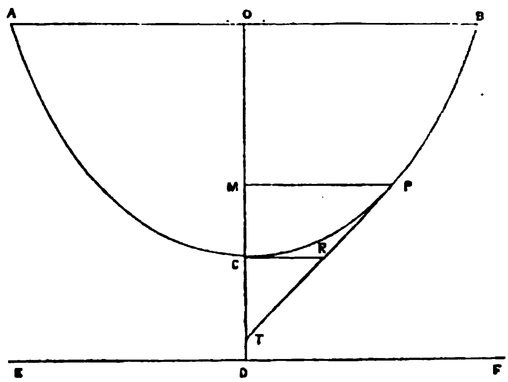
$$\therefore \text{area} = \frac{(r^2 - a^2)^{\frac{3}{2}}}{6a} = \frac{8\pi^3 a^3}{6a} = \frac{4}{3}\pi^3 a^2,$$

and if from this we subtract the area of the circle $AQB = \pi a^2$, we shall have the area of the involute exterior to the circle after the unwrapping of the string $= \frac{4}{3}\pi^3 a^2 - \pi a^2 = \pi a^2\left(\frac{4}{3}\pi^2 - 1\right)$.

10. To find the area of the curve called a catenary.

If a perfectly flexible and inextensible chain or cord ACB, of uniform thickness and weight, be suspended from any two points A and B, so as to hang freely, the curve into which it forms itself is called a *catenary*. Let C be the lowest point of the curve, and CO the vertical axis; then if we put $CM = x$, $MP = y$, $CP = s$, we may find the equation of the curve in the following manner. Consider CP as a rigid body acted upon by the force of gravity, and by the tensions at its extremities C and P in the directions of the tangents CR and

PR; then since these three forces keep the body CP at rest, they will be proportional to the three sides of the triangle MPT, which are parallel to their directions. Now, if the tension of the chain or cord at C be equal to the weight of a length a of the cord, and the weight of the length CP will be as CP or s; therefore



$$\frac{\text{tension at C}}{\text{weight of CP}} = \frac{\text{MP}}{\text{MT}}; \therefore \frac{a}{s} = \frac{dy}{dx} \ldots\ldots (1.)$$

Square (1) and add unity to both sides; then will

$$\frac{a^2+s^2}{s^2} = \frac{dy^2+dx^2}{dx^2} = \frac{ds^2}{dx^2}; \therefore dx = \frac{s\,ds}{\sqrt{(s^2+a^2)}}.$$

Integrating, we have $x = \sqrt{(s^2+a^2)} + \text{C}$; but when $x = 0, s = 0$; therefore $0 = a + \text{C}$, and $\text{C} = -a$; hence the equation of the curve is

$$x + a = \sqrt{(s^2+a^2)}, \text{ or } x^2 + 2ax = s^2 \ldots\ldots (2.)$$

Again, from (1) $\frac{s}{a} = \frac{dx}{dy}$; $\therefore \frac{s^2+a^2}{a^2} = \frac{dx^2+dy^2}{dy^2} = \frac{ds^2}{dy^2}$;

$\therefore \frac{dy}{a} = \frac{ds}{\sqrt{(s^2+a^2)}}$, and $\frac{y}{a} = \log\{s + \sqrt{(s^2+a^2)}\} + \text{C}$.

Now, when $y = 0$, $s = 0$, therefore $0 = \log a + \text{C}$, and $\text{C} = -\log a$;

$$\therefore \frac{y}{a} = \log\frac{s+\sqrt{(s^2+a^2)}}{a}, \text{ or } e^{\frac{y}{a}} = \frac{s+\sqrt{(s^2+a^2)}}{a} \ldots\ldots (3.)$$

But since $e^{-\frac{y}{a}} = \frac{a}{\sqrt{(s^2+a^2)}+s} = \frac{\sqrt{(s^2+a^2)}-s}{a}$; therefore, by subtracting, $e^{\frac{y}{a}} - e^{-\frac{y}{a}} = \frac{2s}{a}$, or $s = \frac{a}{2}\left(e^{\frac{y}{a}} - e^{-\frac{y}{a}}\right) \ldots\ldots (4.)$

Lastly, from (1) we have $\frac{s}{a} = \frac{dx}{dy}$, and therefore (4) will become

$$\frac{dx}{dy} = \frac{1}{2}\left(e^{\frac{y}{a}} - e^{-\frac{y}{a}}\right), \text{ or } dx = \frac{1}{2}\left(e^{\frac{y}{a}} - e^{-\frac{y}{a}}\right)dy.$$

Integrating, we get $x = \frac{a}{2}\left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right) + \text{C}$, and when $x = 0$, $y = 0$; therefore $0 = \frac{1}{2}a\,(1+1) + \text{C}$; whence $\text{C} = -a$, and we get

$$x + a = \frac{a}{2}\left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right) \ldots\ldots (5.)$$

If we put for s its value in (1) we get $\frac{dy}{dx} = \frac{a}{\sqrt{(2ax + x^2)}}$, or

$$dy = \frac{a\,dx}{\sqrt{(2ax + x^2)}}; \text{ hence, } y = a \log \frac{x + a + \sqrt{(2ax + x^2)}}{a} \ldots (6.)$$

To find the area of the curve, we have

$$\int y\,dx = xy - \int x\,dy = xy - a\int \frac{x\,dx}{\sqrt{(2ax + x^2)}} \text{ by (6)}$$

$$= xy - a\sqrt{(2ax + x^2)} + a^2 \log \frac{x + a + \sqrt{(2ax + x^2)}}{a}$$

$= xy - as + ay$, by (2) and (6) $= y(x + a) - as =$ area C P M.

11. Find the area of the spiral whose equation is $r = a\,\theta^n$.

$$\text{Here } \tfrac{1}{2}\int r^2 d\theta = \frac{a^2}{2}\int \theta^{2n} d\theta = \frac{a^2\,\theta^{2n+1}}{2(2n+1)} = \frac{a^2\,\theta^{2n} \times \theta}{2(2n+1)}$$

$$= \frac{a^2}{2(2n+1)}\,\frac{r^2}{a^2}\cdot\left(\frac{r}{a}\right)^{\frac{1}{n}} = \frac{r^{\frac{2n+1}{n}}}{2(2n+1)a^{\frac{1}{n}}} + \text{C}.$$

Let $n = 1$, then the curve is the spiral of Archimedes, and its area

$$= \frac{r^3}{6a} + \text{C}.$$

If the area is measured from the point where $r = r'$, then will the area included by the two radii vectores r and $r' = \frac{r^3 - r'^3}{6a}$.

Let $n = -1$, then the curve is called the hyperbolic spiral, and its

$$\text{area} = \frac{a}{2}(r - r').$$

ADDITIONAL EXAMPLES IN FINDING AREAS.

1. Find the area of the Witch of Agnesi, whose equation is $xy^2 = 4r^2(2r - x)$. *Ans.* $\text{A} = 4\pi r^2$.

2. Find the area of the Cissoid of Diocles, its equation being $y^2(2r - x) = x^3$. *Ans.* $\text{A} = 3\pi r^2$.

3. Find the area of the curve whose equation is $r^2 = a^2 \cos 2\theta$. *Ans.* $\text{A} = a^2$.

4. Find the area of one of the loops of the curve whose equation is $r = a \sin 3\theta$. *Ans.* $\text{A} = \frac{1}{12}\pi a^2$.

5. Let A B $= 2a$, be a given straight line, and let it be bisected perpendicularly in O, by a straight line C D. From A draw the straight line A Q P, cutting C D in Q, and make Q P equal to O Q; find the area of the curve, which is the locus of P. *Ans.* $\text{A} = \frac{1}{2}\pi a^2 + 2a^2$.

6. If from the centre of an hyperbola, whose semiaxes are a and b, perpendiculars be drawn to tangents to the curve, the points of intersection will trace a curve called a *lemniscate;* find its area.

Ans. $\text{A} = ab + (a^2 - b^2)\tan^{-1}\frac{a}{b}$.

7. A straight line $BC = a$ is placed with its extremities on two straight lines AM, AN, at right angles to each other. If in BC there be taken $BP = BA$, the locus of P will be a curve whose area is required. *Ans.* $A = \cdot 0594\, a^2$.

8. Between the sides of a right angle, a straight line is drawn so as to include a given area $= a^2$; if from the right angle a perpendicular be drawn to this line, what will be the area of the curve which is the locus of the intersection? *Ans.* $A = \frac{1}{2} a^2$.

9. Let ACB be a given semicircle whose radius $AO = a$, and CD any ordinate; in the radius OC, take OP a mean proportional between the diameter AB and the ordinate CD; find the area of the curve which is the locus of P. *Ans.* $A = 2a^2$.

10. Find the area of the curve whose equation is $xy^2 = a^3$.
Ans. $A = 2xy$.

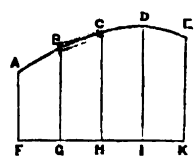
11. The *logarithmic* or *logistic* spiral is a line of such a nature that the angle contained by the fixed axis and the radius vector has a constant ratio to the logarithm of the radius vector.

Ans. If the constant ratio be $1 : \log a$, then $r = a^{\theta}$ is the equation of the spiral, and its area $= \frac{1}{4} M_a (r^2 - r'^2)$, r and r' being any two radii vectores.

4

APPROXIMATION TO THE AREA OF A CURVE.

119. If the coordinates of a sufficient number of points in the curve be given, its area may be determined to any degree of accuracy, by means of the equation of a curve of a given species which passes through the given points, and which nearly coincides with the proposed curve. The general equation of a parabola of any order is of the form $y = A + Bx + Cx^2 + Dx^3 + \ldots + Rx^n$, where the ordinate is always a rational function of the abscissa, and whose area is readily obtained from the formula $\int y\,dx$. Thus, if $y = A + Bx + Cx^2 + Dx^3$ be the equation of the parabolic curve, then its area will be



$$\int y\,dx = Ax + \tfrac{1}{2}Bx^2 + \tfrac{1}{3}Cx^3 + \tfrac{1}{4}Dx^4.$$

And though innumerable other curves may be made to pass through any number of points, yet a parabolic curve is preferable to any other curve, since its area can be determined with such facility.

Let A, B, C, D, E be points in a curve whose coordinates are all given, and let it be required to determine the area AEKFA. Since there are five given points, the equation of the parabolic curve must contain five undetermined coefficients; but to simplify the process, we may take only the three points A, B, C, and find the area of the space ABCHGFA.

Let $x_1 y_1$, $x_2 y_2$, $x_3 y_3$, etc. denote the coordinates of the points A, B, C, etc.; and let $y = A + Bx + Cx^2$ be the equation of the required parabolic curve; then, since the curve passes through the points A, B, C, we have the equations

$$y_1 = A + Bx_1 + Cx_1^2 \ldots\ldots (1.)$$
$$y_2 = A + Bx_2 + Cx_2^2 \ldots\ldots (2.)$$
$$y_3 = A + Bx_3 + Cx_3^2 \ldots\ldots (3.)$$

Subtracting (2) from (1), and (3) from (2); dividing the former remainder by $(x_1 - x_2)$, and the latter by $(x_2 - x_3)$, gives

$$\frac{y_1 - y_2}{x_1 - x_2} = B + C(x_1 + x_2) \ldots (4), \quad \frac{y_2 - y_3}{x_2 - x_3} = B + C(x_2 + x_3) \ldots (5.)$$

Subtracting (5) from (4,) and dividing by $(x_1 - x_3)$, we obtain

$$C = \frac{(y_1 - y_2)(x_2 - x_3) - (y_2 - y_3)(x_1 - x_2)}{(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)}.$$

Hence, also, B and A become known, and thence the equation and area of the curve are determined.

If $x_1 = 0$, and the distance $FG = GH = h$; then we get

$$C = \frac{y_1 - 2y_2 + y_3}{2h^2},\ B = -\frac{3y_1 - 4y_2 + y_3}{2h},\ A = y_1$$

and $$y = y_1 - \frac{3y_1 - 4y_2 + y_3}{2h}x + \frac{y_1 - 2y_2 + y_3}{2h^2}x^2;$$

hence $$\int_0^{2h} y\,dx = \tfrac{1}{3}h(y_1 + 4y_2 + y_3) = \text{area } ABCHGF.$$

In a similar manner, if a parabola be described through the points C, D, E, we get

area $CDEKIH = \frac{1}{3}h(y_3 + 4y_4 + y_5)$, where $HI = IK = h$;

and this, added to the former area, gives the entire area AEKFA equal to

$$\frac{h}{3}\left(y_1 + y_5 + 4y_2 + 4y_4 + 2y_3\right).$$

From this we deduce the following useful rule for finding the area of any portion of a curve, by means of equidistant ordinates.

To the sum of the extreme ordinates, add four times the sum of the even ordinates, and twice the sum of the odd ones; multiply this sum by one-third of the constant distance between the ordinates, and the product will be the area of the figure nearly.

This rule was first given by Simpson, and in his "Mécanique Industrielle," Poncelet has availed himself of it to estimate variable work in general, and especially to determine the work done upon each square inch of the piston of a steam-engine. But the process for determining the equation, and thence the area of a parabolic curve which shall pass through any number of given points, is more general, and may be employed in all cases where the equation of the curve is not known.

Ex. Find the equation and area of a parabolic curve passing through the four points A, B, C, D, whose coordinates are respectively 0, 6; 3, 7; 6, 8; and 10, 9.

Let $y = A + Bx + Cx^2 + Dx^3$ be the required equation; then substituting for x and y in this equation, the given values of the coordinates of the four points, we get

$$6 = A \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$
$$7 = A + 3B + 9C + 27D \ldots\ldots (2.)$$
$$8 = A + 6B + 36C + 216D \ldots\ldots (3.)$$
$$9 = A + 10B + 100C + 1000D \ldots\ldots (4.)$$

From these we obtain, $A = 6,\ B = \frac{131}{420},\ C = \frac{3}{280},\ D = -\frac{1}{840},$

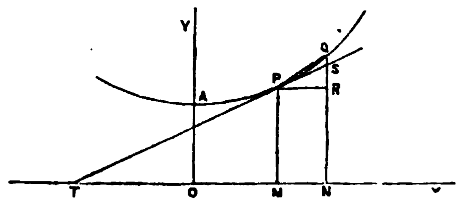
and the equation of the curve is, $y = 6 + \frac{131}{420}x + \frac{3}{280}x^2 - \frac{1}{840}x^3$;

$$\text{whence } \int y\,dx = 6x + \frac{131}{840}x^2 + \frac{1}{280}x^3 - \frac{1}{3360}x^4 + \text{C}.$$

When $x = 0$, then the area $= 0$, and $\text{C} = 0$; and when $x = 10$, the area of the parabolic curve is $= 76 \cdot 190476$, which is a near approximation to the area of the curve A D I F A.

II. *Lengths of Curves.*

120. Let A P Q be a curve, to which T P is a tangent at the point P, it is required to find the length of the arc A P.



The coordinates O M, MP, being, as usual, denoted by x, y, and the arc A P by s; then however near any two points P, Q, in the curve may be taken, we shall have,

$$\text{arc PQ} > \text{chord PQ} < \text{PS} + \text{SQ} \quad \ldots\ldots \quad (\alpha).$$

Now s is evidently a function of x, and if $\text{MN} = h$, $\text{NQ} = y'$, and arc $\text{APQ} = s'$; then by Taylor's theorem we have,

$$s' = s + \frac{ds}{dx}h + \frac{d^2s}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3s}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.},$$

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.};$$

$$\therefore \text{arc PQ} = \frac{ds}{dx}h + \frac{d^2s}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3s}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.}\ldots(1),$$

$$\text{and QR} = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.}$$

$$\text{Again, QS} = \text{QR} - \text{RS} = h\tan\text{QPR} - h\tan\text{SPR}$$

$$= h\left(\frac{\text{QR}}{\text{PR}} - \frac{dy}{dx}\right) = h\left(\frac{\text{QR}}{h} - \frac{dy}{dx}\right)$$

$$= h\left(\frac{d^2y}{dx^2}\cdot\frac{h}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^2}{1.2.3} + \text{etc.}\right);$$

$$\therefore \text{PS}+\text{SQ} = h\sec\text{SPR} + \text{QS}$$

$$= h\left\{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} + \frac{d^2y}{dx^2}\cdot\frac{h}{1.2} + \text{etc.}\right\} \quad \ldots\ldots \quad (2.)$$

$$\text{Lastly, chord PQ} = \sqrt{(\text{PR}^2 + \text{QR}^2)} = \sqrt{(h^2 + \text{QR}^2)}$$

$$= h\sqrt{\left\{\left(1 + \frac{dy^2}{dx^2}\right) + \frac{2\,dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{h}{1.2} + \text{etc.}\right\}}$$

$$= h\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} + \frac{1}{2}\left(1+\frac{dy^2}{dx^2}\right)^{-\frac{1}{2}}\frac{2\,dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{h}{1.2}+\text{etc.}\ldots(3.)$$

Now by (α) the series (1) is less than (2), but greater than (3); therefore, by the lemma (Art. 116),

$$\frac{ds}{dx} = \left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}},\ \text{or}\ ds = dx\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = (dx^2+dy^2)^{\frac{1}{2}};$$

$$\therefore\ s = \int dx\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\ \text{or}\ \int (dx^2+dy^2)^{\frac{1}{2}}\ \ldots\ldots\ (\text{C.})$$

121. To find a general expression for the length of an arc of a curve in terms of polar coordinates, we have $x = r\cos\theta$, and $y = r\sin\theta$; hence, $dx = dr\cos\theta - r\sin\theta\,d\theta$, $dy = dr\sin\theta + r\cos\theta\,d\theta$;

$$\therefore\ s = \int (dx^2+dy^2)^{\frac{1}{2}} = (dr^2+r^2\,d\theta^2)^{\frac{1}{2}}\ \ldots\ldots\ (\text{D.})$$

EXAMPLES.

1. Find the length of the arc of the common parabola.

The equation of the curve is $y^2 = 4\,ax$; hence $y = 2\,a^{\frac{1}{2}}x^{\frac{1}{2}}$, and

$$dy = a^{\frac{1}{2}}x^{-\frac{1}{2}}\,dx,\ \therefore\ ds = \sqrt{(dx^2+dy^2)} = \left(\frac{x+a}{x}\right)^{\frac{1}{2}}dx.$$

Put $\dfrac{x+a}{x} = z^2$; then $x = \dfrac{a}{z^2-1}$ and $dx = -\dfrac{2\,az\,dz}{(z^2-1)^2}$;

$$\text{whence } s = \int z\,dx = -2\,a\int\frac{z^2\,dz}{(z^2-1)^2} = a\int z\cdot\frac{-2\,z\,dz}{(z^2-1)^2}$$

$$= \frac{az}{z^2-1} - a\int\frac{dz}{z^2-1} = \frac{az}{z^2-1} - \frac{a}{2}\int\frac{dz}{z-1} + \frac{a}{2}\int\frac{dz}{z-1}$$

$$= \frac{az}{z^2-1} + \frac{a}{2}\log\frac{z+1}{z-1} + \text{C} = zx + \frac{a}{2}\log\frac{(z+1)^2}{z^2-1} + \text{C}.$$

But $z = \left(\dfrac{x+a}{x}\right)^{\frac{1}{2}}$, $z^2-1 = \dfrac{a}{x}$ and $(z+1)^2 = \dfrac{a+2x+2\,(x^2+ax)^{\frac{1}{2}}}{x}$;

$$\text{therefore } s = (x^2+ax)^{\frac{1}{2}} + \frac{a}{2}\log\frac{a+2x+2\,(x^2+ax)^{\frac{1}{2}}}{a} + \text{C}.$$

If the length of the arc is measured from the vertex, then x and s vanish simultaneously, and therefore $\text{C} = 0$.

Ex. Find the length of the path described by a shot which ranges 6000 feet on a horizontal plane, its greatest height above the plane being 2000 feet.

Here $x = 2000$, $y = 3000$, and $a = \dfrac{y^2}{4x} = 1125$; consequently

$$s = 2500 + \frac{1125}{2}\log 9 = 2500 + \frac{1125}{2}\times 2\cdot 1972246 = 3735\cdot 9388 \text{ feet},$$

and the entire length of the path is therefore $= 7471\cdot 8776$ feet.

2. Find the length of the semi-cubical parabola, whose equation is $a x^2 = y^3$.

Here $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, $dx = \frac{3 y^{\frac{1}{2}}\, dy}{2 a^{\frac{1}{2}}}$; consequently $s = \int (dx^2 + dy^2)^{\frac{1}{2}}$

$$= \int \left(\frac{9 y}{4 a} + 1\right)^{\frac{1}{2}} dy = \frac{1}{2 a^{\frac{1}{2}}} \int (4a + 9y)^{\frac{1}{2}} dy, \text{ or } s = \frac{(4a + 9y)^{\frac{3}{2}}}{27 a^{\frac{1}{2}}} + C.$$

If $y = 0$, then $s = 0$, and $0 = \frac{8 a}{27} + C$; therefore $C = -\frac{8 a}{27}$,

$$\text{and } s = \frac{(4a + 9y)^{\frac{3}{2}}}{27 a^{\frac{1}{2}}} - \frac{8 a}{27}.$$

3. Find the length of the cycloid.

The equation of the curve is $y = r \operatorname{vers}^{-1} \frac{x}{r} + (2 r x - x^2)^{\frac{1}{2}}$;

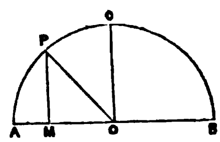
$$\text{hence } dy = \frac{r\, dx}{(2 r x - x^2)^{\frac{1}{2}}} + \frac{r\, dx - x\, dx}{(2 r x - x^2)^{\frac{1}{2}}} = \left(\frac{2 r - x}{x}\right)^{\frac{1}{2}} dx;$$

$$\therefore s = \int (dx^2 + dy^2)^{\frac{1}{2}} = \int \frac{(2 r)^{\frac{1}{2}}\, dx}{x^{\frac{1}{2}}} = (2 r)^{\frac{1}{2}} \int x^{-\frac{1}{2}} dx = 2 (2 r x)^{\frac{1}{2}};$$

hence the arc C P = twice the chord C Q (see fig. Art. 50, *Ex.* 4), and when $x = 2 r$, we have $s = 4 r$; therefore the length of the entire curve is

$$2 s = 8 r = 4 \text{ times the diameter C D}.$$

4. To find the length of the circular arc C P.



Let O M $= x$, M P $= y$, and O A $= r$, then we have $y = \sqrt{(r^2 - x^2)}$, and

$$dy = -\frac{1}{2} . 2 x\, dx\, (r^2 - x^2)^{-\frac{1}{2}} = -\frac{x\, dx}{\sqrt{(r^2 - x^2)}};$$

$$\text{consequently } s = \int (dx^2 + dy^2)^{\frac{1}{2}} = r \int \frac{dx}{\sqrt{(r^2 - x^2)}} = r \int (r^2 - x^2)^{-\frac{1}{2}} dx;$$

$$\therefore s = \int \left(dx + \frac{x^2\, dx}{2 r^2} + \frac{1.3}{2.4} \cdot \frac{x^4\, dx}{r^4} + \frac{1.3.5}{2.4.6} \cdot \frac{x^6\, dx}{r^6} + \ldots\right)$$

$$= x + \frac{1}{3} \cdot \frac{x^3}{2 r^2} + \frac{1}{5} \cdot \frac{1.3}{2.4} \cdot \frac{x^5}{r^4} + \frac{1}{7} \cdot \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{r^6} + \ldots (1),$$

which is the length of the circular arc C P in terms of O M. If angle A O P $= 60^\circ = \frac{\pi}{3}$; then O M $= x = \frac{r}{2}$, and we get arc C P, or

$$\frac{\pi}{6} = r \left(\frac{1}{2} + \frac{1}{2^3 . 2 . 3} + \frac{1.3}{2^6 . 2 . 4 . 5} + \frac{1.3.5}{2^7 . 2 . 4 . 6 . 7} + \text{etc.}\right) = \cdot 5235987\, r;$$

$$\therefore \pi = 3 \cdot 14159\, r = \text{semi-circumference A P B}.$$

The series (1) is not well fitted for computation, as it does not converge with sufficient rapidity unless the arc be very small. We may find others in the following manner:

Let $s = \tan^{-1}x$; then $ds = \frac{dx}{1+x^2}$, which, by common division,

$$\text{gives } ds = dx\,(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots\ldots);$$

$$\therefore s \text{ or } \tan^{-1}x = x\left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \frac{1}{9}x^8 - \frac{1}{11}x^{10} + \ldots\right) \ldots (2),$$

a series which still converges very slowly, unless x is very small.

But if we put $\tan(a+b) = m$, or $a + b = \tan^{-1}m$,

$$\tan a = n, \text{ or } a = \tan^{-1}n,$$

$$\tan b = p, \text{ or } b = \tan^{-1}p,$$

$$\text{then will } \tan^{-1}m = \tan^{-1}n + \tan^{-1}p \ldots\ldots\ldots\ldots (3).$$

$$\text{Again, } \tan b = \tan(a+b-a) = \frac{\tan(a+b) - \tan a}{1 + \tan a \tan(a+b)};$$

$$\therefore p = \frac{m-n}{1+mn} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If $m = 1$, $n = \frac{1}{5}$, then $p = \frac{2}{3}$, $\therefore \tan^{-1}1$ or $\frac{\pi}{4} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{2}{3}$

$m = \frac{2}{3}$, $n = \frac{1}{5}$, $p = \frac{7}{17}$, $\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{7}{17}$

$m = \frac{7}{17}$, $n = \frac{1}{5}$, $p = \frac{9}{46}$, $\tan^{-1}\frac{7}{17} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{9}{46}$

$m = \frac{9}{46}$, $n = \frac{1}{5}$, $p = -\frac{1}{239}$, $\tan^{-1}\frac{9}{46} = \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$m = \frac{1}{239}$, $n = \frac{1}{70}$, $p = -\frac{1}{99}$, $\tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}$

$$\therefore \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} \ldots\ldots \quad (5).$$

By means of (2) and (5) we have

$$\left.\begin{aligned} \frac{\pi}{4} = \frac{4}{5} - \frac{1}{3}\cdot\frac{4}{5^3} + \frac{1}{5}\cdot\frac{4}{5^5} - \frac{1}{7}\cdot\frac{4}{5^7} + \frac{1}{9}\cdot\frac{4}{5^9} - \text{etc.}\ldots \\ -\left(\frac{1}{70} - \frac{1}{3}\cdot\frac{1}{70^3} + \frac{1}{5}\cdot\frac{1}{70^5} - \frac{1}{7}\cdot\frac{1}{70^7} - \text{etc.}\right)\ldots \\ + \frac{1}{99} - \frac{1}{3}\cdot\frac{1}{99^3} + \frac{1}{5}\cdot\frac{1}{99^5} - \frac{1}{9}\cdot\frac{1}{99^7} + \text{etc.}\ldots\ldots \end{aligned}\right\} \ldots (6).$$

This series is well adapted for computation, and it will readily furnish the value of π to any extent that may be required.

Calculation of the Value of π.

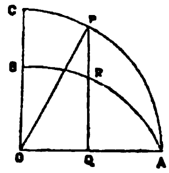
5	$4\cdot0000000000$			
25	$\cdot8000000000 = \frac{4}{5}$; $\therefore$	$\frac{4}{5} =$	$\cdot8000000000$	
25	$\cdot0320000000 = \frac{4}{5^3}$	$-\frac{1}{3}\cdot\frac{4}{5^3} =$		$-\cdot0106666667$
25	$\cdot0012800000 = \frac{4}{5^5}$	$\frac{1}{5}\cdot\frac{4}{5^5} =$	$\cdot0002560000$	
25	$\cdot0000512000 = \frac{4}{5^7}$	$-\frac{1}{7}\cdot\frac{4}{5^7} =$		$-\cdot0000073143$
25	$\cdot0000020480 = \frac{4}{5^9}$	$\frac{1}{9}\cdot\frac{4}{5^9} =$	$\cdot0000002275$	
25	$\cdot0000000819 = \frac{4}{5^{11}}$	$-\frac{1}{11}\cdot\frac{4}{5^{11}} =$		$-\cdot0000000074$
	$\cdot0000000033 = \frac{4}{5^{13}}$	$\frac{1}{13}\,\frac{4}{5^{13}} =$	$\cdot0000000003$	
70	$1\cdot0000000000$			
70	$\cdot0142857143 = \frac{1}{70}$; $\therefore$	$-\frac{1}{70} =$		$-\cdot0142857143$
70	$\cdot0002040816$			
70	$\cdot0000029154 = \frac{1}{70^3}$	$\frac{1}{3}\cdot\frac{1}{70^3} =$	$\cdot0000009718$	
70	$\cdot0000000416$			
	$\cdot0000000006 = \frac{1}{70^5}$	$-\frac{1}{5}\cdot\frac{1}{70^5} =$		$-\cdot0000000001$
99	$1\cdot0000000000$			
99	$\cdot0101010101 = \frac{1}{99}$; $\therefore$	$\frac{1}{99} =$	$\cdot0101010101$	
99	$\cdot0001020304$			
	$\cdot0000010306 = \frac{1}{99^3}$	$-\frac{1}{3}\cdot\frac{1}{99^3} =$		$-\cdot0000003435$
			$\cdot8103582097$	$-\cdot0249600463$
			$\cdot0249600463$	
		$\therefore \frac{\pi}{4} =$	$\cdot7853981634$	
			4	
		$\therefore \pi =$	$3\cdot1415926536$	

Thus the value of π has been calculated to 9 or 10 decimals, by taking only 7 terms of the first series, 3 of the second, and 2 of the third.

5. To find the length of an arc of the ellipse.

The equation of the curve is $a^2 y^2 = b^2 (a^2 - x^2)$, and let a quadrantal arc of a circle be described on the major semi-axis O A. Produce the ordinate Q R to meet the circle in P, and join O P. Let

angle COP $= \theta$; then $x = a \sin \theta$, and $a^2 y^2 = a^2 b^2 \cos^2\theta$, or $y = b \cos \theta$; therefore



$$ds^2 = dx^2 + dy^2 = (a^2 \cos^2\theta + b^2 \sin^2\theta)\, d\theta^2$$
$$= \{a^2 - (a^2 - b^2) \sin^2\theta\}\, d\theta^2 \ldots\ldots\ldots (1).$$
$$= a^2 (1 - e^2 \sin^2\theta)\, d\theta^2, \text{ if } a^2 - b^2 = a^2 e^2.$$

The value of ds deduced from (1) may be expanded in a series in two different ways. The first method consists in developing $(1 - e^2 \sin^2\theta)^{\frac{1}{2}}$ in a series ascending by the powers of $\sin^2\theta$; thus we have

$$ds = a\,(1 - e^2 \sin^2\theta)^{\frac{1}{2}}\, d\theta$$
$$= a\left(1 - \frac{e^2}{2} \sin^2\theta - \frac{e^4}{2.4} \sin^4\theta - \frac{3\,e^6}{2.4.6} \sin^6\theta - \frac{3.5\,e^8}{2.4.6.8} \sin^8\theta - \text{etc.}\right) d\theta;$$
$$\therefore s = a\left\{\int d\theta - \frac{e^2}{2}\int \sin^2\theta\, d\theta - \frac{e^4}{2.4}\int \sin^4\theta\, d\theta - \frac{3\,e^6}{2.4.6}\int \sin^6\theta\, d\theta - \text{etc.}\right\}.$$

Now, by Art. 108, we have generally

$$\int \sin^n\theta\, d\theta = -\frac{1}{n} \sin^{n-1}\theta \cos\theta + \frac{n-1}{n}\int \sin^{n-2}\theta\, d\theta \;..\; (2).$$

Writing 2, 4, 6, etc., successively for n in formula (2), we get

$$\int \sin^2\theta\, d\theta = -\frac{1}{2.2} \sin 2\theta + \frac{1}{2}\,\theta = \text{A, suppose,}$$
$$\int \sin^4\theta\, d\theta = -\frac{1}{2.4} \sin 2\theta \sin^2\theta + \frac{3}{4}\,\text{A} = \text{B},$$
$$\int \sin^6\theta\, d\theta = -\frac{1}{2.6} \sin 2\theta \sin^4\theta + \frac{5}{6}\,\text{B} = \text{C};$$
$$\&\text{c.} \qquad\qquad \&\text{c.}$$

hence $s = a\left(\theta - \dfrac{e^2}{2}\,\text{A} - \dfrac{e^4}{2.4}\,\text{B} - \dfrac{3\,e^6}{2.4.6}\,\text{C} - \dfrac{3.5\,e^8}{2.4.6.8}\,\text{D} - \text{etc.}\right).$

Let $x = a$, or $\theta = \frac{1}{2}\pi$, then the length of the elliptic quadrant BRA is

$$\text{BRA} = \frac{a\,\pi}{2}\left(1 - \frac{e^2}{2^2} - \frac{3\,e^4}{2^2.4^2} - \frac{3^2.5\,e^6}{2^2.4^2.6^2} - \frac{3^2.5^2.7\,e^8}{2^2.4^2.6^2.8^2} - \text{etc.}\right) ..(3).$$

This series does not converge rapidly, unless e is small, and the ellipse differs very little from a circle.

The second method, referred to above, consists in expanding the value of ds in a series of multiple arcs; thus we have from (1),

$$\frac{ds^2}{d\theta^2} = a^2 - (a^2 - b^2) \sin^2\theta = a^2 - (a^2 - b^2)\left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right)$$
$$= a^2 - \frac{a^2 - b^2}{2} + \frac{a^2 - b^2}{2} \cos 2\theta = \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos 2\theta$$
$$= a^2\left\{\frac{a^2 + 2ab + b^2}{4a^2} + \frac{a^2 - 2ab + b^2}{4a^2} + \frac{2(a^2 - b^2)}{4a^2} \cos 2\theta\right\}$$
$$= a^2\left\{\left(\frac{a+b}{2a}\right)^2 + \left(\frac{a-b}{2a}\right)^2 + 2\,\frac{a+b}{2a} \cdot \frac{a-b}{2a} \cos 2\theta\right\}.$$

Let $\dfrac{a+b}{2a} = m$, and $\dfrac{a-b}{2a} = n$; then $\dfrac{ds^2}{d\theta^2} = a^2\,(m^2 + n^2 + 2mn \cos 2\theta.)$

By the exponential theorem (ALGEBRA, Art. 137),

$$e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}=2\left(1-\frac{\theta^2}{1.2}+\frac{\theta^4}{1.2.3.4}-\text{etc.}\right)=2\cos\theta;$$

$\therefore\ 2\cos 2\theta = e^{2\theta\sqrt{-1}}+e^{-2\theta\sqrt{-1}}$; and the last equation gives

$$ds = a\left(m+ne^{2\theta\sqrt{-1}}\right)^{\frac{1}{2}}\left(m+ne^{-2\theta\sqrt{-1}}\right)^{\frac{1}{2}}d\theta$$

$$= am\left(1+\frac{n}{m}e^{2\theta\sqrt{-1}}\right)^{\frac{1}{2}}\left(1+\frac{n}{m}e^{-2\theta\sqrt{-1}}\right)^{\frac{1}{2}}d\theta \ldots\ldots (4).$$

Now, by the binomial theorem,

$$\left(1+\frac{n}{m}e^{2\theta\sqrt{-1}}\right)^{\frac{1}{2}}=1+\frac{1}{2}\cdot\frac{n}{m}e^{2\theta\sqrt{-1}}-\frac{1}{2.4}\cdot\frac{n^2}{m^2}e^{4\theta\sqrt{-1}}+\text{etc.}$$

$$\left(1+\frac{n}{m}e^{-2\theta\sqrt{-1}}\right)^{\frac{1}{2}}=1+\frac{1}{2}\cdot\frac{n}{m}e^{-2\theta\sqrt{-1}}-\frac{1}{2.4}\cdot\frac{n^2}{m^2}e^{-4\theta\sqrt{-1}}+\text{etc.};$$

therefore, by multiplying the former series by the latter, and recollecting that

$$2\cos 2\theta=e^{2\theta\sqrt{-1}}+e^{-2\theta\sqrt{-1}},\ 2\cos 4\theta=e^{4\theta\sqrt{-1}}+e^{-4\theta\sqrt{-1}}-\text{etc.},$$

$$ds = am\left\{\begin{array}{l}1+\dfrac{1}{2^2}\cdot\dfrac{n^2}{m^2}+\dfrac{1}{2^2.4^2}\cdot\dfrac{n^4}{m^4}+\dfrac{1^2.3^2}{2^2.4^2.6^2}\cdot\dfrac{n^6}{m^6}+\&\text{c.}\\ +P\cos 2\theta+Q\cos 4\theta+R\cos 6\theta+\ldots\ldots\end{array}\right\}d\theta,$$

where P, Q, R, etc., are functions of $\frac{n}{m}$.

Integrating this differential expression, and writing for m its value

$$s=\frac{a+b}{2}\left\{\begin{array}{l}\left(1+\dfrac{1}{2^2}\cdot\dfrac{n^2}{m^2}+\dfrac{1}{2^2.4^2}\cdot\dfrac{n^4}{m^4}+\dfrac{1^2.3^2}{2^2.4^2.6^2}\cdot\dfrac{n^6}{m^6}+\text{etc.}\right)\theta\\ \quad+\frac{1}{2}P\sin 2\theta+\frac{1}{4}Q\sin 4\theta+\frac{1}{6}R\sin 6\theta+\text{etc.}\end{array}\right\},$$

which vanishes when $\theta=0$, as it ought, and when $\theta=\frac{1}{2}\pi$, we get arc B R A

$$=\frac{(a+b)\pi}{4}\left\{1+\frac{1}{2^2}\left(\frac{a-b}{a+b}\right)^2+\frac{1}{2^2.4^2}\left(\frac{a-b}{a+b}\right)^4+\frac{1^2.3^2}{2^2.4^2.6^2}\left(\frac{a-b}{a+b}\right)^6+\ldots\right\}(5),$$

and, therefore, the entire elliptic circumference will be four times the value of this series.

Ex. Let the axes be 24 and 18; then $a = 12$, $b = 9$, and arc B R A

$$=\frac{21\pi}{4}\left\{1+\frac{1}{2^2}\cdot\frac{1}{7^2}+\frac{1}{2^2.4^2}\cdot\frac{1}{7^4}+\frac{1^2.3^2}{2^2.4^2.6^2}\cdot\frac{1}{7^6}+\ldots\right\}$$

$= 16{\cdot}5776191198$; hence the periphery of the ellipse is $= 66{\cdot}3104764792$.

EXAMPLES FOR PRACTICE.

1. To find the length of the involute of a circle, its equation being $a\theta=\sqrt{(r^2-a^2)}-a\sec^{-1}\frac{r}{a}$. *Ans.* $s=\frac{1}{2}a\theta^2$.

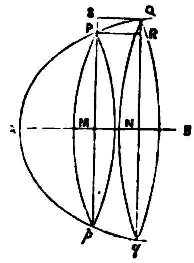
2. Find the length of the curve whose equation is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.
Ans. $s=\frac{3}{2}a^{\frac{1}{3}}x^{\frac{2}{3}}+C$.

3. To find the length of the arc of an hyperbola.
Ans. If l = the length of the asymptote corresponding to the arc s; then, a^2+b^2 being $=a^2e^2$,

$$l - s = \frac{\pi a}{2}\left\{\frac{1}{2e} + \frac{1}{2^2}\cdot\frac{1}{4e^3} + \frac{1^2.3^2}{2^2.4^2}\cdot\frac{1}{6e^5} + \frac{1^2.3^2.5^2}{2^2.4^2\,6^2}\cdot\frac{1}{8e^7} + \ldots\right\}.$$

III. *Volumes of Solids.*

122. Let A P Q be a curve, and let it be required to determine the volume of the solid formed by the revolution of the area A P M about the axis A B.



Let M N $= h$, and draw N Q parallel to P M; through P and Q draw P R and Q S parallel to A B; then the rectangles P N, M Q, will generate cylinders, the radii of whose bases are P M and Q N, and their common altitude is M N. Now, the solid generated by the revolution of the space P Q N M is always less than the cylinder generated by the rectangle Q M, and greater than the cylinder generated by the rectangle P N, however small M N or h may be taken.

Let V and V′ be the volumes of the solids generated by A P M and A Q N; then, if A M $= x$, M P $= y$, N Q $= y'$, we shall have, since both V and y are functions of x,

$$V' = V + \frac{dV}{dx}h + \frac{d^2V}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3V}{dx^3}\cdot\frac{h^3}{1.2.3} + \ldots.$$

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \ldots..$$

$$\therefore\ V' - V = \frac{dV}{dx}h + \frac{d^2V}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3V}{dx^3}\cdot\frac{h^3}{1.2.3} + \text{etc.}\ \ldots.\ (1).$$

Now, the volume of the cylinder generated by the rectangle M Q is

$$\pi y'^2 h = \pi\left(y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \ldots..\right)^2 h$$

$$= \pi\left\{y^2 + 2y\frac{dy}{dx}h + \left(\frac{dy^2}{dx^2} + \frac{y\,d^2y}{dx^2}\right)h^2 + \ldots\right\}h\ ..\ (2).$$

The volume of the cylinder generated by the rectangle P N is $\pi y^2 h$; hence the series (1) is always less than (2), and greater than $\pi y^2 h$, however small h may be taken; therefore, by the lemma (Art. 116),

$$\frac{dV}{dx} = \pi y^2\ \therefore\ dV = \pi y^2\,dx, \text{ and } V = \pi\int y^2\,dx\ \ldots\ldots\ (A).$$

123. If the solid be not one of revolution, but one whose base is a plane figure, as A P M, bounded by a curve and its coordinates, and every section of which parallel to the base is equal and similar to the base; then, if A be the altitude of the solid, we shall have series (1) less than A y' h, and greater than A y h; that is,

$$\frac{dV}{dx}h + \frac{d^2V}{dx^2}\cdot\frac{h^2}{1.2} + \ldots. < A\left(y + \frac{dy}{dx}h + \ldots\right)h > Ayh;$$

hence, by the lemma, $\frac{dV}{dx} = Ay$, or $dV = Ay\,dx$;

$$\therefore\ V = A\int y\,dx = A \times \text{area of the base}\ \ldots.\ (B).$$

124. Since $dV = \pi y^2 \times dx$, and πy^2 = the area of the circular section Pp, the solid of revolution may be considered as generated by the motion of a circle whose radius is y, the centre being always in A B, and its plane perpendicular to it. It is obvious that PM or y must vary so that its extremity P may trace the given curve A P Q, and hence *the differential of the volume is equal to the generating area multiplied by dx.* This theorem is true, whatever be the generating area, provided all the sections parallel to Pp are similar, and can therefore be expressed by the same general equation; hence,

dV, or element of the solid = area of section $Pp \times dx$ ..... (C).

EXAMPLES.

4

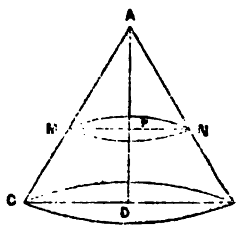
1. To find the volume of a right cone.

Let the height A D = a, and the radius of the base CD = r; then, if x denote A P any variable distance from the vertex, and y the radius P M of the circular section, we have, by similar triangles, $y = \frac{r}{a}x$;

$$\text{whence } dV = \pi y^2 dx = \frac{\pi r^2}{a^2} x^2 dx;$$

$$\therefore V = \frac{\pi r^2}{a^2}\int_0^x x^2 dx = \frac{\pi r^2 x^3}{3a^2} = \pi y^2 . \frac{x}{3}$$

= content of cone A M N.



When $x = a$, then the content of the entire cone ABC is $V = \pi r^2 . \frac{a}{3}$

= area of the base $\times$ $\frac{1}{3}$ of the height.

If r, r' denote the radii of the two ends of the frustum of a right cone, and h the altitude; then by the preceding results we have content of frustum BCMN $= \pi r^2 . \frac{a}{3} - \pi r'^2 . \frac{a-h}{3}$;

but since $a : a - h :: r : r'$; $\therefore a - h = \frac{ar'}{r}$; hence $h = \frac{a(r-r')}{r}$,

and frustum BCMN $= \frac{\pi a}{3} . \frac{r^3 - r'^3}{r} = \frac{\pi}{3} . \frac{a(r-r')}{r} . (r^2 + rr' + r'^2)$

$$= \frac{h}{3}(\pi r^2 + \pi rr' + \pi r'^2) \text{ or } \frac{\pi h}{3}(r^2 + rr' + r'^2).$$

But πr^2 = area of the base, $\pi r'^2$ = area of the top, and $\pi rr'$ is a mean proportional between these two areas; hence the following rule for finding the content of the frustum of a cone.

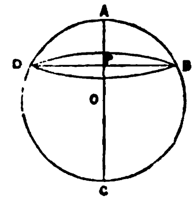
Add together the areas of the two ends, and the mean proportional between them; multiply the sum by one-third of the height, and the product is the content of the frustum.

In a similar manner it may be shown, that the volume of a conical solid, whose base is any given curve, or the volume of a pyramid, is equal to *the area of the base multiplied by one-third of the altitude.*

2. To find the volume of a sphere.

Let $AP = x$, $PD = y$, and the radius $OA = r$; then we have $y^2 = 2rx - x^2$, and hence

$dV = \pi y^2 \, dx = \pi(2rx - x^2)\,dx$; consequently



$$V = 2\pi r \int x\,dx - \pi \int x^2\,dx = \pi r x^2 - \frac{\pi x^3}{3} + C.$$

When $x = 0$, then $V = 0$; therefore $C = 0$, and hence the volume of the segment ADB $= \pi x^2 (r - \frac{1}{3}x) = \frac{\pi}{6} x^2 (6r - 2x)$, which affords the rule for the content of a spherical segment, viz.:—

From three times the diameter of the sphere, take twice the height of the segment; multiply the difference by the square of the height, and by $\frac{1}{6}\pi$ or · 5236 nearly.

When $x = 2r$, the content of the entire sphere is $= \frac{4}{3}\pi r^3 = \frac{\pi}{6}(2r)^3$, which is two-thirds the circumscribed cylinder, its content being $= \pi r^2 \times 2r = 2\pi r^3$.

3. To find the volume of the paraboloid.

The paraboloid is formed by the revolution of a semiparabola about its axis, and hence we have the equation $y^2 = 4ax$; therefore

$$dV = \pi y^2\,dx = 4a\pi x\,dx, \text{ and } V = 2\pi a x^2,$$

$$\text{or, } V = \frac{4\pi a x^2}{2} = \tfrac{1}{2}\pi y^2 x = \tfrac{1}{2} \text{ circumscribed cylinder.}$$

4 4. To find the volumes of the prolate and oblate spheroids.

The prolate spheroid is formed by the revolution of a semiellipse about its major axis, and its equation is $a^2 y^2 = b^2 (a^2 - x^2)$; whence

$$V = \frac{\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{\pi b^2}{a^2}(a^3 - \tfrac{1}{3}a^3) = \frac{2}{3}\pi a b^2;$$

and the volume of the entire solid is $= \frac{4}{3}\pi a b^2$.

The oblate spheroid is formed by the revolution of a semiellipse about its minor axis, and the equation is $b^2 y^2 = a^2 (b^2 - x^2)$; whence

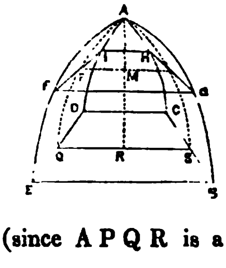
$$V = \frac{\pi a^2}{b^2}\int_0^b (b^2 - x^2)\,dx = \frac{\pi a^2}{b^2}(b^3 - \tfrac{1}{3}b^3) = \frac{2}{3}\pi a^2 b,$$

and the volume of the whole solid $= \frac{4}{3}\pi a^2 b$.

Comparing these results, we find that $\dfrac{\text{prolate spheroid}}{\text{oblate spheroid}} = \dfrac{b}{a}$.

$$\text{Also, } \frac{\text{sphere on major axis}}{\text{prolate spheroid}} = \frac{a^2}{b^2} = \frac{\text{oblate spheroid}}{\text{sphere on minor axis}}.$$

5. The axes of two equal right circular cylinders intersect at right angles; find the volume of the solid common to both.



Let a plane pass through both the axes of the cylinders, dividing the solid into two equal parts, of which the figure $ABCDE$, called a *circular groin*, is one of them. Then the sections of the solid parallel to the square base $BCDE$ are also squares, and those perpendicular to the base, and parallel to one side of it, are semicircles. Let AR be perpendicular to the base, intersecting the section $FGIH$ paralled to it in M.

Let $AM = x$, $MP = y$, $AR = RQ = a$ (since $APQR$ is a

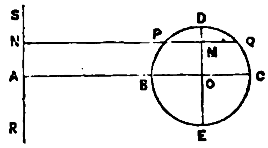
quadrant of a circle); then the area of the section parallel to the base is $= 4\,y^2$; therefore (C)

$$dV = \text{generating area} \times dx = 4\,y^2\,dx = 4\,(2\,a\,x - x^2)\,dx;$$

$$\therefore\ V = 8\,a \int_0^a x\,dx - 4\int_0^a x^2\,dx = 4\,a^3 - \frac{4}{3}\,a^3 = \frac{8\,a^3}{3}.$$

The double of this, viz., $\frac{16}{3}\,a^3$ is the volume of the solid common to both cylinders.

6. To find the content of the solid ring formed by the revolution of a circle round an axis in its own plane.



Let B D C E be the given circle whose centre is O, and R S the given axis in the plane of the circle; draw O A perpendicular to R S, and N P Q parallel to A O, intersecting the diameter D E, which is at right angles to B C in the point M.

Let O M $= x$, M P $= y$, O A $= a$, and O B $= r$; then the area generated by P Q is $= \pi\,(\mathrm{N\,Q}^2 - \mathrm{N\,P}^2) = \pi\,\{(a + y)^2 - (a - y)^2\} = 4\,\pi\,a\,y;$
$\therefore\ dV = 4\,\pi\,a\,y \times dx$, and $V = 4\,\pi\,a \int y\,dx = 4\,\pi\,a \times$ area B O M P.
When $x = r$, then $V = 4\,\pi\,a \times \frac{1}{4}\pi\,r^2 = \pi^2\,a\,r^2$, and the volume of the entire ring $= 2\,\pi^2\,a\,r^2$.

Let the semicircle D B E revolve round the axis R S; then the area generated by P M is $= \pi\,a^2 - \pi\,(a - y)^2 = \pi\,(2\,a\,y - y^2)$; hence we have $V = 2\,a\,\pi \int y\,dx - \pi \int y^2\,dx = 2\,a\,\pi \int y\,dx - \pi \int (2\,r\,x - x^2)\,dx$
$= 2\,a\,\pi \times$ area B O M P $- \pi\,r\,x^2 + \frac{1}{3}\,\pi\,x^3$.
When $x = r$, then $V = \frac{1}{2}\,\pi^2\,a\,r^2 - \frac{2}{3}\,\pi\,r^3 = \frac{1}{6}\,\pi\,r^2\,(3\,\pi\,a - 4\,r)$, and this being doubled, gives the whole volume generated by the semicircle

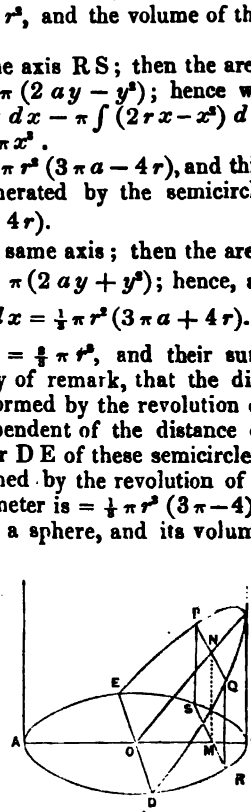
D B E $= \frac{1}{3}\,\pi\,r^2\,(3\,\pi\,a - 4\,r)$.

Let the semicircle D C E revolve round the same axis; then the area generated by M Q is $= \pi\,(a + y)^2 - \pi\,a^2 = \pi\,(2\,a\,y + y^2)$; hence, as before, we have $V = 2\,a\,\pi \int_{-r}^{+r} (2\,a\,y + y^2)\,dx = \frac{1}{3}\,\pi\,r^2\,(3\,\pi\,a + 4\,r)$.

The difference between these volumes is $= \frac{8}{3}\,\pi\,r^3$, and their sum $= 2\,\pi^2\,a\,r^2$, as it ought to be. It is worthy of remark, that the difference between the contents of the solids formed by the revolution of the exterior and interior semicircles is independent of the distance of the axis of rotation from the common diameter D E of these semicircles.

If $a = r$, then the volume of the solid formed by the revolution of a semicircle round a tangent parallel to the diameter is $= \frac{1}{3}\,\pi\,r^3\,(3\,\pi - 4)$; and if $a = 0$, then the solid of revolution is a sphere, and its volume $= \frac{4}{3}\,\pi\,r^3$, as it ought to be.

7. To find the volume of the solid, cut off from a right circular cylinder by a plane passing through the centre of the base, and inclined at an angle α to the plane of the base.



Let the solid be cut by a plane perpendicular to the base of the cylinder, and parallel to the trace D E; the section is a parallelogram P Q R S, and the solid may be considered as generated by the motion of this parallelogram parallel to itself.

Let $OB = r$, $OM = x$; then $SM = \sqrt{(r^2 - x^2)}$, $SP = MN = x \tan \alpha$, and the area of the generating parallelogram $PQRS = 2 \tan \alpha\, x \sqrt{(r^2 - x^2)}$;

$\therefore dV$, or element of the solid $= 2 \tan \alpha\, x\, dx \sqrt{(r^2 - x^2)}$;

$$\therefore V = 2 \tan \alpha \int (r^2 - x^2)^{\frac{1}{2}}\, x\, dx = -\frac{2}{3} \tan \alpha (r^2 - x^2)^{\frac{3}{2}} + C.$$

When $x = 0$, then $V = 0$, and the last equation gives $C = \frac{2}{3} \tan \alpha . r^3$, and the corrected integral is $V = \frac{2}{3} \tan \alpha \{r^3 - (r^2 - x^2)^{\frac{3}{2}}\}$,

and when $x = r$, the entire solid $BCED = \frac{2}{3} r^3 \tan \alpha$.

Examples for Practice.

1. Find the volume generated by the revolution of the *cissoid* round its asymptote. *Ans.* $V = 2 \pi^2 r^3$.

2. Find the volume generated by the revolution of the *witch* round its asymptote. *Ans.* $V = 4 \pi^2 r^3$.

3. Find the volume of the solid generated by the revolution of the cycloid round its base. *Ans.* $V = 5 \pi^2 r^3$.

4. If the cycloid revolve round its axis. *Ans.* $V = \pi r^3 \left(\frac{3\pi^2}{2} - \frac{8}{3}\right)$.

5. Find the volume of the solid generated by the revolution of a parabolic area round its ordinate.

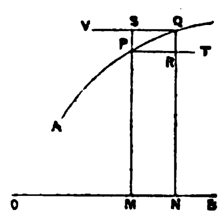
Ans. $V = \frac{8}{15} \pi a^2 b$, where a = the ordinate and b = the abscissa.

6. Determine the volume of an ellipsoid, its equation being

$$\frac{z^2}{c^2} + \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1. \qquad \textit{Ans. } V = \frac{4}{3} \pi abc.$$

IV. *Surfaces of Solids.*

125. Let APQ be a curve, and let it be required to determine the surface described by the revolution of AP about the axis OB. Draw PM and QN, making $MN = h$; and perpendiculars thereto draw PR, QS, and produce them till PT and QV be each equal to the arc PQ; then it is evident that the surface described by PQ is always less than the surface described by QV, and greater than that described by PT, however small MN or h may be taken. But QV and PT describe cylindrical surfaces, respectively equal to $2\pi QN \times PQ$ and $2\pi PM \times PQ$. Now if we put



$AP = s$, $AQ = s'$, $PM = y$, $QN = y'$, the surface described by $AP = S$, and that described by $AQ = S'$; then will

$$S' - S < 2\pi y' (s' - s), \text{ and } S' - S > 2\pi y (s' - s).$$

Now, since S is a function of x, and s is also a function of x, therefore,

$$S' = S + \frac{dS}{dx} h + \frac{d^2S}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3S}{dx^3} \cdot \frac{h^3}{1.2.3} + \ldots\ldots,$$

$$s' = s + \frac{ds}{dx} h + \frac{d^2s}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3s}{dx^3} \cdot \frac{h^3}{1.2.3} + \ldots\ldots,$$

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \ldots\ldots;$$

$$\therefore S' - S = \frac{dS}{dx}h + \frac{d^2S}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3S}{dx^3}\cdot\frac{h^3}{1.2.3} + \ldots\ldots\ldots\ldots(1),$$

$$2\pi y(s' - s) = 2\pi y\left(\frac{ds}{dx}h + \frac{d^2s}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3s}{dx^3}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \ldots\right)\ldots(2),$$

$$2\pi y'(s' - s) = 2\pi\left(y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \ldots\right)\frac{ds}{dx}h + \text{etc.}$$

$$= 2\pi y\left(\frac{ds}{dx} + \frac{d^2s}{dx^2}\cdot\frac{h}{1.2} + \text{etc.}\right)h + 2\pi\frac{dy}{dx}\left(\frac{ds}{dx} + \ldots\right)h^2 + \ldots\ (3).$$

But the series (1) is less than (3) and greater than (2); hence

$$\frac{dS}{dx} = 2\pi y\frac{ds}{dx} \text{ or } dS = 2\pi y\,ds;$$

$$\therefore S = 2\pi\int y\,ds = 2\pi\int y\,(dx^2 + dy^2)^{\frac{1}{2}}\ldots(A).$$

126. Since $dS = 2\pi y.ds$, and $2\pi y$ = the perimeter of the section of the solid through PM; therefore we have dS, or
element of the surface = the perimeter of the section through $PM \times ds$. Hence if the solid be generated by the motion of a plane parallel to itself, the surface may be found by a method similar to that employed for finding the volume of the solid. Thus, if u be the perimeter of the generating plane, and s the arc of the curve made by a plane perpendicular to the generating plane, then $S = \int u\,ds\ldots\ldots\ldots\ldots(B)$.

Examples.

1. To find the surface of a sphere.

The equation of the generating semicircle is $y^2 = 2rx - x^2$; therefore

$$dy = \frac{(r-x)\,dx}{\sqrt{(2rx - x^2)}} \text{ and } ds = \sqrt{(dx^2 + dy^2)} = \frac{r\,dx}{\sqrt{(2rx - x^2)}} = \frac{r\,dx}{y};$$

$$\therefore dS = 2\pi y\,ds = 2\pi r\,dx, \text{ and } S = 2\pi rx + C.$$

But $C = 0$; hence surface of segment $= 2\pi r.x$ = the circumference of a great circle multiplied by the height of the segment.

When $x = 2r$, the surface of the entire sphere $= 4\pi r^2$ = four times the area of a great circle, or equal to the circumference of a great circle multiplied by the diameter, and equal therefore to the curve surface of its circumscribed cylinder.

2. To find the surface of a right cone.

Let $AD = a$ (*fig. p.* 450), $DB = r$, the slant height $AB = b$, $AP = x$, $PN = y$, $AN = s$; then we have, by similar triangles,

$$DB : BA :: PN : AN, \text{ that is } r : b :: y : s = \frac{by}{r};$$

$$\therefore ds = \frac{b}{r}dy, \text{ and } S = 2\pi\int y\,ds = \frac{2\pi b}{r}\int y\,dy = \frac{\pi by^2}{r}.$$

When $y = r$, then the whole surface $= \pi rb$ = the semicircumference of the base, multiplied by the slant height.

3. To find the surface of the circular groin in *Ex.* 5, Art. 124.

Referring to the figure and notation in *Ex.* 5, we have the perimeter of the section F G H I $= 8y$; therefore dS, or element of surface

$$= 8y \times ds = 8y\sqrt{(dx^2+dy^2)} = 8y.\frac{a\,dx}{y} = 8a\,dx (\text{since } y^2 = 2ax - x^2);$$

$$\therefore\ S = 8ax, \text{ and when } x = a, \text{ the entire surface} = 8a^2.$$

4. To find the surface generated by a cycloid revolving round its axis.

The equation of the cycloid is $y = r \operatorname{vers}^{-1}\frac{x}{r} + \sqrt{(2rx - x^2)}$; hence

$$dy = \left(\frac{2r - x}{x}\right)^{\frac{1}{2}} dx, \text{ and } ds = (dx^2 + dy^2)^{\frac{1}{2}} = \left(\frac{2r}{x}\right)^{\frac{1}{2}} dx, \therefore s = 2(2rx)^{\frac{1}{2}}.$$

Hence $dS = 2\pi y\,ds$, and $S = 2\pi\int y\,ds$. Integrating by parts,

$$S = 2\pi y s - 2\pi\int s\,dy = 2\pi y s - 4\pi(2r)^{\frac{1}{2}}\int(2r - x)^{\frac{1}{2}}dx$$
$$= 2\pi y s + \frac{8\pi}{3}(2r)^{\frac{1}{2}}(2r - x)^{\frac{3}{2}} + C.$$

When $x = 0$, $s = 0$, and $S = 0$; therefore $C = -\frac{32\pi r^2}{3}$;

$$\text{and } S = 2\pi y s + \frac{8\pi}{3}(2r)^{\frac{1}{2}}(2r - x)^{\frac{3}{2}} - \frac{32\pi r^2}{3},$$

which, when $x = 2r$, gives $y = \pi r$, $s = 4r$, and the entire surface is

$$8\pi^2 r^2 - \frac{32\pi r^2}{3} = 8\pi r^2\left(\pi - \frac{4}{3}\right).$$

5. To find the surface of the solid in *Ex.* 7, *p.* 452.

Here the generating line is R Q, and if s denote the arc D R, then dS, or element of the surface $= RQ \times ds = x \tan\alpha \times \frac{r\,dx}{\sqrt{(r^2 - x^2)}}$;

$$\therefore\ S = r\tan\alpha\int(r^2 - x^2)^{-\frac{1}{2}}x\,dx = -r\tan\alpha\sqrt{(r^2 - x^2)} + C.$$

When $x = 0$, $S = 0$, then $0 = -r^2\tan\alpha + C$, and $C = r^2\tan\alpha$;

$$\text{therefore } S = r\tan\alpha\,\{r - \sqrt{(r^2 - x^2)}\}.$$

Take $x = r$, and double the result; then the whole surface $= 2r^2\tan\alpha$.

Examples for Practice.

1. Find the surface of a paraboloid. *Ans.* $S = \frac{8\pi a^{\frac{1}{2}}}{3}\left\{(x+a)^{\frac{3}{2}} - a^{\frac{3}{2}}\right\}$.

2. To determine the surface generated by a cycloid revolving round its base, and also the surface when it revolves round the axis.

Ans. $S = \frac{64}{3}\pi r^2$, round the base, and $S = 8\pi r^2\left(\pi - \frac{4}{3}\right)$, round the axis.

3. To determine the surface of a prolate spheroid.

Ans. $S = \frac{2\pi a b}{e}\{\sin^{-1}e + e\sqrt{(1 - e^2)}\}$.

4. To determine the surface of an oblate spheroid.

Ans. $S = 2\pi a^2\left\{1 + \frac{1 - e^2}{2e}\log\frac{1 + e}{1 - e}\right\}$.

END OF VOL. I.

137

LONDON:
Printed by WILLIAM CLOWES and SONS, Stamford Street,
For Her Majesty's Stationery Office.